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Functional CLT and Berry-Esseen estimates for non-homogeneous random walks

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FUNCTIONAL CLT AND BERRY-ESSEEN ESTIMATES FOR NON-HOMOGENEOUS RANDOM WALKS

EDUARDO SAMPAIO PIMENTA

Tese de Doutorado apresentada ao Colegiado da Pós-Graduação em Matemática da Universidade Federal da Bahia como requisito parcial para obtenção do título de doutor em Matemática.

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“Teorema Central do Limite Funcional e Estimativas de Berry-Esseen para Passeios Aleatórios Não-Homogêneos”

Eduardo Sampaio Pimenta

Tese apresentada ao Colegiado do Curso de Pós-graduação em Matemática da Universidade Federal da Bahia, como requisito parcial para obtenção do Título de Doutor em Matemática.

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Prof. Dr. Rangel Baldasso (PUC - RJ)

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“No matter. Try again. Fail again.
Fail Better”

–Samuel Beckett

Resumo

Nesta tese, estabelecemos um Teorema do tipo Trotter-Kato. Mais precisamente, caracterizamos a convergência em distibuição de processos de Feller examinando a convergência de seus geradores. A principal contribuição aqui está em obter estimativas de velocidade quantitativas na topologia vaga para tempos fixos. Como importante aplicação, e deduzimos um teorema central do limite funcional para passeios aleatórios na semi-reta positiva, o qual converge para movimentos Brownianos na semi-reta positiva com condições de fronteira, assim como passeios aleatórios na reta convergindo para o *Snapping Out Brownian motion*.

Palavras-chave: Teorema central do limite funcional, Geradores, estimativas de Berry-Esseen, movimento Browniano geral, Snapping Out Brownian motion

Abstract

In this thesis, we establish a Trotter-Kato type theorem. More precisely, we characterize the convergence in distribution of Feller processes by examining the convergence of their generators. The main contribution here is to obtain quantitative rate estimates in the vague topology for fixed times. As an important application, a central functional limit theorem is derived for random walks on the positive half-line, which converges to Brownian motions on the positive half-line with boundary conditions, as well as random walks on the line converging to the Snapping Out Brownian motion.

Keywords: Functional central limit theorem, Generator, Berry-Esseen estimates, general Brownian motion, snapping-out Brownian motion

Contents

Chapter 1

The General Framework

1.1 Introduction

The subject of functional central limit theorems (functional CLTs) originated from the now standard Donsker Theorem and Invariance Principles for Brownian motion. Since then, a substantial body of literature has emerged, focusing on invariance principles for various types of random walks (including those on random media) and their convergence to standard Brownian motion. However, the convergence to Brownian-type processes, such as the skew Brownian motion, sticky Brownian motion, elastic Brownian motion, and others, has received much less attention up to the present day.

As some rare examples, in 1981 [Harrison and Shepp, 1981], Harrison and Sheep showed the convergence of a specific queue to sticky Brownian motion. In 1991, Amir [Amir, 1991] established the convergence of rescaled discrete-time random walks with deterministic waiting times, also to sticky Brownian motion. More recently, in 2021, Erhard, Franco, and da Silva [Erhard et al., 2021] proved a functional central limit for the slow bond random walk. The limit of this process is given by the *snapping out Brownian motion*, a Brownian-type process first introduced in 2016 by Lejay [Lejay, 2016]. Somewhat related to the subject, in a recent paper [Kosygina et al., 2022], Kosygina, Mountford, and Peterson have shown the convergence of the one-dimensional cookie random walk (a walk that takes a decision based on the local time of the present position) towards what they called a *Brownian motion perturbed at extrema*, which is a stochastic process $W(t)$ solving a functional equation relating $W(t)$, its maxima and minima, and a standard Brownian motion.

In this work, we present a criterion to ensure a functional central limit theorem for Feller processes, based on the rate of convergence of their generators. This criterion comes with a corresponding Berry-Esseen type estimate. Convergence in distribution via the convergence of generators is not a new topic. A classical result in the book by Ethier and Kurtz [Ethier and Kurtz, 1986, Theorem 6.1 on page 28] can be stated as follows:

Theorem 1.1. [Ethier and Kurtz, 1986, Theorem 6.1, page 28] Consider a sequence $\{\mathsf{T}_\mathsf{n}(t) : t \geq 0\}$ of strongly continuous contraction semigroups, where each T_n is defined on a Banach space X_n and has generator L_n. Denote by π_n the projection of another Banach space X onto X_n where a strongly continuous semigroup T with generator L is defined. Then $\mathsf{T}_\mathsf{n}(t)\pi_n f$ converges to $\mathsf{T}(t)f$ for each t if and only if for every f in a core for L there exists f_n in a core for L_n such that $f_n \to f$ and $\mathsf{L}_\mathsf{n} f_n \to \mathsf{L} f$.

The convergence assumed in Theorem 1.1 is in the sense of Definition 1.1 which will be defined in sequence.

Our main result complements Theorem 1.1 and shows that a rate of convergence in the convergence of the generators implies a rate of convergence of the corresponding semigroups. This in turn will imply a rate of convergence in the vague topology of the law induced by T_n to the one induced by T. Our result therefore establishes a sort of *weak Berry-Esseen estimate*. The primary applications we have in mind are boundary problems, where we show that, in many cases, it is crucial that the function f_n does not necessarily coincide with $\pi_n f$ but can be chosen to address issues arising from the boundary conditions. As an interesting application of this general framework, we establish a functional central limit theorem for a wide class of random walks on the positive integers, which converge to the most general Brownian motion on the positive half-line, and we show as well that the slow bond random walk converges to the snapping-out Brownian motion in a simpler way than the one in the work or Erhard, Franco and Silva [Erhard et al., 2021].

This chapter and the next one are based on the work [Erhard et al., 2024] and the last chapter presents a toy model that gives insights into a new problem.

1.2 The Functional Setup

This section is reserved for presenting and rigorously proving the general framework used in the applications explored throughout this doctoral dissertation.

Before proceeding, let us fix some notation: Throughout the text, by $\mathbb{N}_0$, we mean the set of natural numbers with 0, that is $\{0, 1, 2, \dots\}$. Additionally, for any two functions f and g, by $f \lesssim g$ we mean that there exists a positive constant c such that $f(n) \leq cg(n)$ for every $n \in \mathbb{N}$. Finally, the constant does not depend on n but is allowed to depend on other parameters.

For each $n \in \mathbb{N}$ we consider $\mathfrak{B}_\mathfrak{n} := (\mathfrak{B}_\mathfrak{n}, \|\cdot\|)$ and $\mathfrak{B} := (\mathfrak{B}, \|\cdot\|)$ to be Banach spaces, where by $\|\cdot\|$ we mean their norm. For each $t \geq 0$ fixed, let $\mathsf{T}_\mathsf{n}(t) : \mathfrak{B}_\mathfrak{n} \to \mathfrak{B}_\mathfrak{n}$ and $\mathsf{T}(t) : \mathfrak{B} \to \mathfrak{B}$ strongly continuous contraction semigroups and we denote by L_n and L their infinitesimal generators with domains $\mathfrak{D}(\mathsf{L}_\mathsf{n})$ and $\mathfrak{D}(\mathsf{L})$ respectively. Finally, the notation $f \in \mathfrak{D}(\mathsf{L}^2)$ means that the element f and the generator applied to this element, $\mathsf{L}f$, both belong to the domain $\mathfrak{D}(\mathsf{L})$, formally, $\mathfrak{D}(\mathsf{L}^2) := \{f \in \mathfrak{D}(\mathsf{L}) : \mathsf{L}f \in \mathfrak{D}(\mathsf{L})\}$.

Define $\pi_n : \mathfrak{B} \to \mathfrak{B}_\mathfrak{n}$ to be a bounded linear operator indexed in $n \in \mathbb{N}$, here called a *natural projection*. Additionally, for each $n \in \mathbb{N}$, define $\Xi_n : \mathfrak{B} \to \mathfrak{B}_\mathfrak{n}$, a bounded family of linear operators, called *the correction operators*. Moreover, let $\|\cdot\|_{\text{OP}}$ denote the operator norm.

Definition 1.1. Let $\mathfrak{B}$ and $\mathfrak{B}_\mathfrak{n}$ Banach spaces, indexed in $n \in \mathbb{N}$. We say that $f_n \in \mathfrak{B}_\mathfrak{n}$ converges to $f \in \mathfrak{B}$ if $\|\pi_n f - f_n\| \to 0$ as $n \to \infty$.

We further introduce the operator $\Phi_n := \pi_n + \Xi_n$ which combines the natural projections and correction operators into a single mapping. Toward establishing the foundation for proving the main theorem, we consider the following set of hypotheses:

(A1) If $f \in \mathfrak{B}$, then $\Phi_n f \in \mathfrak{D}(\mathsf{L}_\mathsf{n})$;

(A2) There exist sequences $s_1(n) \downarrow 0$, $s_2(n) \downarrow 0$ and $s_3(n) \downarrow 0$ satisfying for any $f \in \mathfrak{D}(\mathsf{L}^2)$

$$\|\pi_n \mathsf{L}f - \mathsf{L}_\mathsf{n}\Phi_n f\| \leq s_1(n)\|f\| + s_2(n)\|\mathsf{L}f\| + s_3(n)\left\|\mathsf{L}^2 f\right\| ;$$

(A3) There exist sequences $r_1(n) \downarrow 0$ and $r_2(n) \downarrow 0$ such that, for any $f \in \mathfrak{D}(\mathsf{L})$, we have that

$$\|\Xi_n f\| \leq r_1(n)\|f\| + r_2(n)\|\mathsf{L}f\| .$$

As we will see next, these conditions provide an upper bound on the distance between the semigroups generated by L and $\mathsf{L_n}$. This result serves as a quantitative refinement of the work by [Trotter, 1958] and a later improvement in [Kato, 1978] where they establish equivalence between convergence of semigroups and convergence of generators.

Theorem 1.2. Under hypotheses (A1) - (A3), for any $f \in \mathfrak{D}\left(\mathsf{L}^2\right)$ and for any t in a compact interval $[0, b]$, we have that

$$\begin{aligned}\|\pi_n \mathsf{T}(t) f - \mathsf{T_n}(t)\pi_n f\| \lesssim \max\{s_1(n), r_1(n)\}\, \|f\| + \max\{s_2(n), r_1(n), r_2(n)\}\, \|\mathsf{L}f\| \\ + \max\{s_3(n), r_2(n)\}\, \left\|\mathsf{L}^2 f\right\| .\end{aligned}$$

where the constant of proportionality depends only on b.

Proof. Firstly, note that, from the conditions (A2) with (A3), we have for any $f \in \mathfrak{D}\left(\mathsf{L}^2\right)$

$$\begin{aligned}\|\Phi_n \mathsf{L} f - \mathsf{L_n}\Phi_n f\| &\le \|\Xi_n \mathsf{L} f\| + \|\pi_n \mathsf{L} f - \mathsf{L_n}\Phi_n f\| \\ &\le r_1(n)\, \|\mathsf{L}f\| + r_2(n)\, \left\|\mathsf{L}^2 f\right\| + s_1(n)\, \|f\| + s_2(n)\, \|\mathsf{L}f\| + s_3(n)\, \left\|\mathsf{L}^2 f\right\| \\ &\le s_1(n)\, \|f\| + 2\max\{r_1(n), s_2(n)\}\, \|\mathsf{L}f\| + 2\max\{s_3(n), r_2(n)\}\, \left\|\mathsf{L}^2 f\right\| .\end{aligned} \tag{1.2.1}$$

Fix $t \ge 0$ and define $g_{s,t} := \mathsf{T}(t-s)f$ for $0 \le s \le t$. Taking the derivative with respect to s, it follows that

$$\partial_s g_{s,t} = \partial_s \mathsf{T}(t-s) f = -\mathsf{L}\mathsf{T}(t-s) f = -\mathsf{L} g_{s,t} . \tag{1.2.2}$$

Since $\{\mathsf{T}(t) : t \ge 0\}$ is a semigroup, we have that

$$\Phi_n f = \Phi_n \mathsf{T}(0) f = \Phi_n g_{s,s} .$$

From equation (1.2.2) and the assumption that Φ_n is a bounded operator, one can check that

$$\begin{aligned}\mathsf{T_n}(t)\Phi_n f &= \mathsf{T_n}(0)\Phi_n g_{0,t} + \int_0^t \partial_s \mathsf{T_n}(s)\Phi_n g_{s,t}\, ds \\ &= \mathsf{T_n}(0)\Phi_n g_{0,t} + \int_0^t \left[\mathsf{L_n}\mathsf{T_n}(s)\Phi_n g_{s,t} - \mathsf{T_n}(s)\Phi_n \mathsf{L} g_{s,t}\right] ds\end{aligned}$$

where the first identity follows from The Fundamental Theorem of Calculus while the second one is due to the chain rule. Also note that the hypothesis (A1) ensures that $\mathsf{L_n}\Phi_n g_{s,t}$ is well defined.

Bearing the above equation in mind, the semigroups in consideration, $\{\mathsf{T}(t) : t \ge 0\}$ and $\{\mathsf{T_n}(t) : t \ge 0\}$ are contraction semigroups, that is, for every $t > 0$,

$$\|\mathsf{T}(t)\|_{\mathrm{OP}} \le 1 , \quad \text{and} \quad \|\mathsf{T}(t)\|_{\mathrm{OP}} \le 1 ,$$

and since $\mathsf{T}(t)\mathsf{L} = \mathsf{L}\mathsf{T}(t)$ and $\mathsf{T_n}(t)\mathsf{L_n} = \mathsf{L_n}\mathsf{T_n}(t)$ for all $n \in \mathbb{N}$, and in addition considering equation (1.2.1), for all $t \geq 0$ and for every $f \in \mathfrak{D}(\mathsf{L}^2)$ it follows

$$
\begin{aligned}
\|\mathsf{T}_\mathsf{n}(t)\Phi_n f - \Phi_n \mathsf{T}(t)f\| &= \left\| \int_0^t \mathsf{T}_\mathsf{n}(s)\left[\mathsf{L}_\mathsf{n}\Phi_n g_{s,t} - \Phi_n \mathsf{L} g_{s,t}\right] ds \right\| \\
&\leq \int_0^t \|\mathsf{T}_\mathsf{n}(s)\|_{\mathsf{OP}} \cdot \|\mathsf{L}_\mathsf{n}\Phi_n g_{s,t} - \Phi_n \mathsf{L} g_{s,t}\| \, ds \\
&\leq \int_0^t \left[s_1(n)\|\mathsf{T}(t-s)f\| + 2\max\{r_1(n), s_2(n)\}\|\mathsf{L}\mathsf{T}(t-s)f\|\right] ds \\
&\qquad + 2\int_0^t \max\{s_3(n), r_2(n)\}\left\|\mathsf{L}^2\mathsf{T}(t-s)f\right\| ds \\
&\leq \int_0^t \left[s_1(n)\|\mathsf{T}(t-s)f\| + 2\max\{r_1(n), s_2(n)\}\|\mathsf{T}(t-s)\mathsf{L}f\|\right] ds \\
&\qquad + 2\int_0^t \max\{s_3(n), r_2(n)\}\left\|\mathsf{T}(t-s)\mathsf{L}^2 f\right\| ds \\
&\leq 2t\left(s_1(n)\|f\| + \max\{s_2(n), r_1(n)\}\|\mathsf{Lf}\| + \max\{s_3(n), r_2(n)\}\left\|\mathsf{L}^2 f\right\|\right). \qquad (1.2.3)
\end{aligned}
$$

Finally, recalling that $\Phi_n = \pi_n + \Xi_n$, from the triangle inequality it follows that

$$
\begin{aligned}
\|\mathsf{T}_\mathsf{n}(t)\pi_n f - \pi_n \mathsf{T}(t)f\| &= \|\mathsf{T}_\mathsf{n}(t)\left(\Phi_n - \Xi_n\right)f - \left(\Phi_n - \Xi_n\right)\mathsf{T}(t)f\| \\
&\leq \|\mathsf{T}_\mathsf{n}(t)\Phi_n f - \Phi_n \mathsf{T}(t)f\| + \|\mathsf{T}_\mathsf{n}(t)\Xi_n f\| + \|\Xi_n \mathsf{T}(t)f\|. \qquad (1.2.4)
\end{aligned}
$$

Again, since $\{\mathsf{T}_\mathsf{n}(t) : t \geq 0\}$ and $\{\mathsf{T}(t) : t \geq 0\}$ are contraction semigroups, by invoking hypothesis (A3) one can check that

$$
\begin{aligned}
\|\mathsf{T}_\mathsf{n}(t)\Xi_n f\| + \|\Xi_n \mathsf{T}(t)f\| &\leq \|\mathsf{T}_\mathsf{n}(t)\|_{\mathsf{OP}}\|\Xi_n f\| + \|\Xi_n \mathsf{T}(t)f\| \\
&\leq r_1(n)\|f\| + r_2(n)\|\mathsf{L}f\| + r_1(n)\|\mathsf{T}(t)f\| + r_2(n)\|\mathsf{L}\mathsf{T}(t)f\| \\
&\leq r_1(n)\|f\| + r_2(n)\|\mathsf{L}f\| + r_1(n)\|\mathsf{T}(t)\|_{\mathsf{OP}}\|f\| + r_2(n)\|\mathsf{T}(t)\mathsf{L}f\| \\
&\leq 2r_1(n)\|f\| + r_2(n)\|\mathsf{L}f\| + r_2(n)\|\mathsf{T}(t)\|_{\mathsf{OP}}\|\mathsf{L}f\| \\
&\leq 2\left(r_1(n)\|f\| + r_2(n)\|\mathsf{L}f\|\right). \qquad (1.2.5)
\end{aligned}
$$

Thus, by gluing (1.2.3), (1.2.4) and (1.2.5), the result follows. □

1.3 The Probability Setup

So far, we have a result on functional analysis. From this point onward, we restrict our focus to the setting of probability measures. Let us then define the space of trajectories in which the stochastic processes will take place.

Consider (S, ρ) to be a separable locally compact (but possibly not complete) metric space and consider a function $f : \mathsf{S} \to \mathbb{R}$. For us, by $\lim_{x\to\infty} f(x) = 0$ we mean that, for any fixed $x_0 \in \mathsf{S}$,

$$
\lim_{x:\rho(x,x_0)\to\infty} f(x) = 0\,. \qquad (1.3.1)
$$

In other words, if a point is arbitrarily far from any fixed point, the function evaluated at this point vanishes. It is important to note that, in this context, the choice of x_0 is not relevant.

Denote by $\bar{\mathsf{S}}$ the completion of the metric space (S, ρ) with respect to the metric ρ, and by $\partial\mathsf{S} := \bar{\mathsf{S}} \setminus \mathsf{S}$ we mean the boundary of the completion. Let Δ denote an extra point isolated from S such that the distance between Δ and any other $x \in \mathsf{S}$ is positive, for instance, $\rho(\Delta, x) \geq 1$. We call it the *cemetery* point of S and such point has the property that any test function vanishes when evaluated at it.

From now on, we are interested on the following function subspace:

Definition 1.2. We denote by $\mathcal{C}_0(\mathsf{S})$ the space of functions $f : \bar{\mathsf{S}} \cup \Delta \to \mathbb{R}$ vanishing at infinity, if f satisfy that

- $\lim_{x\to\infty} f(x) = 0\,,$
- for any $x_0 \in \partial\mathsf{S}$, it holds that $\lim_{x\to x_0} f(x) = 0\,,$
- $f(\Delta) = 0.$

Remark 1.1. Observe that Definition 1.2 ensures that f is continuous over $\bar{\mathsf{S}} \cup \Delta$ and therefore, measurable. Indeed, for any point on the boundary, every function belonging to that set vanishes, and as a result, the isolated point does not generate any jump for those functions.

To formally state the theorem concerning the space of subprobability measures, we introduce additional hypotheses that serve as the foundation for constructing a base for the topology in which our analysis will take place.

(B1) From now on, the Banach space $\mathfrak{B}$ is given by $\mathcal{C}_0(\mathsf{S})$ equipped with the uniform topology and the natural projection π_n is the restriction to a subset S_n of S, that is, $\pi_n f = f|_{\mathsf{S}_n}$.

(B2) There is a sequence of functions $\{f_k : k \geq 0\}$ in $\mathcal{C}_0(\mathsf{S})$ such that span$(\{f_k : k \geq 0\})$ is dense. Moreover, for each $k \in \mathbb{N}$, one can find a sequence $\{f_{j,k} : j, k \geq 0\} \subset \mathfrak{D}\left(\mathsf{L}^2\right)$ such that $f_{j,k} \to f_k$ when $j \to \infty$, in the uniform topology.

(B3) There exist sequences $h_1(j)$ and $h_2(j)$ such that, for all $k, j \geq 0$,

$$\|\mathsf{L} f_{k,j}\| \leq h_1(j)\,\|f_k\| \quad \text{and} \quad \left\|\mathsf{L}^2 f_{k,j}\right\| \leq h_2(j)\,\|f_k\| \tag{1.3.2}$$

satisfying that

$$\sum_{j\geq 0} \frac{h_i(j)}{2^j} < \infty, \qquad \text{for } i = 1, 2\,. \tag{1.3.3}$$

Additionally, the sequences $\{f_k : k \geq 0\}$ and $\{f_{k,j} : k, j \geq 0\}$ satisfy

$$\sum_{j\geq 0} \frac{\|f_k\|}{2^k} < \infty\,, \tag{1.3.4}$$

and

$$\sum_{j,k\geq 0} \frac{\|f_{j,k}\|}{2^{k+j}} < \infty\,. \tag{1.3.5}$$

Remark 1.2. Let us briefly summarize the conditions outlined above:

- The first condition (B1), establishes a connection between the discrete and continuous spaces, specifically by considering a sequence of embedded discrete spaces that converge toward the continuous counterpart.
- The second condition, (B2), will play an important role in defining the metric we will use on the space of sub-probability measures, as we shall see.

- Finally, for the last condition (B3), observe that in (1.3.2), the sequences $h_1(j)$ and $h_2(j)$ do not depend on $k \in \mathbb{N}$. Therefore, applying the generator ensures uniformity in $k \in \mathbb{N}$. Moreover, these sequences are well-behaved in the sense that they do not grow too fast, as explicitly shown in (1.3.4) and (1.3.5). The key point here is to ensure that span$(\{f_k : k \geq 0\})$ is dense, and in particular, by normalizing the functions f_k, we can always guarantee that (1.3.4) holds. It is noteworthy that (1.3.5) is a direct consequence of (1.3.2), (1.3.3) and (1.3.4). This structure is used to simplify the presentation later on.

Lemma 1.1. Suppose that condition (B2) holds. Then the space $\mathfrak{D}\left(\mathsf{L}^2\right)$ is dense on $\mathcal{C}_0(\mathsf{S})$.

Proof. Let $f \in \mathcal{C}_0(\mathsf{S})$. From the hypothesis (B2), there exists $\{f_k : k \geq 0\} \subset \mathcal{C}_0(\mathsf{S})$ such that the space generated by it, span$(\{f_k : k \geq 0\})$, is dense in $\mathcal{C}_0(\mathsf{S})$. Fix an arbitrary $\varepsilon > 0$. Then, there exist constants $\{a_k\}_{k=1}^N$ such that

$$\left\| f - \sum_{k=1}^N a_k f_k \right\| \leq \frac{\varepsilon}{2}.$$

Again from hypothesis (B2), for every $k \in \{1, \cdots, N\}$ fixed, there exists $j_0 := j_0(k, \varepsilon)$ such that for every $j \geq j_0$ it holds $\|f_k - f_{k,j}\| \leq \frac{\varepsilon}{2N}$. Define $g_j := \sum_{k=1}^N a_k f_{k,j} \in \mathfrak{D}\left(\mathsf{L}^2\right)$. Thus, for $j > j_0$

$$\|f - g_j\| \leq \left\| f - \sum_{k=1}^N a_k f_k \right\| + \left\| \sum_{k=0}^N a_k [f_k - f_{k,j}] \right\|$$

This shows the density of $\mathfrak{D}\left(\mathsf{L}^2\right)$ in $\mathcal{C}_0(\mathsf{S})$. □

Let us now introduce a suitable metric on the space of sub-probability measures. Let $\mathcal{B}$ denote the Borel σ-algebra on (S, d), and let $\mathcal{M}_{\leq 1}(\mathsf{S})$ represent the set of all sub-probability measures defined on the measurable space $(\mathsf{S}, \mathcal{B})$.

Definition 1.3. let $\{f_k : k \geq 0\} \subset \mathcal{C}_0(\mathsf{S})$ and $\{f_{j,k} : j, k \geq 0\} \subset \mathfrak{D}\left(\mathsf{L}^2\right)$ be as in (B2). We define

$$\mathrm{d}(\mu, \nu) := \sum_{j,k \geq 0} \frac{1}{2^{j+k}} \left(\left| \int f_{j,k} \mathrm{d}\mu - \int f_{j,k} \mathrm{d}\nu \right| \wedge 1 \right)$$

for any $\mu, \nu \in \mathcal{M}_{\leq 1}(\mathsf{S})$.

Remark 1.3. At first glance, defining the distance d using a doubly indexed sequence $\{f_{k,j} : k, j \geq 0\}$ instead of the more common singly indexed sequence might seem unconventional. However, this notation aligns more naturally with our intended applications. Additionally, we emphasize that each $f_{k,j}$ belongs to the domain of L^2, a detail that will be essential in the forthcoming arguments.

Proposition 1.1. Assume that S is a Polish space. The function $\mathrm{d} : \mathcal{M}_{\leq 1}(\mathsf{S}) \times \mathcal{M}_{\leq 1}(\mathsf{S}) \to [0, \infty)$ defined in Definition 1.3 is a metric on the set $\mathcal{M}_{\leq 1}(\mathsf{S})$ and convergence with respect to the metric d is equivalent to vague convergence. Moreover, if $\lim_{n\to\infty} \mathrm{d}(\mu_n, \mu) = 0$ and μ is a probability measure, then μ_n converges weakly to μ.

Proof. It is immedate that d is a metric. Moreover, from Lemma 1.1, we have that $\mathfrak{D}\left(\mathsf{L}^2\right)$ is dense in $\mathcal{C}_0(\mathsf{S})$, it follows that convergence under d implies convergence on the vague topology. Indeed, for any $f \in \mathcal{C}_0(\mathsf{S})$, from the density, given $\varepsilon > 0$, there exists a function $g \in \mathfrak{D}\left(\mathsf{L}^2\right)$ such that $\|f - g\| < \varepsilon$. Thus

$$\begin{aligned}\left|\int_{\mathsf{S}} f \mathrm{d}\mu_n - \int_{\mathsf{S}} f \mathrm{d}\mu\right| &= \left|\int_{\mathsf{S}} f \mathrm{d}\mu_n - \int_{\mathsf{S}} g \mathrm{d}\mu_n\right| + \left|\int_{\mathsf{S}} g \mathrm{d}\mu_n - \int_{\mathsf{S}} g \mathrm{d}\mu\right| + \left|\int_{\mathsf{S}} g \mathrm{d}\mu - \int_{\mathsf{S}} f \mathrm{d}\mu\right| \\ &\leq \int_{\mathsf{S}} |f - g| \mathrm{d}\mu_n + \left|\int_{\mathsf{S}} g \mathrm{d}\mu_n - \int_{\mathsf{S}} g \mathrm{d}\mu\right| + \int_{\mathsf{S}} |g - f| \mathrm{d}\mu \\ &\leq \mu_n(\mathsf{S}) \cdot \varepsilon + \mu(\mathsf{S}) \cdot \varepsilon + \left|\int_{\mathsf{S}} g \mathrm{d}\mu_n - \int_{\mathsf{S}} g \mathrm{d}\mu\right|\end{aligned}$$

since $\mu_n(\mathsf{S}) \leq 1$ and $\mu(\mathsf{S}) = 1$, from the hypothesis, that is, $\mathrm{d}(\mu_n, \mu) \to 0$, the result holds. Conversely, since we are working with vague topology and $f_{j,k} \in \mathcal{C}_0(\mathbb{S})$, the convergence under d holds.

Now, suppose that we have the convergence over d. Therefore we have convergence under vague topology. Since S is complete and separable, a single probability measure is tight in S, see [Billingsley, 1999, Theorem 1.4, page 10]. Thus, for any $\varepsilon > 0$, there exists a compact $\mathsf{K} := \mathsf{K}(\varepsilon) \subset \mathsf{S}$ such that $\mu(\mathsf{K}) > 1 - \frac{\varepsilon}{2}$.

Fix $f \in \mathcal{C}_b(\mathsf{S})$ a bounded continuous function. Given $\delta > 0$, define $g := g_\delta$ as follows

$$g(x) := f(x) \cdot \left[1 - \tfrac{\rho(x,\mathsf{K})}{\delta}\right]^+ \in \mathcal{C}_0(\mathbb{S}),$$

where by m^+ we mean the positive part of the m. Thus

$$\begin{aligned}\left|\int_{\mathsf{S}} f \, \mathrm{d}\mu_n - \int_{\mathsf{S}} f \mathrm{d}\mu\right| &\leq \left|\int_{\mathsf{S}} f \mathrm{d}\mu_n - \int_{\mathsf{S}} g \mathrm{d}\mu_n\right| + \left|\int_{\mathsf{S}} g \mathrm{d}\mu_n - \int g \mathrm{d}\mu\right| + \left|\int_{\mathsf{S}} g \mathrm{d}\mu - \int_{\mathsf{S}} f \mathrm{d}\mu\right| \\ &\leq \int_{\mathsf{S}} |(f - g)| \, \mathrm{d}\mu_n + \left|\int g(\mathrm{d}\mu - \mathrm{d}\mu_n)\right| + \int_{\mathsf{S}} |(g - f)| \, \mathrm{d}\mu \\ &\leq 2\,\|f\| \cdot [\mu_n(\mathsf{K}^c) + \mu(\mathsf{K}^c)] + \left|\int_{\mathsf{S}} g(\mathrm{d}\mu - \mathrm{d}\mu_n)\right|.\end{aligned}$$

Observe that the last term vanishes as n goes to infinity because we have assumed convergence under d and consequently, we have vague convergence.

To finish the proof, we must check that there is no escape of mass, that is, μ_n concentrate, almost all its mass, inside a compact set, say K.

Define

$$h(x) := \left[1 - \tfrac{\rho(x,\mathsf{K})}{\delta}\right]^+$$

and $\mathsf{K}^\delta := \{y : \rho(\mathsf{K}, y) < \delta\}$. Since h is also in $\mathcal{C}_0(\mathsf{S})$, there exists $n_0 := n_0(\varepsilon)$ such that, for $n \geq n_0$,

$$\begin{aligned}\frac{\varepsilon}{2} \geq \left|\int_{\mathsf{S}} h \mathrm{d}\mu - \int_{\mathsf{S}} h \mathrm{d}\mu_n\right| &\geq \left|\int_{\mathsf{K}^\delta} h \mathrm{d}\mu\right| - \left|\int_{\mathsf{K}^\delta} h \mathrm{d}\mu_n\right| \\ &\geq \int \mathbb{1}_{\mathsf{K}^\delta} \mathrm{d}\mu - \int_{\mathsf{K}} \mathrm{d}\mu_n \\ &= \mu(\mathsf{K}^\delta) - \mu_n(\mathsf{K}),\end{aligned}$$

and hence, $\frac{\varepsilon}{2} - \mu(\mathsf{K}^\delta) \geq -\mu_n(\mathsf{K})$. Summing 1 and making $\delta \to 0$, we achieve $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} \geq \mu_n(\mathsf{K}^c)$, and we conclude the proof. □

We state now our last hypothesis which ensures regularity at the initial time:

(B4) The semigroup T associated to the generator L is Lipschitz in the sense that, for each $t>0$, there exists a constant $M:=M(t)>0$ such that

$$|\mathsf{T}(t)f(x)-\mathsf{T}(t)f(y)|\leq M\cdot\|f\|\,\mathbf{d}(x,y),\quad \forall x,y\in\mathsf{S}\qquad \forall f\in\mathcal{C}_0(\mathsf{S})\,.$$

We are now prepared to present the first principal result of this work. Recall that $\mathsf{S}_n\subset\mathsf{S}$. This result is referred to as the *weak Berry-Esseen* estimate due to its reliance on the vague topology. Specifically, it establishes rates of convergence within the framework of a weaker topological structure.

Theorem 1.3 (Berry-Esseen estimate with respect to d and functional CLT)**.** Assume hypotheses (A1) - (A3) and (B1) - (B4). Let $\{\mathsf{X}(t):t\in[0,T]\}$ and $\{\mathsf{X}_\mathsf{n}(t):t\in[0,T]\}$ be Feller processes on S and S_n associated to the generators L and L_n, respectively, assumed to start at the points $x_n\in\mathsf{S}_n$ and $x\in\mathsf{S}$, respectively, where $d(x_n,x)\leq i(n)$ for some function $i(n)$ such that $i(n)\downarrow 0$.

Fix some time $t>0$ and denote by μ and μ_n the probability distribution on $\bar{\mathsf{S}}\cup\Delta$ induced by $\mathsf{X}(t)$ and $\mathsf{X}_n(t)$, respectively, starting from the points $x_n\in\mathsf{S}_n$ and $x\in\mathsf{S}$. Then

$$\mathbf{d}(\mu_n,\mu)\lesssim\max\{i(n),s_1(n),s_1(n),s_2(n),s_3(n),r_1(n),r_2(n)\}\tag{1.3.6}$$

Moreover, we also have pathwise convergence $\mathsf{X}_n\Rightarrow\mathsf{X}$ in the Skorohod $\mathsf{D}_\mathsf{S}[0,\infty)$.

Remark 1.4. The main novelty in the statement above is the *weak Berry-Esseen* estimate (1.3.6). The convergence in the Skorohod space $\mathsf{D}_\mathsf{S}[0,\infty)$ is actually a corolary of (1.3.6) and [Ethier and Kurtz, 1986, Theorem 2.11, page 172] as we will see.

Proof. Let $\{f_k:k\geq 0\}\subset\mathcal{C}_0(\mathsf{S})$ be a dense family and, for each k fixed, consider the sequence $f_{j,k}\in\mathfrak{D}\left(\mathsf{L}^2\right)$ such that $f_{j,k}\to f_k$ whenever $j\to\infty$ in the uniform topology satisfying hypothesis (B2). To simplify, denote by

$$\begin{aligned}a(n)&=\max\{r_1(n),s_1(n)\}\,,\\ b(n)&=\max\{s_1(n),r_1(n),r_2(n)\}\,,\\ c(n)&=\max\{s_2(n),r_2(n)\}\,.\end{aligned}$$

Firstly, note that

$$\begin{aligned}&\sum_{j,k\geq 0}\frac{1}{2^{j+k}}\Big(a(n)\,\|f_k\|+b(n)h_1(n)\,\|f_k\|+c(n)h_2(n)\,\|f_k\|\Big)\\ &\quad=a(n)\sum_{j,k\geq 0}\frac{\|f_k\|}{2^{j+k}}+b(n)\sum_{j,k\geq 0}\frac{h_1(j)}{2^j}\frac{\|f_k\|}{2^k}+c(n)\sum_{j,k\geq 0}\frac{h_2(j)}{2^j}\frac{\|f_k\|}{2^k}\\ &\quad=a(n)\sum_{j,k\geq 0}\frac{\|f_k\|}{2^{j+k}}+b(n)\sum_{j\geq 0}\frac{h_1(j)}{2^j}\sum_{k\geq 0}\frac{\|f_k\|}{2^k}+c(n)\sum_{j\geq 0}\frac{h_2(j)}{2^j}\sum_{k\geq 0}\frac{\|f_k\|}{2^k}\\ &\quad\overset{(1.3.3),(1.3.4),(1.3.5)}{\lesssim}\max\{a(n),b(n),c(n)\}\,.\end{aligned}\tag{1.3.7}$$

Also, it is noteworthy that

$$\begin{aligned}\left|\int f_{k,j}\mathbf{d}\mu_n-\int f_{k,j}\mathbf{d}\mu\right|&=|\mathsf{T}_\mathsf{n}(t)\pi_n f_{j,k}(x_n)-\mathsf{T}(t)f_{j,k}(x)|\\ &\leq|\mathsf{T}_\mathsf{n}(t)\pi_n f_{j,k}(x_n)-\pi_n\mathsf{T}(t)f_{j,k}(x_n)|+|\pi_n\mathsf{T}(t)f_{j,k}(x_n)-\mathsf{T}(t)f_{j,k}(x)|\\ &\overset{(B4)}{\lesssim}\max\left\{i(n),\|\mathsf{T}_\mathsf{n}(t)\pi_n f_{j,k}-\pi_n\mathsf{T}(t)f_{j,k}\|\right\}\,.\end{aligned}\tag{1.3.8}$$

Let us obtain now, the Berry-Esseen estimate for this convergence:

$$
\begin{aligned}
\mathbf{d}(\mu_n,\mu) &= \sum_{j,k\geq 0} \frac{1}{2^{k+j}} \left(\left| \int f_{k,j} \mathbf{d}\mu_n - \int f_{k,j} \mathbf{d}\mu \right| \wedge 1 \right) \\
&\overset{(1.3.8)}{\lesssim} \max\left\{ i(n), \sum_{j,k\geq 0} \frac{1}{2^{j+k}} \Big(\|\mathsf{T}_\mathsf{n}(t)\pi_n f_{j,k} - \pi_n \mathsf{T}(t) f_{j,k}\| \wedge 1 \Big) \right\} \\
&\overset{\text{Thm } 1.2}{\lesssim} \max\left\{ i(n), \sum_{j,k\geq 0} \frac{1}{2^{j+k}} \Big(a(n)\,\|f_{j,k}\| + b(n)\,\|\mathsf{L} f_{j,k}\| + c(n)\,\big\|\mathsf{L}^2 f_{j,k}\big\| \Big) \wedge 1 \right\} \\
&\overset{(B3)}{\lesssim} \max\left\{ i(n), \sum_{j,k\geq 0} \frac{1}{2^{j+k}} \Big(a(n)\,\|f_k\| + b(n) h_1(j)\,\|f_k\| + c(n) h_2(j)\,\|f_k\| \Big) \wedge 1 \right\} \\
&\overset{(1.3.7)}{\lesssim} \max\{i(n), a(n), b(n), c(n)\}\,.
\end{aligned}
$$

Finally, since $X_n(0) \Rightarrow X(0)$ and $\mathbf{d}(\mu_n,\mu) \to 0$ as $n \to 0$, the convergence $X_n \Rightarrow X$ in the Skorohod topology is a consequence of [Ethier and Kurtz, 1986, Theorem 2.11, page 173] and we can conclude the proof. □

Chapter 2

A functional Central Limit Theorem for the General Brownian motion on the half-line

2.1 Introduction

This chapter is dedicated to obtaining a Donsker-type theorem for what is known in the literature as the general Brownian motion on the half line and guarantees convergence in the Skorohod space. The most general Brownian motion on the positive half-line was studied by Feller, and a comprehensive overview on it can be found in Knight's book, see [Knight, 1981, Theorem 6.2, p. 157]. To put it simply, it is defined as a class of Feller processes on the positive half line such that its excursions to zero are the same as those of a standard Brownian motion, which can be shown to be a mixture of the reflected Brownian motion, absorbed Brownian motion, and killed Brownian motion. Its generator is given by one half of the continuous Laplacian acting on the domain of $\mathcal{C}^2$-functions decaying at infinity and satisfying $c_1 f(0) - c_2 f'(0) + \frac{c_3}{2} f''(0) = 0$ with $c_1 + c_2 + c_3 = 1$, $c_i \geq 0$. Given three non-negative parameters c_1, c_2, c_3 that sum to one, we also occasionally denote by $B(c_1, c_2, c_3)$ the corresponding general Brownian motion.

The discrete class of models considered here involves continuous-time random walks on $(\frac{1}{N}\mathbb{N}) \cup \{\Delta\}$, where $\mathbb{N} = 0, 1, \ldots$ and the state Δ is usually referred to as the *cemetery*. The walk follows the usual symmetric walk on $1, 2, \ldots$ with jump rates to nearest neighbors everywhere equal to $1/2$. However, at state 0, we introduce the following: the rate to jump to state 1 is A/n^{α}, and the rate to jump to the cemetery is B/n^{β}, where $\alpha, A, \beta, B \geq 0$, and n is the scaling parameter. Additionally, if the walk reaches the cemetery, it remains there indefinitely.

The chosen values of parameters α, A, β, B then determine the limiting Brownian-type process of the random walk. We show here that for any choice of $c_1, c_2, c_3 \geq 0$ such that $c_1 + c_2 + c_3 = 1$ and $c_1 \neq 1$ there are classes of choices of α, A, β, B such that the above random walk converges to $B(c_1, c_2, c_3)$. We additionally show that by making a small shift to the right of the random walk, we have convergence to the killed Brownian motion, which corresponds to $c_1 = 1$.

Each type of Brownian motion has different properties and behaviors, which makes them useful in different applications. For example, reflected Brownian motion can model a financial asset that cannot have negative values, while absorbed Brownian motion can model the extinction of a biological population. Elastic Brownian motion can model the behavior of a particle that is attracted to 0 but has long-range repulsion from some boundary point and finally, sticky Brownian motion can model a particle that sticks to a point, killed Brownian motion corresponds to a continuous walk that jumps directly to the cemetery when it gets arbitrarily close to the origin. All of these processes are particular cases of the above-mentioned general Brownian motion on the half-line

2.2 Model and further notation

Denote by $\mathsf{B}_0(t)$ the Brownian motion on $[0,\infty)$ absorbed upon reaching zero. By Δ we represent the cemetery state, that is, for any test function, we have that $f(\Delta)=0$. Finally, by $\mathfrak{T}_0$ we mean the hitting time of zero.

Definition 2.1. [Knight, 1981, page 153] A *general Brownian motion* on the positive half-line is a diffusion process W on the set $\mathbb{G}:=\{\Delta\}\cup[0,\infty)$ such that the absorbed process $\{\mathsf{W}(t\wedge\mathfrak{T}_0): t\wedge\mathfrak{T}_0\geq 0\}$ on $[0,\infty)$ has the same distribution as $\mathsf{B}_0(t)$ for any starting point $x\geq 0$.

It is important to note that the definition specifies the process behavior as an absorbed Brownian motion upon reaching zero. Beyond this point, the process ceases to encode any meaningful information. In other words, once the process reaches this state, its subsequent behavior becomes irrelevant, allowing it to follow any trajectory, including vanishing entirely.

To gain a deeper understanding of the general Brownian motion, a characterization of its properties was established by analyzing its behavior at certain boundaries, specifically through the study of the generator associated to the process.

Theorem 2.1. [Knight, 1981, Theorem 6.2, page 157] Any general Brownian motion W on $[0,\infty)$ has generator $\mathsf{L}:=\frac{1}{2}\frac{d^2}{dx^2}$ with corresponding domain

$$\mathfrak{D}\left(\mathsf{L}^2\right):=\left\{f\in\mathcal{C}_0^2(\mathbb{G}): f''\in\mathcal{C}_0(\mathbb{G}) \text{ and } c_1 f(0)-c_2 f'(0)+\frac{c_3}{2}f''(0)=0\right\} \tag{2.2.1}$$

for some $c_i\geq 0$ such that $c_1+c_2+c_3=1$ and $c_1\neq 1$.

We will now discuss the above process:

- The case $c_2=1$ corresponds to the *reflected Brownian motion* which has the same distribution as the modulus of a standard Brownian motion.
- On the other hand, $c_3=1$ represents the *absorbed Brownian motion*, characterized by the distribution of a standard Brownian motion stopped upon reaching zero.
- The case $c_1=1$, represents the *killed Brownian motion*.

As indicated above, the scenario $c_1=1$ is excluded from Feller's Theorem 2.1, which explicitly requires $c_1\neq 1$. Indeed, this exclusion arises because, for $c_1=1$ the domain (2.2.1) is not a dense set in $\mathcal{C}_0(\mathbb{G})$, where $\mathbb{G}=\{\Delta\}\cup[0,\infty)$. Consequently, it cannot be the domain of a generator. However, if we remove the origin, considering $\mathbb{G}_0=\{\Delta\}\cup(0,\infty)$ instead, it does define a Feller process because now the set of test functions is assumed to converge to zero at the origin (c.f. Definition 1.2 and refer to [Chung and Zhao, 1995, Chapter 2] for details).

The killed BM can be interpreted as a process that jumps immediately to the cemetery state Δ upon "reaching the origin". Actually, it never actually reaches the origin but approaches it arbitrarily closely. This distinction explains why $c_1=1$ is not included in Theorem 2.1: since the killed BM does not touch zero, it cannot satisfy the condition of Definition 2.1.

Although the absorbed and killed Brownian motions are distinct processes, they share similarities in their nature. For the sake of clarity and improved exposition, we will treat the killed Brownian motion as a particular case of the general Brownian motion on the positive half-line.

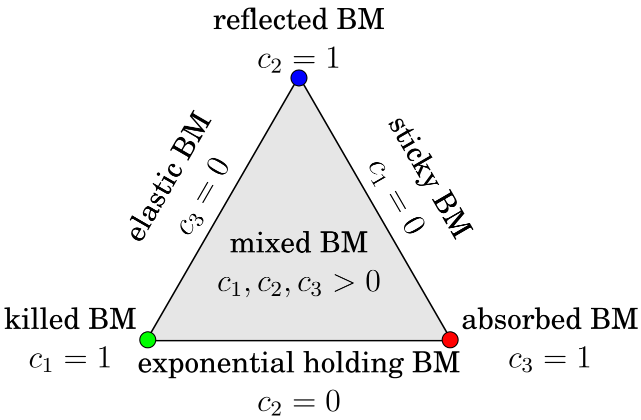

Figure 2.1: Description of the general Brownian motion on the half-line according to the chosen values of $c_1, c_2, c_3 \geq 0$ on the simplex $c_1 + c_2 + c_3 = 1$. Note that the killed BM, which formally corresponds to $c_1 = 1$, is not rigorously a case of the general BM, as we can see in Theorem 2.1.

- The case $c_1 = 0$ corresponds to the *sticky Brownian motion*.

The sticky Brownian motion serves as an interpolation between the absorbed Brownian motion and the reflected Brownian motion. It exhibits the property of spending a positive Lebesgue measure of time at zero, but never remaining at zero for any non-degenerate time interval. For a thorough and comprehensive overview, the reader is encouraged to consult [Borodin, 1989, Warren, 1999].

- The case $c_3 = 0$ corresponds to the *elastic Brownian motion*, also known as *partially reflected Brownian motion*.

This process is a mixture of the reflected BM and killed BM. It can be also constructed in terms of the local time at zero: we toss an exponential random variable τ whose parameter is related to c_1 and c_2, and a path of the reflected BM. Once the local time at zero of the reflected BM reaches the value τ, the process goes to the cemetery and stays there forever (see [Lejay, 2016] and references therein).

- The case $c_2 = 0$ corresponds to the *exponential holding Brownian motion*.

Its behavior is the following: once it visits zero, it stays there for an exponentially distributed amount of time and then is killed, that is, it jumps to the cemetery and stays there forever. See [Knight, 1981] on the exponential holding BM. This case allows us to interpret the case $c_1 = 1$ as a kind of explosion: it is an extreme case of the exponential holding BM, where the parameter of the exponential clock associated to jumps from the origin to the cemetery is infinite, leading to an instantaneous jump.

- The case $c_1, c_2, c_3 > 0$ is a mixture of these behaviours, and we will refer to any such process as the *mixed Brownian motion*.

Finally, it is instructive to mention that there are actually only two behaviours at zero. Namely, how much the BM sticks at zero, and how much the BM is attracted to the cemetery. This is in agreement with the fact that there are three parameters c_1, c_2 and c_3, but only two degrees of freedom, since these parameters are restricted to the simplex $c_1 + c_2 + c_3 = 1$. See Figure 2.1 for an illustration of the general BM in terms of the choices of c_1, c_2 and c_3.

Bearing in mind the general Brownian motion and the approximation Theorems 1.2, and 1.3, we are interesteding showing a Donsker-type Theorem, that is, showing the GBM as a continuous counterpart of some random walk with constraints defined on the positive half-line and then, find a suitable basis to obtain a weak Berry-Esseen estimate.

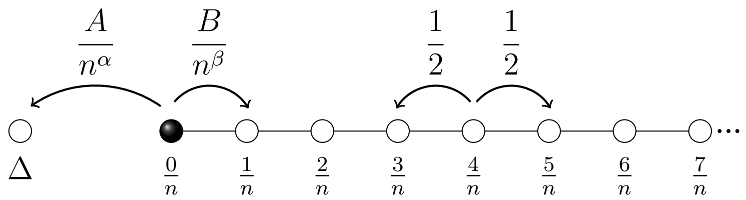

Figure 2.2: Jump rates for the *boundary random walk*.

The *boundary random walk* is the Feller process depending on positive parameters A, B, α and β defined on the state space $\mathbb{G}_n := \left(\frac{1}{n}\mathbb{N}\right) \cup \{\Delta\}$ for $n \in \mathbb{N}$, whose generator L_n acts on functions $f : \mathbb{G}_n \to \mathbb{R}$ as follows:

$$\mathsf{L}_n f(x) := \begin{cases} \frac{1}{2}\left[f\left(\frac{x+1}{n}\right) - f\left(\frac{x}{n}\right)\right] + \frac{1}{2}\left[f\left(\frac{x-1}{n}\right) - f\left(\frac{x}{n}\right)\right], & x = \frac{1}{n}, \frac{2}{n}, \cdots \\ \frac{A}{n^\alpha} f(0) + \frac{B}{n^\beta}\left[f\left(\frac{1}{n}\right) - f(0)\right], & x = \frac{0}{n}, \end{cases}$$

The next theorem is our second main result, and as we will see follows from Theorem 1.3. The cases stated below are illustrated in Figure 2.3 which characterizes phase transitions depending on parameters A, B, α, and β.

Theorem 2.2 (Functional CLT for the boundary random walk)**.** Fix $u, t > 0$. Let $\{X_n(t) : t \geq 0\}$ be the boundary random walk of parameters $\alpha, \beta, A, B \geq 0$ sped up by n^2 (that is, whose generator is $n^2\mathsf{L}_n$), starting from the point $\frac{\lfloor un \rfloor}{n} \in \mathbb{G}_n \subset \mathbb{G}$ and denote by $\mu_n = \mu_n(t)$ the distribution of X_n at time $t > 0$. Recall the metric $\mathbf{d}$ defined in 1.3 and denote by $\mu = \mu(t)$ the distribution at time $t > 0$ of the limit process in each of following cases. Then:

1. If $\alpha = \beta + 1$ and $\beta \in [0, 1)$, then $\{X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\text{ebm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}}[0, \infty)$, where X^{ebm} is the elastic BM on $\mathbb{G} = \{\Delta\} \cup \mathbb{R}_{\geq 0}$ of parameters

$$c_1 = \frac{B}{A+B}, \quad c_2 = \frac{A}{A+B} \quad \text{and} \quad c_3 = 0$$

starting from the point u. Moreover,

   (a) if $\beta \in (0, 1)$, then $\mathbf{d}(\mu_n, \mu) \lesssim \max\{n^{-\beta}, n^{\beta-1}\}$,

   (b) if $\beta = 0$, then $\mathbf{d}(\mu_n, \mu) \lesssim n^{-1}$.

2. If $\alpha \in (2,+\infty]$ and $\beta = 1$, then $\{X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\mathrm{sbm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}}[0,\infty)$, where X^{sbm} is the sticky BM on $\mathbb{R}_{\geq 0}$ of parameters
$$c_1 = 0, \quad c_2 = \frac{B}{B+1}, \quad \text{and} \quad c_3 = \frac{1}{B+1}$$
starting from the point u. Moreover, in this case,
$$\mathbf{d}(\mu_n,\mu) \lesssim \max\{n^{2-\alpha}, n^{-1}\}.$$

3. If $\alpha = 2$ and $\beta \in (1,+\infty]$, then $\{X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\mathrm{ehbm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}}[0,\infty)$, where X^{ehbm} is the exponential holding BM on $\mathbb{G} = \{\Delta\} \cup \mathbb{R}_{\geq 0}$ of parameters
$$c_1 = \frac{A}{1+A}, \quad c_2 = 0 \quad \text{and} \quad c_3 = \frac{1}{1+A}$$
starting from the point u. Moreover, for $\beta \in (2,\infty)$,
$$\mathbf{d}(\mu_n,\mu) \lesssim \max\{n^{2-\beta}, n^{-1}\}.$$

4. If $\alpha > \beta + 1$ and $\beta \in [0,1)$, then $\{X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\mathrm{rbm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$, where X^{rbm} is the reflected BM on $\mathbb{R}_{\geq 0}$, of parameters
$$c_1 = 0, \quad c_2 = 1 \quad \text{and} \quad c_3 = 0$$
starting from the point u. Moreover, for $\beta \in (0,1)$,

   (a) if $1 + \beta < \alpha < 2$, then $\mathbf{d}(\mu_n,\mu) \lesssim \max\{n^{-\beta}, n^{-\alpha+\beta+1}, n^{\alpha-2}\}$,
   (b) if $\alpha = 2$, then $\mathbf{d}(\mu_n,\mu) \lesssim \max\{n^{-\beta}, n^{\beta-1}\}$,
   (c) if $\alpha > 2$, then $\mathbf{d}(\mu_n,\mu) \lesssim \max\{n^{2-\alpha}, n^{-\beta}, n^{\beta-1}\}$.

5. If $\alpha \in (2,\infty]$ and $\beta \in (1,+\infty]$, then $\{X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\mathrm{abm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$, where X^{abm} is the absorbed BM on $\mathbb{R}_{\geq 0}$, of parameters
$$c_1 = 0, \quad c_2 = 0 \quad \text{and} \quad c_3 = 1$$
starting from the point u. Moreover, for $\alpha > 2$ and $\beta > 2$,
$$\mathbf{d}(\mu_n,\mu) \lesssim \max\{n^{2-\alpha}, n^{2-\beta}, n^{-1}\}.$$

6. If $\alpha = 2$ and $\beta = 1$, then $\{X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\mathrm{mbm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}}[0,\infty)$, where X^{mbm} is the mixed BM on $\mathbb{G} = \{\Delta\} \cup \mathbb{R}_{\geq 0}$ of parameters
$$c_1 = \frac{A}{1+A+B}, \quad c_2 = \frac{B}{1+A+B} \quad \text{and} \quad c_3 = \frac{1}{1+A+B}$$
starting from the point u. Moreover, $\mathbf{d}(\mu_n,\mu) \lesssim n^{-1}$.

Since the natural state space of the killed BM is $\mathbb{G} = \{\Delta\} \cup (0,\infty)$, which does not include the origin, we need a different setup to have the convergence of the boundary random walk towards the killed BM. This is the content of the next result. Let $\tau_n : \mathbb{G} \to \mathbb{G}$ be the shift to the right of $1/n$ given by
$$\tau_n(\Delta) = \Delta \quad \text{and} \quad \tau_n(u) = u + \tfrac{1}{n} \quad \text{for } u \in [0,\infty).$$

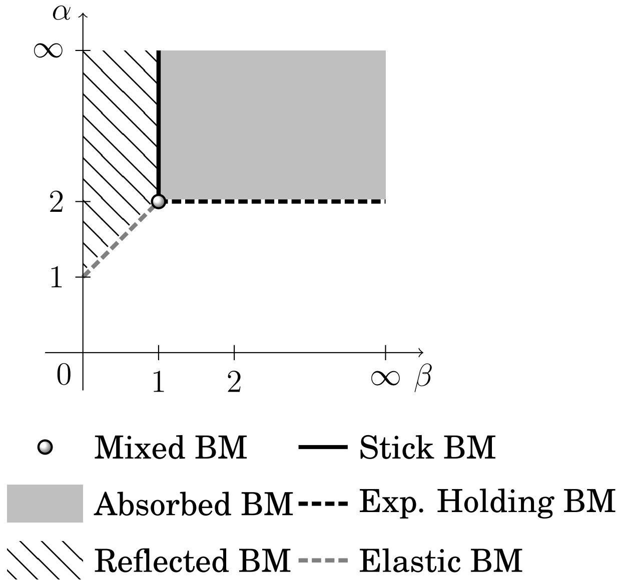

Figure 2.3: Possible limits for the boundary random walk according to the ranges of $\alpha, \beta \in [0, \infty]$. Note that it includes the cases where $\alpha = \infty$ or $\beta = \infty$, which correspond to $A = 0$ and $B = 0$ respectively. Speed of convergence is provided for all choices of α and β, except for the strip $1 < \beta < 2$.

Theorem 2.3 (Convergence of the shifted boundary RW to the killed BM)**.** If $\alpha < 1 + \beta$ for $\beta \in [0, 1]$, or $\beta > 1$, then $\{\tau_n X_n(t) : t \geq 0\}$ converges weakly to $\{X^{\text{kbm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}_0}[0, \infty)$, where in this case X^{kbm} is the killed BM on $\mathbb{G}_0 = \{\Delta\} \cup (0, +\infty)$, which is formally the general BM of parameters

$$c_1 = 1, \quad c_2 = 0 \quad \text{and} \quad c_3 = 0\,.$$

2.3 Proof of the functional central limit Theorem for the boundary random walk

This section is dedicated to proving Theorems 2.2 and 2.3 as well as the tools needed for the proof.

Define

$$\mathcal{A}_{\geq 0} := \Big\{ p(x)e^{-x^2} : p : \mathbb{R}_{\geq 0} \to \mathbb{R} \text{ is a polynomial} \Big\}\,.$$

We claim that the linear vector space generated by $\mathcal{A}_{\geq 0}$ is dense in $\mathcal{C}_0(\mathbb{R}_{\geq 0})$. In order to show that, consider the general space

$$\mathcal{A} := \Big\{ p(x)e^{-x^2} : p : \mathbb{R} \to \mathbb{R} \text{ is a polynomial} \Big\}\,.$$

Thus

Lemma 2.1. The set $\operatorname{span}(\mathcal{A})$ is dense in $\mathcal{C}_0(\mathbb{R})$.

Proof. Suppose by contradiction that $\operatorname{span}(\mathcal{A})$ is not dense. Then, by the Hahn-Banach Theorem, there exists a non-zero functional $\Lambda : \mathcal{C}_0(\mathbb{R}) \to \mathbb{R}$ such that $\Lambda|_{\operatorname{span}(\mathcal{A})} \equiv 0$. By the Riesz-Markov Theorem, there exists a measure μ such that

$$\Lambda(f) := \int f \, \mathrm{d}\mu$$

for all $f \in \mathcal{C}_0(\mathbb{R})$. By the Jordan decomposition theorem, there exist μ^+, μ^- positive real-valued measures such that $\mu = \mu^+ - \mu^-$ where at least one of the two measures is finite.

For any $g \in \mathrm{span}(\mathcal{A})$, it follows that $g(x) = p(x)e^{-x^2}$ for some polynomial p, and

$$\begin{aligned} 0 = \Lambda(g) &= \int g(x) \mathbf{d}\mu(x) \\ &= \int p(x)e^{-x^2} \mathbf{d}(\mu^+(x) - \mu^-(x)) \\ &= \int p(x)e^{-x^2} \mathbf{d}\mu^+ - \int p(x)e^{-x^2} \mathbf{d}\mu^- , \end{aligned}$$

and hence,

$$\int p(x)e^{-x^2} \, \mathbf{d}\mu^+ \;=\; \int p(x)e^{-x^2} \, \mathbf{d}\mu^- . \tag{2.3.1}$$

Let $\nu^+(A) = \int_A e^{-x^2}\mathbf{d}\mu^+$ and $\nu^-(A) = \int_A e^{-x^2}\mathbf{d}\mu^-$ be real-valued measures, and define $\rho^+, \rho^- : \mathbb{C} \to \mathbb{R}$ through $\rho^\pm(z) := \int e^{zx}\mathbf{d}\nu^\pm(x)$. Observe that $\rho^\pm$ are well-defined since

$$\begin{aligned} |\rho^+(z)| &= \left| \int_{\mathbb{R}} e^{zx} \, \mathbf{d}\nu^+(x) \right| \\ &= \left| \int_{\mathbb{R}} e^{zx} e^{-x^2} \, \mathbf{d}\mu^+(x) \right| \\ &\leq \int_{\mathbb{R}} \left| e^{zx - x^2} \right| \mathbf{d}\mu^+(x) . \end{aligned}$$

and similarly for ρ^-. Since the decay of e^{-x^2} dominates the growth of e^{zx}, we have that $|\rho^\pm(z)|$ is finite for every $z \in \mathbb{C}$ once the integral converges absolutely, and then, is well-defined. Writing the exponential in its power series, we have that

$$\frac{d}{dz}e^{zt} \;=\; \frac{d}{dz}\sum_{k=0}^{\infty}\frac{(zt)^k}{k!} \;=\; \sum_{k=0}^{\infty}\frac{d}{dz}\frac{(zt)^k}{k!} \;=\; \frac{k(zt)^k}{k!}\frac{d}{dz}(zt) \;=\; \sum_{k=1}^{\infty}\frac{(zt)^k}{k!}t = te^{zt} \tag{2.3.2}$$

Thus, from equation (2.3.2) one can check that

$$\frac{d}{dz}\rho^\pm(z) \;=\; \frac{d}{dz}\int_{\mathbb{R}} e^{zt}\mathbf{d}\nu^\pm(t) \;=\; \int_{\mathbb{R}} \frac{d}{dz}e^{zt}\mathbf{d}\nu^\pm(t) \;=\; \int_{\mathbb{R}} te^{zt}\mathbf{d}\nu^\pm(t) \tag{2.3.3}$$

In view of (2.3.1) and (2.3.3), we have that

$$\frac{d^{(n)}}{dz^{(n)}}\rho^+(0) \;=\; \frac{d^{(n)}}{dz^{(n)}}\rho^-(0)$$

We now make a comparison between the power series of ρ^+ and ρ^-

$$\rho^+ z(0) \;=\; \sum_{n=0}^{\infty}\frac{d}{dz}\rho^+(0)z^n \;=\; \sum_{n=0}^{\infty}\frac{d}{dz}\rho^-(0)z^n = \rho^-(0) ,$$

and therefore, $\rho^+(z) = \rho^-(z)$ for any $z \in \mathbb{C}$.

Therefore, for all $s \in \mathbb{R}$, $\rho^+(is) = \rho^-(is)$ for all $s \in \mathbb{R}$, and hence $\nu^+ = \nu^-$, which guarantees that $\Lambda \equiv 0$, a contradiction. Hence $\mathrm{span}(\mathcal{A})$ must be dense in $\mathcal{C}_0(\mathbb{R})$. □

As an immediate consequence of Lemma 2.1, we have

Corollary 2.1. The set $\mathrm{span}(\mathcal{A}_{\geq 0})$ is dense in $\mathcal{C}_0(\mathbb{R}_{\geq 0})$.

Proof of Theorem 2.2. In order to prove Theorem 2.2, we must check the conditions (A1) - (A3) and (B1) - (B4) and then apply Theorems 1.2 and 1.3.

We begin by verifying the hypotheses (B1) through (B4). Satisfying these conditions entails constructing a basis for the metric d. Notably, this verification is sufficiently general and does not require partitioning the analysis into distinct cases for each parameter α, β, A and B. Let $\widetilde{f}_k : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be defined by $\widetilde{f}_k(x) := x^k e^{-x^2}$ for all $k \geq 0$, which are illustrated in Figure 2.4. By Corollary 2.1 we know that $\mathrm{span}(\{\widetilde{f}_k\}_{k\geq 0})$ is dense in $\mathcal{C}_0(\mathbb{R}_{\geq 0})$ and an elementary calculation gives that $\|\widetilde{f}_0\| = 1$ and $\|\widetilde{f}_k\| = \left(\frac{k}{2}\right)^{\frac{k}{2}} e^{-\frac{k}{2}}$, for all $k \geq 1$. Then, the family of functions defined by

$$f_k := \frac{\widetilde{f}_k}{\|\widetilde{f}_k\|}, \quad \forall\, k \geq 0\,, \tag{2.3.4}$$

still has the property that its span is dense in $\mathcal{C}_0(\mathbb{R}_{\geq 0})$ and it satisfies (1.3.4).

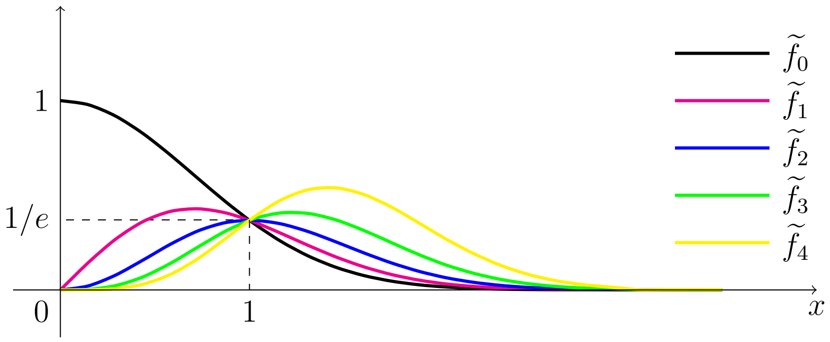

Figure 2.4: Illustration of the functions $\widetilde{f}_k : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $\widetilde{f}_k(x) := x^k e^{-x^2}$.

Our goal now is to find sequences $f_{k,j} \in \mathfrak{D}(\mathsf{L}^2)$ fulfilling hypotheses (B2) and (B3). Note that the functions f_k are smooth and, for $k \geq 5$, the function itself and its first four derivatives at zero are zero. Recalling (2.2.1), this property immediately ensures that $f_k \in \mathfrak{D}(\mathsf{L}^2)$ for $k \geq 5$. Accordingly, we define

$$f_{k,j} := f_k \quad \text{for } k \geq 5\,.$$

To treat the case $k \leq 4$, we define the shift operator τ_j by

$$\tau_j(f)(x) = f\Big(x - \frac{4}{j}\Big)\mathbf{1}_{[x \geq \frac{4}{j}]}(x)\,.$$

Define now an extension of $\tau_j(f_k)$ to the whole line through a reflection around the y-axis, that is,

$$g_{k,j}(x) := \begin{cases} (\tau_j)(f_k)(x)\,, & \text{if } x \geq 0 \\ (\tau_j)(f_k)(-x)\,, & \text{if } x < 0 \end{cases} \quad \text{for } k \in \{1,2,3,4\} \text{ and } j \geq 0\,,$$

which are continuous, but not smooth at the point $4/j$. To remedy this, consider the $\mathcal{C}^\infty$-approximation of identity $\varphi_j : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_j(x) := \begin{cases} \dfrac{1}{c_j} \exp\left(-\dfrac{1}{1-(jx)^2}\right), & \text{if } |jx| < 1 \\ 0\,, & \text{otherwise} \end{cases}$$

where

$$c_j := \int_{\mathbb{R}} \exp\Big(-\frac{1}{1-(jx)^2}\Big)\mathbb{1}_{\{|jx|\leq 1\}}\mathrm{d}x$$

is the normalizing constant. Note that $\varphi_j(x) = j\varphi_1(jx)$, which yields the following relation between the derivatives

$$\frac{d^{(i)}}{dx^{(i)}}\varphi_j(x) \;=\; j^{i+1}\frac{d^{(i)}}{dx^{(i)}}\varphi_1(jx)\,, \quad \forall\, j>0\,. \tag{2.3.5}$$

Define now

$$f_{k,j}(x) \;:=\; (g_{k,j} * \varphi_j)(x) \quad \text{for } k\in\{1,2,3,4\} \text{ and } j\geq 0\,,$$

which is smooth and a simple but tedious calculation shows that $f_{k,j} \to f_k$ uniformly as $j\to\infty$. Since the boundary conditions are satisfied for $f_{j,k}$ and $\mathsf{L}f_{j,k}$, we obtain that $f_{k,j}\in\mathfrak{D}(\mathsf{L}^2)$ for any $k\in\{1,2,3,4\}$ and any $j\geq 0$.

Denote $\Big\|\frac{d^{(i)}}{dx^{(i)}}\varphi_1\Big\| = A_i$ for $i\in\mathbb{N}$. Observe that

$$\begin{aligned}
\left|\frac{d^{(i)}}{dx^{(i)}}f_{k,j}(x)\right| \;&=\; \left|\frac{d^{(i)}}{dx^{(i)}}\int_{-\frac{1}{j}}^{\frac{1}{j}} g_{k,j}(x-y)\varphi_j(y)\,dy\right| \\
&=\; \left|\int_{-\frac{1}{j}}^{\frac{1}{j}} g_{k,j}(x-y)\frac{d^{(i)}}{dx^{(i)}}\varphi_j(y)dy\right| \\
&\leq\; 2j^i\,\|g_{k,j}\|\,A_i\,,
\end{aligned}$$

where we made use of the scaling relation (2.3.5) to obtain the last inequality. Since $\|g_{k,j}\| = \|f_k\| = 1$, we obtain that

$$\|\mathsf{L}f_{k,j}\| \leq j^2 A_2 \quad \text{and} \quad \big\|\mathsf{L}^2 f_{k,j}\big\| \leq j^4 A_4\,, \quad \forall\, k\geq 1, j\geq 0\,.$$

Now, it only remains to construct $f_{0,j}$ such that it verifies conditions (B2) and (B3). To that end, let $\mathsf{P}:\mathbb{R}_{\geq 0}\to\mathbb{R}$ be a polynomial such that both $e^{-x^2}+\mathsf{P}$ and $(e^{-x^2}+\mathsf{P})''$ satisfy the boundary condition (2.2.1) and such that additionally $\mathsf{P}(0)=0$. To continue, define the $\mathcal{C}^\infty$-bump function $b_1:\mathbb{R}\to\mathbb{R}_{\geq 0}$ by

$$b_1(t) \;=\; 1 - \frac{\ell(t^2-1)}{\ell(t^2-1)+\ell(2-t^2)}$$

where

$$\ell(t) \;=\; \begin{cases} e^{-\frac{1}{t}}, & \text{if } t>0 \\ 0, & \text{if } t\leq 0\,, \end{cases}$$

and for $j\geq 1$ define $b_j(t) = b_1(jt)$. The function $0\leq b_j\leq 1$ is equal to one in an interval of size $1/j$ around the origin and zero outside the interval $[-\sqrt{2}/j, \sqrt{2}/j]$. Finally, define

$$f_{0,j}(x) \;:=\; e^{-x^2} + (b_j\mathsf{P})(x)\,.$$

The fact that $f_{0,j}$ vanishes for $x\geq\sqrt{2}/j$ and that $f_{0,j}(0)=1$ together with its continuity guarantee that $f_{0,j}\to e^{-x^2}$ as $j\to\infty$ in the uniform topology. Since the polynomial $e^{-x^2}+\mathsf{P}$ satisfies the aforementioned boundary condition, we also have $f_{0,j}\in\mathfrak{D}(\mathsf{L}^2)$ for all positive integers j. Note that $b_j(x) = b_1(jx)$, yielding for all $i\geq 0$ and all $j\geq 1$

$$\frac{d^{(i)}}{dx^{(i)}}b_j(x) \;=\; j^i\frac{d^{(i)}}{dx^{(i)}}b_1(jx)\,.$$

Since the generator of the general Brownian motion on the half-line is given by $\frac{1}{2}\Delta$, it is immediate that

$$\mathsf{L}e^{-x^2} \lesssim x^2e^{-x^2} \quad \text{and} \quad \mathsf{L}^2e^{-x^2} \lesssim x^4e^{-x^2}. \tag{2.3.6}$$

Now, since $b_j \equiv 0$ outside the compact set $[-\frac{\sqrt{2}}{j}, \frac{\sqrt{2}}{j}]$ and P is a polynomial, for all $i \geq 1$ one has that $\left\|\frac{d^{(i)}}{dx^{(i)}}b_j\mathsf{P}\right\| \lesssim j^i$. Hence, using equation (2.3.6), the product rule and the bound above, we can obtain upper bounds for the generator L and L^2 norm $\|\mathsf{L}f_{0,j}\| \lesssim \big(\|f_2\| + j^2\big)\|f_0\| \lesssim j^2\|f_0\|$ and $\|\mathsf{L}^2f_{0,j}\| \lesssim \big(\|f_4\| + j^4\big)\|f_0\| \lesssim j^4\|f_0\|$, where we used that all the f_k' s were normalized. Hence, we ensured that conditions (B2) and (B3) are met.

The Lipschitz hypothesis (B4) relies on the knowledge about the semigroup of the limiting process. The limiting processes mentioned in Theorem 1.3 are, the reflected, absorbed, mixed, sticky, elastic, exponential holding and killed Brownian motion. All of them have explicit formulas for their semigroups (which are obtained from the semigroup of the standard Brownian motion by simple modifications), which can be found in the book [Borodin and Salminen, 2002, Appendix 1, starting at page 119]. From these formulas it can be checked that (B4) holds for each one of those semigroups.

We will verify the condition (B4) only for the cornerstone processes on the simplex: The reflected Brownian motion, the absorbed Brownian motion and killed Brownian motion, the other cases will be omitted.

Lemma 2.2. The semigroup $\{\mathsf{P}^{\mathrm{rbm}}(t) : t \geq 0\}$ of the reflected Brownian motion given by

$$\mathsf{P}^{\mathrm{rbm}}(t)f(x) := \int_0^\infty \frac{1}{\sqrt{2\pi t}}\left[e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}}\right]f(y)\mathrm{d}y, \qquad \text{for } x \in [0,\infty)$$

associated to the generator $\mathsf{L}_{\mathrm{rbm}}$ described by (2.1) where $c_2 = 1$ is Lipschitz.

Proof. Denote by

$$\begin{aligned} p_t(x,y) &:= \frac{1}{\sqrt{2\pi t}}\left[e^{-\frac{(x-y)^2}{2t}} + e^{-\frac{(x+y)^2}{2t}}\right], \\ q_t(x,y) &:= \frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}, \end{aligned}$$

the transition semigroup of the reflected Brownian motion and an auxiliary function. Observe that showing that $\mathsf{P}^{\mathrm{rbm}}(t)$ is Lipschitz is equivalent to ensuring that there exists a positive constant K such that

$$|\partial_x\mathsf{P}^{\mathrm{rbm}}(t)f(x)| \leq K, \qquad \forall x \in \mathbb{R}\setminus\{0\},\ \forall t \geq 0. \tag{2.3.7}$$

Thus

$$\begin{aligned} \partial_x\mathsf{P}^{\mathrm{rbm}}(t)f(x) &= \int_0^\infty \partial_x p_t(x,y)f(y)\mathrm{d}y \\ &= \int_0^\infty \partial_x[q_t(x,y) + q_t(x,-y)]f(y)\mathrm{d}y \\ &= \int_0^\infty \left[\frac{(x-y)}{t}q_t(x,y) + \frac{(x+y)}{t}q_t(x,-y)\right]f(y)\mathrm{d}y. \end{aligned}$$

Finally, note that

$$\begin{aligned}\left|\int_0^{\infty} \frac{(x-y)}{t} q_t(x,y)\mathrm{d}y\right| &\le \int_0^{\infty} \frac{|x-y|}{t} q_t(x,y)\mathrm{d}y \\ &= \int_0^{\infty} \frac{|x-y|}{t} \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\mathrm{d}y \\ &\le \frac{1}{t}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} |y| e^{-\frac{(y)^2}{2t}}\mathrm{d}y \\ &\le \frac{1}{t}\mathbb{E}[|Y|]\end{aligned}$$

where Y has the same distribution as a Gaussian $\mathcal{N}(0,t)$. Thus, from a triangular inequality,

$$|\partial_x \mathsf{P}^{\mathrm{rbm}}(t)f(x)| \le \|f\| \left|\int_0^{\infty} \frac{(x-y)}{t} q_t(x,y)\mathrm{d}y\right| + \|f\| \left|\int_0^{\infty} \frac{(x+y)}{t} q_t(x,-y)\mathrm{d}y\right| ,$$

which shows that (2.3.7) holds, and the result follows finishing the proof. □

Lemma 2.3. The semigroup $\{\mathsf{P}^{\mathrm{kbm}}(t) : t \ge 0\}$ of the killed Brownian motion over $\mathbb{G}_0$ given by

$$\mathsf{P}^{\mathrm{kbm}}(t)f\,(x) := \int_0^{\infty} \frac{1}{\sqrt{2\pi t}} \Big[e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\Big] f(y)\mathrm{d}y, \qquad \text{for } x \in (0,\infty)$$

associated to the generator $\mathsf{L}_{\mathrm{kbm}}$ is Lipschitz.

Proof. The proof follows similarly to the one in 2.2. □

Consider

$$Erf(z) := \tfrac{2}{\sqrt{z}} \int_0^z e^{-u^2}\mathrm{d}\,u$$

the error function and, by $Erfc(z) := 1 - Erf(z)$ we denote the complementary error function. The absorbed Brownian motion has the semigroup $\{\mathsf{P}^{\mathrm{abm}}(t) : t \ge 0\}$ given by

$$\mathsf{P}^{\mathrm{kbm}}(t)f\,(x) := \int_0^{\infty} \frac{1}{\sqrt{2\pi t}} \Big[e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\Big] f(y)\mathrm{d}y, \qquad \text{for } x \in (0,\infty)$$

and

$$\mathbb{P}_x(X^{\mathrm{abm}}(t) = 0) := \begin{cases} Erfc\left(\frac{x}{\sqrt{2t}}\right), & x > 0 \\ 1, & x = 0\,. \end{cases}$$

Since the semigroups $\{\mathsf{P}^{\mathrm{kbm}}(t) : t \ge 0\}$ and $\{\mathsf{P}^{\mathrm{abm}}(t) : t \ge 0\}$ are almost the same unless by the boundary, Lemma 2.3 also implies that the semigroup $\{\mathsf{P}^{\mathrm{abm}}(t) : t \ge 0\}$ of the absorbed Brownian motion, associated to the generator $\mathsf{L}_{\mathrm{abm}}$ described via (2.1) where $c_3 = 1$, is lipschitz.

Remark 2.1. Hypothesis (B4) has been used just once, in the proof of Theorem 1.3, and its importance relies on the fact that the random walk and its limiting process may not have the same starting point. For instance, in the setup of Theorem 2.2, the boundary random walk starts from $\lfloor un\rfloor/n$, whereas its Brownian counterpart starts from $u > 0$. If we assume that the discrete process and the limiting process start from the same point $u \in \mathsf{S}$, hypothesis (B4) can be dropped from Theorem 1.3. This is possible in the setup of Theorem 2.2, for instance, if we assume that the scaling parameter is given by $n = n(k) = 2^k$ and the initial point u is a positive integer.

It remains now, to verify the remaining conditions (A1) - (A3). We must find the correction operator Ξ_n, which is very model dependent. Before we study each model separately, we start with some generalities. We will always assume that $(\Xi_n f)(\Delta) = 0$, and moreover all functions considered here satisfy $f(\Delta) = 0$. Thus, the generator of the random walk with boundary conditions sped up by n^2, applied to the function $\Phi_n f$ at zero is given by

$$\begin{aligned} n^2 \mathsf{L}_n \Phi_n f\left(\tfrac{0}{n}\right) \;=\; & -\tfrac{A}{n^{\alpha-2}} f\left(\tfrac{0}{n}\right) + \tfrac{B}{n^{\beta-2}}\left[f\left(\tfrac{1}{n}\right) - f\left(\tfrac{0}{n}\right)\right] \\ & -\tfrac{A}{n^{\alpha-2}} \Xi_n f(0) + \tfrac{B}{n^{\beta-2}}\left[\Xi_n f\left(\tfrac{1}{n}\right) - \Xi_n f(0)\right]. \end{aligned} \tag{2.3.8}$$

Outside of zero, a Taylor expansion yields

$$\begin{aligned} f\left(\tfrac{x+1}{n}\right) - f\left(\tfrac{x}{n}\right) &= \tfrac{1}{n} f'\left(\tfrac{x}{n}\right) + \tfrac{1}{2n^2} f''\left(\tfrac{x}{n}\right) + \tfrac{1}{3!n^3} f'''\left(\tfrac{x}{n}\right) + \tfrac{1}{4!n^4} f''''(\theta) \\ f\left(\tfrac{x-1}{n}\right) - f\left(\tfrac{x}{n}\right) &= -\tfrac{1}{n} f'\left(\tfrac{x}{n}\right) + \tfrac{1}{2n^2} f''\left(\tfrac{x}{n}\right) - \tfrac{1}{3!n^3} f'''\left(\tfrac{x}{n}\right) + \tfrac{1}{4!n^4} f''''(\eta), \end{aligned} \tag{2.3.9}$$

for some $\theta \in (x/n, (x+1)/n)$ and $\eta \in ((x-1)/n, x/n)$. Thus

$$\begin{aligned} n^2 \mathsf{L}_n \Phi_n f\left(\tfrac{x}{n}\right) \;&=\; \tfrac{n^2}{2}\left[f\left(\tfrac{x+1}{n}\right) + f\left(\tfrac{x-1}{n}\right) - 2f\left(\tfrac{x}{n}\right)\right] \\ &\quad + \tfrac{n^2}{2}\Big[\Xi_n f\left(\tfrac{x+1}{n}\right) + \Xi_n f\left(\tfrac{x-1}{n}\right) - 2\Xi_n f\left(\tfrac{x}{n}\right)\Big] \\ &\overset{(2.3.9)}{=} \tfrac{1}{2}\Delta f\left(\tfrac{x}{n}\right) + \tfrac{1}{2\cdot 4!n^2}\Big[f''''(\theta) + f''''(\eta)\Big] + n^2 \mathsf{L}_n \Xi_n f\left(\tfrac{x}{n}\right) \\ &= \tfrac{1}{2}\pi_n \Delta f\left(\tfrac{x}{n}\right) + \big\|\Delta^2 f\big\| \cdot O\left(\tfrac{1}{n^2}\right) + n^2 \mathsf{L}_n \Xi_n f\left(\tfrac{x}{n}\right), \end{aligned} \tag{2.3.10}$$

As previously noted, the correction operator exhibits significant dependence on the model's parameters. Consequently, the proof will be organized into multiple cases based on these parameters, with some cases further subdivided into subcases for clarity.

The elastic BM: the case $\beta \in [0,1), \alpha = \beta + 1$

Recall that in this case we set

$$c_1 = \frac{B}{A+B}, \quad c_2 = \frac{A}{A+B} \quad \text{and} \quad c_3 = 0\,.$$

Denote the generator of the elastic Brownian motion by L_{ebm}. Its domain is given by

$$\mathfrak{D}\left(\mathsf{L}_{\text{ebm}}\right) \;=\; \left\{f \in \mathcal{C}_0^2(\mathbb{G}) \;:\; \tfrac{A}{A+B} f(0) - \tfrac{B}{A+B} f'(0) \;=\; 0\right\}.$$

Let $f \in \mathfrak{D}(\mathsf{L}_{\text{ebm}}^2)$, which yields the boundary conditions $Af(0) = Bf'(0)$ and $Af''(0) = Bf'''(0)$. Using this together with $\alpha = 1 + \beta$ in Equation (2.3.8) in addition to a Taylor expansion, yields that

$$\begin{aligned} n^2 \mathsf{L}_n \Phi_n f\left(\tfrac{0}{n}\right) \;&=\; -\tfrac{A}{n^{\alpha-2}} f\left(\tfrac{0}{n}\right) + \tfrac{B}{n^{\beta-2}}\left[\tfrac{f'(0)}{n} + \tfrac{f''(0)}{2!n^2} + \tfrac{f'''(0)}{3!n^3} + \tfrac{f''''(\eta)}{4!n^4}\right] \\ &\quad - \tfrac{A}{n^{\alpha-2}} \Xi_n f(0) + \tfrac{B}{n^{\beta-2}}\Big[\Xi_n f\left(\tfrac{1}{n}\right) - \Xi_n f(0)\Big] \\ &= \tfrac{B}{n^{\beta}}\left[\tfrac{f''(0)}{2!} + \tfrac{A}{B}\tfrac{f''(0)}{3!n} + \tfrac{f''''(\eta)}{4!n^2}\right] && (2.3.11) \\ &\quad - \tfrac{A}{n^{\beta-1}} \Xi_n f(0) + \tfrac{B}{n^{\beta-2}}\Big[\Xi_n f\left(\tfrac{1}{n}\right) - \Xi_n f(0)\Big] && (2.3.12) \end{aligned}$$

for some $0 \leq \eta \leq 1/n$.

Note that for $\beta > 0$ the parcel (2.3.11) is vanishing, and it is at this point where the correction operator enters the game. Let us analyze it in cases:

- $\beta \neq 0$,
- $\beta = 0$.

Subcase $\beta \neq 0$

Assume first that $\beta \neq 0$; we will discuss the situation $\beta = 0$ at the ending of this case. Define

$$\Xi_n f\left(\tfrac{x}{n}\right) \;:=\; -\frac{\frac{1}{2}f''(0)}{An^{1-\beta}\left(1+\frac{1}{n}g\left(\frac{x}{n}\right)\right)}\,, \tag{2.3.13}$$

where g is some arbitrary nonnegative Lipschitz function of constant $K>0$ satisfying $g(0)=0$ and $g(u)\to\infty$ as $u\to\infty$. Note that the condition on the growth of g is only necessary to assure that Ξ_n belongs to the domain of L_n which is given by $\mathfrak{D}(\mathsf{L}_n) = \big\{f : \{\Delta\}\cup\frac{1}{n}\mathbb{N}\to\mathbb{R}$ such that $\lim_{\frac{x}{n}\to\infty} f\left(\frac{x}{n}\right)=0$ and $f(\Delta)=0\big\}$. We then have that

$$\begin{aligned} n^2\mathsf{L}_n\Phi_n f\left(\tfrac{0}{n}\right) \;&=\; -\tfrac{A}{n^{\beta-1}}\Xi_n f\,(0) + \tfrac{B}{n^{\beta-2}}\Big[\Xi_n f\left(\tfrac{1}{n}\right) - \Xi_n f\,(0)\Big]\\ &= \tfrac{1}{2}f''(0) + \tfrac{Bn}{A}\left[\tfrac{1}{1+\frac{1}{n}g\left(\frac{1}{n}\right)} - 1\right]\tfrac{1}{2}f''(0)\\ &= \tfrac{1}{2}f''(0) + \tfrac{Bn}{A}\left[\tfrac{\frac{1}{n}g\left(\frac{1}{n}\right)}{1+\frac{1}{n}g\left(\frac{1}{n}\right)}\right]\tfrac{1}{2}f''(0)\\ &= \mathsf{L}_{\text{ebm}}f\,(0) + \tfrac{BK}{A}\|\mathsf{L}_{\text{ebm}}f\|\cdot O(\tfrac{1}{n}) \end{aligned}$$

because $g(0)=0$ and g is K-Lipschitz. Plugging it into (2.3.11) – (2.3.12) yields

$$\left|\left(\pi_n\mathsf{L}_{\text{ebm}}f - n^2\mathsf{L}_n\Phi_n f\right)(0)\right| \;\lesssim\; \left(\tfrac{1}{n^\beta}+\tfrac{1}{n}+\tfrac{1}{n^{1+\beta}}\right)\|\mathsf{L}_{\text{ebm}}f\| + \tfrac{1}{n^{2+\beta}}\left\|\mathsf{L}^2_{\text{ebm}}f\right\|\,. \tag{2.3.14}$$

For $\frac{x}{n}\in\mathbb{G}_n\backslash\{0\}$, equation (2.3.10) indicates that we need to estimate

$$\begin{aligned} n^2\mathsf{L}_n\Xi_n f\left(\tfrac{x}{n}\right) &= \tfrac{\mathsf{L}_{\text{ebm}}f(0)n^2}{An^{1-\beta}}\left[\tfrac{\frac{1}{n}\left(g\left(\frac{x+1}{n}\right)-g\left(\frac{x}{n}\right)\right)}{\left(1+\frac{1}{n}g\left(\frac{x+1}{n}\right)\right)\left(1+\frac{1}{n}g\left(\frac{x}{n}\right)\right)} + \tfrac{\frac{1}{n}\left(g\left(\frac{x-1}{n}\right)-g\left(\frac{x}{n}\right)\right)}{\left(1+\frac{1}{n}g\left(\frac{x-1}{n}\right)\right)\left(1+\frac{1}{n}g\left(\frac{x}{n}\right)\right)}\right]\\ &= K\|\mathsf{L}_{\text{ebm}}\|\cdot O\left(\tfrac{1}{n^{1-\beta}}\right)\,, \end{aligned}$$

where we used again that g is K-Lipschitz. Plugging it into (2.3.10), we conclude that

$$\left|\left(\pi_n\mathsf{L}_{\text{ebm}}f - n^2\mathsf{L}_n\Phi_n f\right)\left(\tfrac{x}{n}\right)\right| \;\lesssim\; \tfrac{1}{n^{1-\beta}}\|\mathsf{L}_{\text{ebm}}f\| + \tfrac{1}{n^2}\left\|\mathsf{L}^2_{\text{ebm}}f\right\| \tag{2.3.15}$$

uniformly in $\frac{x}{n}\in\mathbb{G}_n\backslash\{0\}$. Putting together (2.3.14) and (2.3.15), we infer that

$$\begin{aligned} \left\|\pi_n\mathsf{L}_{\text{ebm}}f - n^2\mathsf{L}_n\Phi_n f\right\| \;&\lesssim\; \max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n},\tfrac{1}{n^{1-\beta}}\right\}\|\mathsf{L}_{\text{ebm}}f\|\\ &\quad + \max\left\{\tfrac{1}{n^2},\tfrac{1}{n^{2+\beta}}\right\}\left\|\mathsf{L}^2_{\text{ebm}}f\right\|\\ &= \max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n^{1-\beta}}\right\}\|\mathsf{L}_{\text{ebm}}f\| + \tfrac{1}{n^2}\left\|\mathsf{L}^2_{\text{ebm}}f\right\|\,. \end{aligned} \tag{2.3.16}$$

In view of (2.3.16), we have assured hypothesis (A2) and it is only missing to check (A3). From (2.3.13), we immediately get that

$$\|\Xi_n f\| \;\lesssim\; \tfrac{1}{n^{1-\beta}}\|\mathsf{L}_{\text{ebm}}\|\,,$$

showing that (H3) holds. Hence, Theorem 1.3 yields

$$\mathbf{d}(\mu_n,\mu) \;\lesssim\; \max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n^{1-\beta}},\tfrac{1}{n},\tfrac{1}{n^2}\right\} \;=\; \max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n^{1-\beta}}\right\}$$

and that $\{\mathsf{X}_\mathsf{n}(t): t\geq 0\}$ weakly converges to $\{X^{\text{ebm}}(t): t\geq 0\}$ under the J_1-Skorohod topology of $\mathsf{D}_\mathbb{G}[0,\infty)$. We thus can conclude this subcase.

Subcase $\beta = 0$

We come to the case $\beta = 0$. In this subcase we define

$$\Xi_n f\left(\tfrac{x}{n}\right) := -\frac{(1-B)\frac{1}{2}f''(0)}{An\left(1+\frac{1}{n}g\left(\frac{x}{n}\right)\right)}.$$

Analogous arguments as above yield that $\mathbf{d}(\mu_n,\mu) \lesssim \frac{1}{n}$. We omit the details.

The sticky BM: the case $\beta = 1, \alpha \in (2,\infty)$

Recall that in this case we set

$$c_1 = 0, \quad c_2 = \frac{B}{B+1} \quad \text{and} \quad c_3 = \frac{1}{B+1}.$$

We denote the generator of the sticky Brownian motion by $\mathsf{L}_{\mathrm{sbm}}$. Its domain is

$$\mathfrak{D}\left(\mathsf{L}_{\mathrm{sbm}}\right) = \left\{f \in \mathcal{C}_0^2(\mathbb{G}) : -\tfrac{B}{B+1}f'(0) + \tfrac{1}{2}\tfrac{1}{B+1}f''(0) = 0\right\}.$$

As we shall see in a moment, no correction will be necessary, and therefore we define $\Xi_n \equiv 0$. Let $f \in \mathfrak{D}(\mathsf{L}^2_{\mathrm{sbm}})$, which yields the boundary conditions $Bf'(0) = \frac{1}{2}f''(0)$ and $Bf'''(0) = \frac{1}{2}f''''(0)$. Keeping this conditions in mind and also that $\beta = 1$ and $\alpha \in (2,\infty)$, we obtain from Equation (2.3.8) and a Taylor expansion, for some $0 \leq \eta \leq 1/n$, that

$$\begin{aligned}
n^2\mathsf{L}_n\Phi_n f\,(0) &= -\tfrac{A}{n^{\alpha-2}}f\,(0) + Bf'(0) + \tfrac{B}{2n}f''(0) + \tfrac{B}{3!n^2}f'''(0) + \tfrac{B}{4!n^3}f''''(\eta) \\
&= -\tfrac{A}{n^{\alpha-2}}f\,(0) + \left(\tfrac{1}{2} + \tfrac{B}{2n}\right)f''(0) + \tfrac{1}{2\cdot 3!n^2}f''''\,(0) + \tfrac{B}{4!n^3}f''''\,(\eta) \\
&= \mathsf{L}_{\mathrm{sbm}}f(0) + \|f\|\cdot O\big(\tfrac{1}{n^{\alpha-2}}\big) + \|\mathsf{L}_{\mathrm{sbm}}f\|\cdot O\big(\tfrac{1}{n}\big) + \big\|\mathsf{L}^2_{\mathrm{sbm}}f\big\|\cdot O\big(\tfrac{1}{n^2}\big),
\end{aligned}$$

Thus,

$$\left|\left(\pi_n\mathsf{L}_{\mathrm{sbm}}f - n^2\mathsf{L}_n\Phi_n f\right)(0)\right| = \|f\|\cdot O\left(\tfrac{1}{n^{\alpha-2}}\right) + \|\mathsf{L}_{\mathrm{sbm}}f\|\cdot O\big(\tfrac{1}{n}\big) + \big\|\mathsf{L}^2_{\mathrm{sbm}}f\big\|\cdot O\big(\tfrac{1}{n^2}\big).$$

Recalling (2.3.10) and our choice of $\Xi_n \equiv 0$, it yields

$$\left\|\pi_n\mathsf{L}_{\mathrm{sbm}}f - n^2\mathsf{L}_n\Phi_n f\right\| \lesssim \tfrac{1}{n^{\alpha-2}}\|f\| + \tfrac{1}{n}\|\mathsf{L}_{\mathrm{sbm}}f\| + \tfrac{1}{n^2}\big\|\mathsf{L}^2_{\mathrm{sbm}}f\big\|. \tag{2.3.17}$$

Thus, Theorem 1.3 shows that

$$\mathbf{d}(\mu_n,\mu) \lesssim \max\left\{\tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n^2}, \tfrac{1}{n}\right\} = \max\left\{\tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n}\right\},$$

and also that $\{X_{tn^2} : t \geq 0\}$ converges weakly to $\{X^{\mathrm{sbm}}(t) : t \geq 0\}$ under the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$. Hence, we can conclude this case.

The exponential holding BM: the case $\alpha = 2$ and $\beta \in (1,\infty]$

Recall that in this case we set

$$c_1 = \frac{A}{A+1}, \quad c_2 = 0 \quad \text{and} \quad c_3 = \frac{1}{A+1}$$

We denote by $\mathsf{L}_{\mathrm{ehbm}}$ the generator of the exponential holding Brownian motion. Its domain is given by

$$\mathfrak{D}\left(\mathsf{L}_{\mathrm{ehbm}}\right) = \left\{f \in \mathcal{C}^2(\mathbb{G}) : \tfrac{A}{A+1}f\,(0) + \tfrac{1}{2}\tfrac{1}{A+1}f''(0) = 0\right\}.$$

For $\alpha = 2$ and $\beta \in (1,\infty)$, it is necessary to analyze the second parameter β across two distinct regions: one where it is possible to achieve a rate of convergence, and another where such rate of convergence cannot be attained. Again, the proof is divided into two cases:

- $\beta > 2$,
- $\beta \in (1, 2]$.

Subcase $\beta > 2$

Assume for now $\beta > 2$. Consider the correction operator identically null, that is, $\Xi_n \equiv 0$. Let $f \in \mathfrak{D}(\mathsf{L}^2_{\text{ehbm}})$, which yields the boundary conditions $Af(0) = -\frac{1}{2}f''(0)$ and $Af''(0) = -\frac{1}{2}f''''(0)$. Then,

$$\begin{aligned} n^2\mathsf{L}_n\Phi_n f(0) &= -Af(0) + \tfrac{B}{n^{\beta-2}}\Big[f\left(\tfrac{1}{n}\right) - f(0)\Big] \\ &= \tfrac{1}{2}f''(0) + \tfrac{B}{n^{\beta-2}}\Big[f\left(\tfrac{1}{n}\right) - f(0)\Big] \\ &= \mathsf{L}_{\text{ehbm}}f(0) + \|f\| \cdot O\Big(\tfrac{1}{n^{\beta-2}}\Big), \end{aligned} \tag{2.3.18}$$

which implies

$$\left|\left(\pi_n\mathsf{L}_{\text{ehbm}}f - n^2\mathsf{L}_n\Phi_n f\right)(0)\right| \;\leq\; \tfrac{2}{n^{\beta-2}}\,\|f\|$$

and consequently taking (2.3.10) into account

$$\Big\|\pi_n\mathsf{L}_{\text{ehbm}}f - n^2\mathsf{L}_n\Phi_n f\Big\| \;\lesssim\; \tfrac{1}{n^{\beta-2}}\,\|f\| + \tfrac{1}{n^2}\,\Big\|\mathsf{L}^2_{\text{ehbm}}f\Big\|\,. \tag{2.3.19}$$

Thus, Theorem 1.3 implies that

$$\mathbf{d}(\mu, \mu_n) \;\lesssim\; \max\left\{\tfrac{1}{n^{\beta-2}}, \tfrac{1}{n^2}, \tfrac{1}{n}\right\} \;=\; \max\left\{\tfrac{1}{n^{\beta-2}}, \tfrac{1}{n}\right\},$$

and also that $\{X_{tn^2} : t \geq 0\}$ converges weakly to $\{X^{\text{ehbm}}(t) : t \geq 0\}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{G}}[0, \infty)$.

Subcase $\beta \in (1, 2]$

We turn to the case $\beta \in (1, 2]$. Note that as a consequence of (2.3.18), we get

$$\Big\|\pi_n\mathsf{L}_{\text{ehbm}}f - n^2\mathsf{L}_n\pi_n f\Big\| \to 0.$$

Applying *mutatis mutandis* [Ethier and Kurtz, 1986, Theorem 6.1, page 28] and also [Ethier and Kurtz, 1986, Theorem 2.11, page 172] one can conclude the convergence towards the exponential holding BM. However, since the rate of convergence relies on the first derivative of f, we are not allowed to apply Theorem 1.3, and no speed of convergence could be provided in this case.

The reflected BM: the case $\beta \in [0, 1), \alpha > \beta + 1$

Denote by L_{rbm} the generator of the reflected Brownian motion, whose domain is

$$\mathfrak{D}(\mathsf{L}_{\text{rbm}}) \;:=\; \left\{f \in \mathcal{C}^2_0(\mathbb{G}) : f'(0) = 0\right\}.$$

Let $f \in \mathfrak{D}\left(\mathsf{L}^2_{\text{rbm}}\right)$, then $f'(0) = f'''(0) = 0$. Thus,

$$\begin{aligned} n^2\mathsf{L}_n\Phi_n f(0) = -An^{2-\alpha}f(0) &+ \tfrac{B}{2n^\beta}f''(0) + \tfrac{B}{4!n^{2+\beta}}f''''(\eta) \\ &- \tfrac{A}{n^{\alpha-2}}\Xi_n f(0) + \tfrac{B}{n^{\beta-2}}\left[\Xi_n f\left(\tfrac{1}{n}\right) - \Xi_n f(0)\right], \end{aligned} \tag{2.3.20}$$

for some $0 \leq \eta \leq 1/n$. Let $\beta[0, 1)$, the analysis of the behavior of the correction above will be systematically divided into three distinct subcases, namely:

- $1+\beta<\alpha<2\,,$
- $\alpha=2\,,$
- $\alpha>2\,.$

Each of them will be detailed in the following sections.

Subcase $\beta\in[0,1),\beta+1<\alpha<2$

Assume the following subcase $\beta\in[0,1),\beta+1<\alpha<2$. Let

$$\Xi_n f\left(\tfrac{x}{n}\right) \;:=\; \widehat{\Xi}_n f\left(\tfrac{x}{n}\right)+\widetilde{\Xi}_n f\left(\tfrac{x}{n}\right)\,,$$

where

$$\widehat{\Xi}_n f\left(\tfrac{x}{n}\right) \;=\; -\frac{Af\left(0\right)}{Bn^{\alpha-\beta-1}}h\left(\tfrac{x}{n}\right) \quad \text{and} \quad \widetilde{\Xi}_n f\left(\tfrac{x}{n}\right) \;=\; -\frac{\frac{1}{2}f''(0)}{An^{2-\alpha}\left(1+\frac{1}{n}g\left(\frac{x}{n}\right)\right)}\,,$$

where it is assumed that h is a fixed smooth compactly supported function satisfying $h(0)=h''(0)=0$, $h'(0)=1$, while g is a fixed nonnegative smooth compact supported function satisfying $g(0)=0$.

As we shall see in a moment, $n^2\mathsf{L}_n\widehat{\Xi}_n f\left(\frac{0}{n}\right)$ plays the role of canceling the exploding term $-An^{2-\alpha}$ in (2.3.20), while $n^2\mathsf{L}_n\widetilde{\Xi}_n f\left(\frac{0}{n}\right)$ converges to $\frac{1}{2}f''(0)=\mathsf{L}_{\text{rbm}}(0)$, thus "correcting" the limit of the generator at zero. Furthermore, the discrete Laplacian of both functions outside $0/n$ will be uniformly asymptotically null. First of all, note that

$$\|\Xi_n f\| \;\lesssim\; \tfrac{\|f\|}{n^{\alpha-\beta-1}}+\tfrac{\|f''\|}{n^{2-\alpha}} \tag{2.3.21}$$

which converges to zero since $\beta+1<\alpha<2$, verifying hypothesis (A3). Our goal now is to check (A2). Since $h(0)=0$, and it is smooth, it yields that

$$\begin{aligned} n^2\mathsf{L}_n\widehat{\Xi}_n f\left(\tfrac{0}{n}\right) \;&=\; -\,An^{2-\alpha}\widehat{\Xi}_n f\left(0\right)+Bn^{2-\beta}\left[\widehat{\Xi}_n f\left(\tfrac{1}{n}\right)-\widehat{\Xi}_n f\left(0\right)\right]\\ &=Bn^{2-\beta}\widehat{\Xi}_n f\left(\tfrac{1}{n}\right)\\ &\quad-\,Af\left(0\right)n^{3-\alpha}\left[h(0)+h'(0)\tfrac{1}{n}+\tfrac{h''(0)}{2!}\tfrac{1}{n^2}++\tfrac{h'''(\theta)}{3!}\tfrac{1}{n^3}\right]\,, \end{aligned}$$

for some $\theta\in[0,1/n]$. Since $h(0)=h''(0)=0$ and $h'(0)=1$, we conclude that

$$n^2\mathsf{L}_n\widehat{\Xi}_n f\left(\tfrac{0}{n}\right) \;=\; -\,Af\left(0\right)n^{2-\alpha}+\|f\|\cdot O(\tfrac{1}{n^\alpha})\,.$$

On the other hand,

$$\begin{aligned} n^2\mathsf{L}_n\widetilde{\Xi}_n f\left(\tfrac{0}{n}\right) \;&=\; -\,An^{2-\alpha}\widetilde{\Xi}_n f\left(0\right)+Bn^{2-\beta}\left[\widetilde{\Xi}_n f\left(\tfrac{1}{n}\right)-\widetilde{\Xi}_n f\left(0\right)\right]\\ &=\tfrac{1}{2}f''(0)+\tfrac{B}{2A}n^{\alpha-\beta}f''(0)\frac{\frac{1}{n}g\left(\frac{1}{n}\right)}{1+\frac{1}{n}g\left(\frac{1}{n}\right)}\\ &=\mathsf{L}_{\text{rbm}}f\left(0\right)+\|\mathsf{L}_{\text{rbm}}f\|\cdot O(\tfrac{1}{n^{2-\alpha+\beta}}) \end{aligned}$$

since $g(0)=0$ and g is smooth. Therefore, recalling (2.3.20),

$$\begin{aligned} &\left|\pi_n\mathsf{L}_{\text{rbm}}f\left(0\right)-n^2\mathsf{L}_n\Phi_n f\left(0\right)\right|\\ &\lesssim\; \tfrac{1}{n^\alpha}\|f\|+\left(\tfrac{1}{n^{2-\alpha+\beta}}+\tfrac{1}{n^\beta}\right)\|\mathsf{L}_{\text{rbm}}f\|+\tfrac{1}{n^{2+\beta}}\|\mathsf{L}^2_{\text{rbm}}f\|\,. \end{aligned}$$

Let us deal with the convergence outside zero. By the usual convergence of the discrete Laplacian towards the continuous Laplacian, it is easy to check that, for $\frac{x}{n}\in\mathbb{G}_n\backslash\{0\}$,

$$\left|n^2\mathsf{L}_n\widehat{\Xi}_n f\left(\tfrac{x}{n}\right)\right| \;\lesssim\; \frac{|f(0)|}{n^{\alpha-\beta-1}}\Big[\|h''\|+\tfrac{\|h''''\|}{n^2}\Big] \;\lesssim\; \frac{\|f\|}{n^{\alpha-\beta-1}}\,.$$

On the other hand, also for for $\frac{x}{n}\in\mathbb{G}_n\backslash\{0\}$, we have

$$\begin{aligned} n^2\mathsf{L}_n\widetilde{\Xi}_n f\left(\tfrac{x}{n}\right) \;&=\; -\tfrac{f''(0)n^\alpha}{2A}\left[\frac{\frac{1}{n}\left(g(\frac{x}{n})-g\left(\frac{x+1}{n}\right)\right)}{\left(1+\frac{1}{n}g(\frac{x}{n})\right)\left(1+\frac{1}{n}g\left(\frac{x+1}{n}\right)\right)}+\frac{\frac{1}{n}\left(g(\frac{x}{n})-g\left(\frac{x-1}{n}\right)\right)}{\left(1+\frac{1}{n}g(\frac{x}{n})\right)\left(1+\frac{1}{n}g\left(\frac{x-1}{n}\right)\right)}\right]\\ &=\; \|f''\|\cdot O(\tfrac{1}{n^{2-\alpha}})\,. \end{aligned}$$

Putting together all those bounds with (2.3.20) and (2.3.10), we finally get

$$\begin{aligned} \left\|\pi_n\mathsf{L}_{\text{rbm}}f-n^2\mathsf{L}_n\Phi_n f\right\| \;\lesssim\; &\max\left\{\tfrac{1}{n^\alpha},\tfrac{1}{n^{\alpha-\beta-1}}\right\}\|f\|\\ &+\max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n^{2-\alpha+\beta}},\tfrac{1}{n^{2-\alpha}}\right\}\|\mathsf{L}_{\text{rbm}}f\| \\ &+\max\left\{\tfrac{1}{n^{2+\beta}},\tfrac{1}{n^2}\right\}\left\|\mathsf{L}^2_{\text{rbm}}f\right\|\,. \end{aligned} \tag{2.3.22}$$

In view of (2.3.22) and (2.3.21), we can apply Theorem 1.3, hence giving us that

$$\begin{aligned} \mathbf{d}(\mu_n,\mu) \;&\lesssim\; \max\left\{\tfrac{1}{n},\tfrac{1}{n^\alpha},\tfrac{1}{n^\beta},\tfrac{1}{n^{\alpha-\beta-1}},\tfrac{1}{n^{2-\alpha}},\tfrac{1}{n^{2+\beta}},\tfrac{1}{n^2}\right\}\\ &=\; \max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n^{\alpha-\beta-1}},\tfrac{1}{n^{2-\alpha}}\right\} \end{aligned}$$

and that $\{X_n(t)\,:\,t\geq\}$ weakly converges to $\{X^{\text{rbm}}(t)\,:\,t\geq 0\}$ under the J_1-Skorohod topology on $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$, ending this subcase.

Subcase $\beta\in[0,1),\alpha=2$

Let us start the subcase $\beta\in[0,1),\alpha=2$. Unlike in the previous subcase, here the parcel $-An^{2-\alpha}f(0)$ coming from (2.3.20) does not explode, being a constant. In this situation we define

$$\Xi_n f\left(\tfrac{x}{n}\right) \;:=\; -\frac{\left(\frac{1}{2}f''(0)+Af(0)\right)}{Bn^{1-\beta}}h\left(\tfrac{x}{n}\right)$$

where h is a fixed smooth compactly supported function satisfying $h(0)=h''(0)=0$ and $h'(0)=1$. Note that

$$\|\Xi_n f\| \;\lesssim\; \tfrac{\|f\|+\|f''\|}{n^{1-\beta}} \tag{2.3.23}$$

which converges to zero since $\beta\in[0,1)$. Moreover

$$\begin{aligned} n^2\mathsf{L}_n\Xi_n f\left(\tfrac{0}{n}\right) \;&=\; -An^{2-\alpha}\Xi_n f(0)+Bn^{2-\beta}\left[\Xi_n f\left(\tfrac{1}{n}\right)-\Xi_n f(0)\right]\\ &=\Big(\tfrac{1}{2}f''(0)+Af(0)\Big)\Big[h'(0)+\tfrac{h''(\theta)}{2!n}\Big]\\ &=\tfrac{1}{2}f''(0)+Af(0)+(\|f\|+\|f''\|)\cdot O(\tfrac{1}{n})\,. \end{aligned}$$

Plugging this bound into (2.3.20), we get

$$\begin{aligned} \left\|\pi_n\mathsf{L}_{\text{rbm}}f-n^2\mathsf{L}_n\Phi_n f\right\| \;\lesssim\; &\max\left\{\tfrac{1}{n},\tfrac{1}{n^{1-\beta}}\right\}\|f\|\\ &+\max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n}\right\}\|\mathsf{L}_{\text{rbm}}f\|\\ &+\max\left\{\tfrac{1}{n^{2+\beta}},\tfrac{1}{n^2}\right\}\left\|\mathsf{L}^2_{\text{rbm}}f\right\|\,. \end{aligned}$$

In view of the inequality above and (2.3.23), we can apply Theorem 1.3, hence giving us that

$$\begin{aligned} \mathbf{d}(\mu_n,\mu) \;&\lesssim\; \max\left\{\tfrac{1}{n},\tfrac{1}{n^\beta},\tfrac{1}{n^{1-\beta}},\tfrac{1}{n^{2+\beta}},\tfrac{1}{n^2}\right\}\\ &=\; \max\left\{\tfrac{1}{n^\beta},\tfrac{1}{n^{1-\beta}}\right\} \end{aligned}$$

and $\{\mathsf{X}_{\mathsf{n}}(t):t\geq 0\}$ weakly converges to $\{X^{\text{rbm}}(t):t\geq 0\}$ under the J_1-Skorohod topology on $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$, ending this subcase.

Subcase $\beta \in [0,1), \alpha > 2$

Here the parcel $-An^{2-\alpha} f(0) = \|f\| \cdot O(1/n^{\alpha-2})$ coming from (2.3.20) vanishes as n goes to infinity. We then define

$$\Xi_n f\left(\tfrac{x}{n}\right) := -\frac{\frac{1}{2}f''(0)}{Bn^{1-\beta}} h\left(\tfrac{x}{n}\right)$$

where, as before, h is a fixed smooth, compactly supported function satisfying $h(0) = h''(0) = 0$ and $h'(0) = 1$. Analogously to what we did before in the previous subcases,

$$\|\Xi_n f\| \lesssim \frac{\|f''\|}{n^{1-\beta}} \tag{2.3.24}$$

and

$$\begin{aligned} n^2 \mathsf{L}_n \Xi_n f\left(\tfrac{0}{n}\right) &= -An^{2-\alpha}\Xi_n f(0) + Bn^{2-\beta}\left[\Xi_n f\left(\tfrac{1}{n}\right) - \Xi_n f(0)\right] \\ &= \tfrac{1}{2} f''(0) + \|f''\| \cdot O(\tfrac{1}{n}). \end{aligned}$$

Plugging this bound into (2.3.20), we get

$$\begin{aligned} \left\|\pi_n \mathsf{L}_{\text{rbm}} f - n^2 \mathsf{L}_n \Phi_n f\right\| \lesssim\ & \frac{1}{n^{\alpha-2}} \|f\| \\ & + \max\left\{\tfrac{1}{n^\beta}, \tfrac{1}{n}, \tfrac{1}{n^{1-\beta}}\right\} \|\mathsf{L}_{\text{rbm}} f\| \\ & + \max\left\{\tfrac{1}{n^{2+\beta}}, \tfrac{1}{n^2}\right\} \left\|\mathsf{L}^2_{\text{rbm}} f\right\|. \end{aligned}$$

Denote by μ the distribution of the reflected BM at time $t > 0$. In view of the inequality above and (2.3.24), we can apply Theorem 1.3, hence giving us that

$$\mathbf{d}(\mu_n, \mu) \lesssim \max\left\{\tfrac{1}{n}, \tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n^{1-\beta}}, \tfrac{1}{n^\beta}, \tfrac{1}{n^{2+\beta}}, \tfrac{1}{n^2}\right\} = \max\left\{\tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n^{1-\beta}}, \tfrac{1}{n^\beta}\right\}$$

and $\{X_n(t) : t \geq\}$ weakly converges to $\{X^{\text{rbm}}(t) : t \geq 0\}$ under the J_1-Skorohod topology on $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$, ending this subcase and completing the case $\beta \in [0,1)$ and $1 + \beta < \alpha$.

The absorbed BM: the case $\beta > 1, \alpha > 2$

Denote the generator of the absorbed Brownian motion by L_{abm}. Its domain is

$$\mathfrak{D}\left(\mathsf{L}_{\text{abm}}\right) := \{f \in \mathcal{C}^2_0(\mathbb{G}) : f''(0) = 0\}.$$

To rigorously address this case, it is necessary to analyze two distinct subcases, which are outlined as follows:

- $\alpha > 2$ and $\beta > 2$,
- $\alpha > 2$ and $\beta \in (1, 2]$.

Subcase $\alpha > 2$ **and** $\beta > 2$

Consider a null correction $\Xi_n \equiv 0$. Then

$$n^2 \mathsf{L}_n \Phi_n f(0) = -\frac{Af(0)}{n^{\alpha-2}} + \frac{B}{n^{\beta-2}}\left[f\left(\tfrac{1}{n}\right) - f(0)\right],$$

thus combining this with (2.3.10)

$$\left\|\pi_n \mathsf{L}_{\text{abm}} f - n^2 \mathsf{L}_n \Phi_n f\right\| \lesssim \max\left\{\tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n^{\beta-2}}\right\} \|f\| + \tfrac{1}{n^2} \left\|\mathsf{L}^2_{\text{abm}} f\right\|. \tag{2.3.25}$$

Denote by μ the probability measure of the absorbed BM at time $t > 0$. In view of (2.3.25), we can apply Theorem 1.3, which gives us

$$\mathbf{d}(\mu_n, \mu) \lesssim \max\left\{\tfrac{1}{n}, \tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n^{\beta-2}}, \tfrac{1}{n^2}\right\} = \max\left\{\tfrac{1}{n}, \tfrac{1}{n^{\alpha-2}}, \tfrac{1}{n^{\beta-2}}\right\}$$

and $\{\mathsf{X}_n(t) : t \geq 0\}$ weakly converges to $\left\{\mathsf{X}^{\text{abm}}(t) : t \geq 0\right\}$ under the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{R}_{\geq 0}}[0,\infty)$.

Subcase $\alpha > 2$ and $\beta \in (1, 2]$

Similarly to what happened in the exponential holding BM in the strip $1 < \beta < 2$, we can check that $\|\pi_n \mathsf{L}_{\text{abm}} f - n^2 \mathsf{L}_n \pi_n f\| \to 0$, so we can apply *mutatis mutandis* [Ethier and Kurtz, 1986, Theorem 6.1, page 28] and [Ethier and Kurtz, 1986, Theorem 2.11, page 172] to deduce the convergence towards the Absorbed BM. However, since the rate of convergence relies on the first derivative of f, we cannot apply Theorem 1.3, hence no speed of convergence is provided in this case.

The mixed BM: the case $\alpha = 2$ and $\beta = 1$

Consider

$$c_1 = \frac{A}{1+A+B} \qquad c_2 = \frac{B}{1+A+B} \qquad \text{and} \qquad c_3 = \frac{1}{1+A+B}\,.$$

Let L_{mbm} be the generator of the mixed Brownian motion, whose domain is given by

$$\mathfrak{D}\left(\mathsf{L}_{\text{mbm}}\right) := \Big\{ f \in \mathcal{C}_0^2(\mathbb{G}) : \tfrac{Af(0)}{1+A+B} - \tfrac{Bf'(0)}{1+A+B} + \tfrac{\frac{1}{2}f''(0)}{(1+A+B)} = 0 \Big\}.$$

In this case, also no correction is needed, so define $\Xi_n \equiv 0$ to be the null operator. Since $f \in \mathfrak{D}(\mathsf{L}^2_{\text{mbm}})$, then we get the boundary conditions $Bf'(0) = Af(0) + \frac{1}{2}f''(0)$ and $Bf'''(0) = Af''(0) + \frac{1}{2}f''''(0)$. For $\alpha = 2$ and $\beta = 1$, we have by applying the boundary conditions and from (2.3.8), that

$$\begin{aligned} n^2 \mathsf{L}_n \Phi_n f(0) &= (Bf'(0) - Af(0)) + \tfrac{B}{2!n} f''(0) + \tfrac{B}{3!n^2} f'''(0) + \tfrac{B}{4!n^3} f''''(\eta) \\ &= \tfrac{1}{2} f''(0) + \left(\tfrac{A}{3!n^2} + \tfrac{B}{2!n}\right) f''(0) + \tfrac{1}{2\cdot 3!n^2} f''''(0) + \tfrac{B}{4!n^3} f''''(\eta) \\ &= \mathsf{L}_{\text{mbm}} f(0) + \|\mathsf{L}_{\text{mbm}} f\| \cdot O\!\left(\tfrac{1}{n}\right) + \left\|\mathsf{L}^2_{\text{mbm}} f\right\| \cdot O\!\left(\tfrac{1}{n^2}\right) \end{aligned}$$

for some $0 \le \eta \le 1/n$. Thus

$$|\left(\pi_n \mathsf{L}_{\text{mbm}} f - n^2 \mathsf{L}_n \Phi_n f\right)(0)| \le \|\mathsf{L}_{\text{mbm}} f\| \cdot O\!\left(\tfrac{1}{n}\right) + \left\|\mathsf{L}^2_{\text{mbm}} f\right\| \cdot O\!\left(\tfrac{1}{n^2}\right).$$

By the above bound,

$$\left\|\pi_n \mathsf{L}_{\text{mbm}} f - n^2 \mathsf{L}_n \Phi_n f\right\| \lesssim \tfrac{1}{n} \|\mathsf{L}_{\text{mbm}} f\| + \tfrac{1}{n^2} \left\|\mathsf{L}^2_{\text{mbm}} f\right\|.$$

Denote by μ the probability measure of the mixed BM at time $t > 0$. Thus, by the previous inequality we can invoke Theorem 1.3, concluding that

$$\mathbf{d}(\mu_n, \mu) \lesssim \max\left\{\tfrac{1}{n}, \tfrac{1}{n^2}\right\} = \tfrac{1}{n}$$

and that $\{\mathsf{X}_\mathsf{n}(t) : t \ge 0\}$ weakly converges to $\{X^{\text{mbm}}(t) : t \ge 0\}$ under the J_1-Skorohod topology on $\mathsf{D}_{\mathbb{G}}[0, \infty)$. ☐

It now remains to prove Theorem 2.3. Before delving into the details we recall that the topology is different here, because $\mathsf{S} = (0, \infty)$. In this scenario, the functions in the space $\mathcal{C}_0(\mathsf{S})$ must converge to zero at zero, see Definition 1.2.

Since the topology is different from the previous setup, we need to check again hypothesis (G2). Recall the definition of the functions f_k in (2.3.4).

Proposition 2.1. Let $\mathcal{B} = \{f_i : i \ge 1\}$. Then $\text{span}(\mathcal{B})$ is dense on $\mathcal{C}_0\big((0, \infty)\big)$.

Proof. Immediate from Corollary 2.1 and the fact that $f_i(0) = 0$ for any $i \ge 1$. ☐

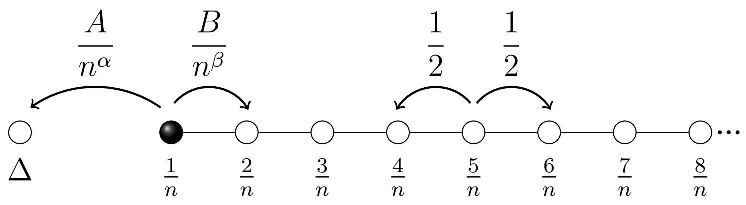

Figure 2.5: Jump rates for the *shifted* boundary random walk.

Proof of Theorem 2.3. The Proposition 2.1 ensures (G2). To ease arguments, note that $\tau_n X_n$ is the same as considering the boundary random walk X_n on the state space in Figure 2.5. Denote by X^{kbm} the killed Brownian motion of coefficients $c_1 = 0$, $c_2 = 1$, $c_3 = 0$ and let L_{kbm} be its the generator, whose domain is given by

$$\mathfrak{D}(\mathsf{L}_{\text{kbm}}) := \Big\{ f \in \mathcal{C}_0^2(\mathbb{G}) : f(0) = 0 \Big\}.$$

Let $f \in \mathfrak{D}(\mathsf{L}_{\text{kbm}})$. By the definition of $\mathcal{C}_0(\mathbb{G})$ and since $\mathsf{L}_{\text{kbm}} f \in \mathcal{C}_0(\mathbb{G})$, we infer that $f''(0) = 0$. By doing Taylor expansions around zero and using that $f(0) = f''(0) = 0$, we get

$$\begin{aligned} n^2 \mathsf{L}_n f\left(\tfrac{1}{n}\right) &= \tfrac{An^2}{n^\alpha}\Big[f(\Delta) - f\left(\tfrac{1}{n}\right)\Big] + \tfrac{Bn^2}{n^\beta}\Big[f\left(\tfrac{2}{n}\right) - f\left(\tfrac{1}{n}\right)\Big] \\ &= -An^{2-\alpha}\Big[f(0) + f'(0)\tfrac{1}{n} + \tfrac{f''(0)}{2!}\tfrac{1}{n^2} + \tfrac{f'''(\theta_1)}{3!}\tfrac{1}{n^3}\Big] \\ &\quad + Bn^{2-\beta}\Big[f'(0)\tfrac{2}{n} + \tfrac{f''(0)}{2!}\left(\tfrac{2}{n}\right)^2 + \tfrac{f'''(\theta_2)}{3!}\left(\tfrac{2}{n}\right)^3 - f'(0)\tfrac{1}{n} - \tfrac{f''(0)}{2!}\tfrac{1}{n^2} - \tfrac{f'''(\theta_3)}{3!}\tfrac{1}{n^3}\Big] \\ &= f'\left(\tfrac{0}{n}\right)\Big[Bn^{1-\beta} - An^{1-\alpha}\Big] + O(\max\{\tfrac{1}{n^{1+\beta}}, \tfrac{1}{n^{1+\alpha}}\}) \end{aligned} \tag{2.3.26}$$

where $\theta_1, \theta_2, \theta_3 \in [0, \frac{1}{n}]$. On the other hand, if our correction Ξ_n is such that $\Xi_n f(\Delta) = 0$,

$$\begin{aligned} n^2 \mathsf{L}_n \Xi_n f\left(\tfrac{1}{n}\right) &= \tfrac{An^2}{n^\alpha}\Big[\Xi_n f(\Delta) - \Xi_n f\left(\tfrac{1}{n}\right)\Big] + \tfrac{Bn^2}{n^\beta}\Big[\Xi_n f\left(\tfrac{2}{n}\right) - \Xi_n f\left(\tfrac{1}{n}\right)\Big] \\ &= -An^{2-\alpha}\Xi_n f\left(\tfrac{1}{n}\right) + Bn^{2-\beta}\Big[\Xi_n f\left(\tfrac{2}{n}\right) - \Xi_n f\left(\tfrac{1}{n}\right)\Big]. \end{aligned} \tag{2.3.27}$$

Since $\mathsf{L}_{\text{kbm}} f(0) = \frac{1}{2} f''(0) = 0$, our goal is to find a correction Ξ_n such that (2.3.27) cancels the non-vanishing terms in (2.3.26).

Subcase $\beta > 1$

In this case, the term $Bn^{1-\beta}$ in (2.3.26) vanishes as $n \to \infty$, and we only need to deal with $An^{1-\alpha}$. Define

$$\Xi_n f\left(\tfrac{x}{n}\right) := -\frac{f'(0) \cdot n^{-1}}{1 + \frac{g\frac{x-1}{n}}{n}}$$

for $\frac{x}{n} \geq \frac{1}{n}$, where $g : \mathbb{G} \to \mathbb{R}$ is a nonnegative compactly supported smooth function such that $g(0) = 0$. It is now straightforward to check that $\|\pi_n \mathsf{L}_{\text{kbm}} f - n^2 \mathsf{L}_n \Phi_n f\| \to 0$. Thus, applying *mutatis mutandis* [Ethier and Kurtz, 1986, Theorem 6.1, page 28] and [Ethier and Kurtz, 1986, Theorem 2.11, page 172], we conclude that $\{X_n(t) : t \geq 0\}$ weakly converges to $\{X^{\text{kbm}}(t) : t \geq 0\}$ under the J_1-Skorohod topology on $\mathsf{D}_{\mathbb{G}}[0, \infty)$. Note that $\|\Xi_n f\| = \|f'\| O(1/n)$, so we cannot apply our Theorem 1.2 and no speed of convergence is provided.

Subcase $\beta \in [0,1], \alpha < 1+\beta$

In this scenario, the parcel $Bn^{1-\beta}$ does not vanish as $n \to \infty$, and we need to define an extra correction to deal with it. Define

$$\Xi_n f\left(\tfrac{x}{n}\right) \;:=\; -\frac{f'(0)\cdot n^{-1}}{1+\frac{g\left(\frac{x-1}{n}\right)}{n}} + \frac{\frac{B}{A}f'(0)\cdot n^{\alpha-\beta-1}}{1+\frac{g\left(\frac{x-1}{n}\right)}{n}}$$

for $\frac{x}{n} \geq \frac{1}{n}$ and let g be the same function as in the previous case. Note that the condition $\alpha < 1+\beta$ guarantees that the correction above goes to zero as $n \to \infty$. It is now straightforward to check that $\|\pi_n \mathsf{L}_{\text{kbm}} f - n^2 \mathsf{L}_n \Phi_n f\| \to 0$, leading us to conclude that $\{X_n(t) : t \geq 0\}$ weakly converges to $\{X^{\text{kbm}}(t) : t \geq 0\}$ under the J_1-Skorohod topology on $\mathsf{D}_{\mathbb{G}}[0,\infty)$. □

Remark 2.2. Note that the region corresponding to the parameters (β, α) for which the shifted boundary random walk converges to the killed BM covers the white region in the first quadrant of Figure 2.3 and also the regions related to the exponential holding BM and absorbed BM. This is natural to interpret once we realize that in the killed BM setting there is an equivalence between the origin and the cemetery Δ.

Chapter 3

Weak *Berry-Esseen* estimates for the slow bond random walk

3.1 Introduction

Throughout this chapter, our goal is to construct the shifted 1-dimensional Snapping-Out Brownian motion as the scaling limit of the slow bond random walk as well as to exhibit a *weak Berry Esseen* estimate.

Rigorously, consider X_t^{slow}, the process having its exchange rates given by α/n^β whenever the particle tries to jump from the positive half-line to the negative one, and vice versa, see the figure 3.1. We can define it as the Feller process whose generator $\mathsf{L_n}$ acts on local functions $f : \mathbb{Z}_n \to \mathbb{R}$ via

$$\mathsf{L_n} f\left(\tfrac{x}{n}\right) = \theta^n_{x,x+1} \nabla^n_{x,x+1} f\left(\tfrac{x}{n}\right) + \theta^n_{x,x-1} \nabla^n_{x,x-1} f\left(\tfrac{x}{n}\right) , \tag{3.1.1}$$

where

$$\theta^n_{x,x+1} = \theta^n_{x+1,x} = \begin{cases} \frac{\alpha}{n^\beta} & \text{if } x = -1 \\ 1 & \text{otherwise.} \end{cases}$$

and $\nabla^n_{x,y} : \mathcal{C}_0(\mathbb{Z}_n) \to \mathcal{C}_0(\mathbb{Z}_n)$ is defined by

$$\nabla^n_{x,y} f\left(\tfrac{x}{n}\right) := f\left(\tfrac{y}{n}\right) - f\left(\tfrac{x}{n}\right)$$

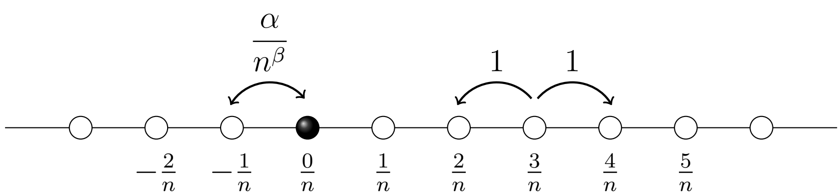

Figure 3.1: Jump rates for the *slow bond random walk*.

An important observation is that this random walk is the same as the one studied by Erhard et al in [Erhard et al., 2021]. In their work, the authors showed that the slow bond random walk weakly converges to the snapping-out Brownian motion and they exhibit *Berry-Esseen* estimates. Now, let us briefly introduce the snapping-out Brownian motion:

The *elastic (or partially reflected) Brownian motion* on $[0,\infty)$ is a continuous stochastic process that serves as an intermediate model between the absorbed Brownian motion and the reflected Brownian motion on $[0,\infty)$. This process can be described as a reflected Brownian motion that is *killed* at a stopping time determined by an exponential distribution. Specifically, for a given positive parameter κ, we sample a random variable $Y \sim \exp(\kappa)$, which is independent of the reflected Brownian motion. The process evolves until the local time of the reflected Brownian motion at zero reaches Y, at which point it is killed (absorbed) at the origin. The process is associated to the Robin boundary condition and it works as a "basic brick" for constructing the snapping-out Brownian motion. For a broader discussion, including connections to the d-dimensional setting, mixed boundary value problems, and Laplacian transport phenomena, we refer to the reader the survey [Grebenkov, 2006] as well as [Feller, 1954] for a comprehensive overview.

The *snapping out Brownian motion* (SNOB) process on $\mathbb{G} = (-\infty, 0^-] \cup [0^+, \infty)$, with parameter κ, is a Feller process recently constructed in [Lejay, 2016] by *gluing* pieces of the elastic Brownian motion with parameter 2κ. When the 2κ-elastic Brownian motion is killed at zero, the process is restarted at 0^+ or 0^- with equal probability $1/2$. An equivalent way to define this process is to consider the κ-elastic Brownian motion and, upon being killed at 0^+ (or equivalently at 0^-), restart it on the opposite side, 0^- (or 0^+, respectively).

Alternatively, the snapping-out Brownian motion (SNOB) can be formulated via its *resolvent operator*, providing a rigorous analytical framework for describing the process. A formal characterization of the SNOB has been established as follows:

Proposition 3.1. [Lejay, 2016, Proposition 1, page 7] The resolvent family $(G_\alpha)_{\alpha>0}$ of the SNOB is a solution to

$$\left(\alpha - \frac{1}{2}\Delta\right) G_\alpha f(x) = f(x), \qquad \text{for } x \in \mathbb{G} \tag{3.1.2}$$

with

$$\begin{cases} \nabla G_\alpha f(0^+) = \nabla G_\alpha f(0^-)\,, \\ \nabla G_\alpha f(0) = \frac{\kappa}{2}(G_\alpha f(0^+) - G_\alpha f(0^-))\,, \end{cases} \tag{3.1.3}$$

for any bounded, continuous function f on $\mathbb{G}$ that vanishes at infinity.

It is noteworthy that this proposition allows us to identify the infinitesimal generator of the snapping-out Brownian motion. Furthermore, we can interpret the points 0^+ and 0^- as sides of a semi-permeable barrier which arises for example in diffusion magnetic resonance imaging [Fieremans et al., 2010], or chemistry [Singer et al., 2008]. Additionally, the boundary conditions given by Equation (3.1.3) provide a mathematical framework to describe the interaction of the process with the boundary, encapsulating both reflection and killing effects.

As previously mentioned, [Erhard et al., 2021] has proved an explicit form for the semigroup of the snapping-out Brownian motion as well as *Berry-Esseen* estimates for the convergence under the dual bounded Lipschitz metric. The result can be stated as follows:

Proposition 3.2. [Erhard et al., 2021, Proposition 2.3] Let $(\mathsf{P}^{\text{snob}}(t))_{t\geq 0} : \mathcal{C}_0(\mathbb{G}) \to \mathcal{C}_0(\mathbb{G})$ be the semigroup of the SNOB with parameter κ. Then, for any $f \in \mathcal{C}_0(\mathbb{G})$, we have that $\mathsf{P}^{\text{SNOB}}(t) f(u)$ is the solution of the partial differential equation

$$\begin{cases} \partial_t v(t,u) = \frac{1}{2}\Delta v(t,u), \quad u \neq 0\,, \\ \partial_u v(t,o^+) = \partial_u v(t,0^-) = \frac{\kappa}{2}[v(t,0^+) - v(t,0^-)] \quad ,t>0 \\ v(0,u) = f(u), \quad u \in \mathbb{R}\,. \end{cases}$$

Moreover, the semigroup $(\mathsf{P}^{\text{snob}}(t))_{t\geq 0} : \mathcal{C}_0(\mathbb{G}) \to \mathcal{C}_0(\mathbb{G})$ is given by

$$\begin{aligned}\mathsf{P}^{\text{snob}}(t)f(u) &= \tfrac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\mathrm{d}y \\ &+ \tfrac{1}{\sqrt{2\pi t}} e^{\kappa u} \int_u^{\infty} e^{-\kappa z} \int_0^{\infty} \Big[\Big(\tfrac{z-y+\kappa t}{2t}\Big) e^{-\frac{(z-y)^2}{2t}} + \Big(\tfrac{z+y+\kappa t}{2t}\Big) e^{-\frac{(z+y)^2}{2t}}\Big] f_{\text{odd}}(y)\mathrm{d}y\mathrm{d}z \end{aligned} \tag{3.1.4}$$

for $u > 0$ and

$$\begin{aligned}\mathsf{P}^{\text{snob}}(t)f(u) &= \tfrac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\text{even}}(y)\mathrm{d}y \\ &- \tfrac{1}{\sqrt{2\pi t}} e^{\kappa u} \int_u^{\infty} e^{-\kappa z} \int_0^{\infty} \Big[\Big(\tfrac{z-y+\kappa t}{2t}\Big) e^{-\frac{(z-y)^2}{2t}} + \Big(\tfrac{z+y+\kappa t}{2t}\Big) e^{-\frac{(z+y)^2}{2t}}\Big] f_{\text{odd}}(y)\mathrm{d}y\mathrm{d}z \end{aligned} \tag{3.1.5}$$

for $u < 0$, where $f_{\text{even}}(x) := \frac{1}{2}(f(x) + f(-x))$ and $f_{\text{odd}}(x) := \frac{1}{2}(f(x) - f(-x))$ are the even and the odd parts of f, respectively.

The second result they proved has its foundation on the space of probability measures equipped with a specific metric: The bounded Lipschitz functions $BL(\mathsf{S})$ over a metric space (S, ρ) is the set of real functions on S such that

$$\|f\| := \sup_{u\in\mathsf{S}} |f(u)| < \infty \qquad \text{and}$$

$$\|f\|_L := \sup_{\substack{u,v\in\mathsf{S}\\ u\neq v}} \frac{|f(u) - f(v)|}{\rho(u,v)} < \infty\,.$$

and $BL(\mathsf{S})$ is a normed linear space with norm $\|f\|_{BL} := \|f\| + \|f\|_L$. The *dual bounded Lipschitz metric* $\mathbf{d}_{BL}$ is a metric over the set of probability measures $\mathcal{P}(\mathsf{S})$ given by

$$\mathbf{d}_{BL}(\mu,\nu) := \sup_{\substack{f\in BL(\mathsf{S})\\ \|f\|_{BL}\leq 1}} \left| \int f\mathrm{d}\mu - \int f\mathrm{d}\nu \right|\,.$$

Theorem 3.1. [Erhard et al., 2021, Berry-Esseen estimates, Theorem 2.4] Fix $t > 0$ and $u \neq 0$. Denote by $\mu^{\text{slow}}_{tn^2}$ the probability measure on $\mathbb{R}$ induced by the slow bond random walk $n^{-1}X^{\text{slow}}_{tn^2}$ starting from the site $\lfloor un \rfloor \in \mathbb{Z}$. Denote by μ^{snob}_t and μ^{rbm}_t the probability measures on $\mathsf{S} = \mathbb{G}$ induced by Snapping-Out Brownian motion B^{snob}_t and reflected Brownian motion B^{rbm}_t, respectively, and denote by μ_t the probability measure on $\mathsf{S} = \mathbb{R}$ induced by the Brownian motion B_t. All the previous Brownian motions are assumed to start from u. We have, for the bounded Lipschitz norm, that

- If $\beta \in [0,1)$, then
$$\mathbf{d}_{\text{BL}}(\mu^{\text{slow}}_{tn^2}, \mu_t) \lesssim n^{\beta-1}\,.$$
- If $\beta = 1$, then for any $\delta > 0$,
$$\mathbf{d}_{\text{BL}}(\mu^{\text{slow}}_{tn^2}, \mu^{\text{snob}}_t) \lesssim n^{-1/2+\delta}\,.$$
- If $\beta \in (1,\infty]$, then
$$\mathbf{d}_{\text{BL}}(\mu^{\text{slow}}_{tn^2}, \mu^{\text{rbm}}_t) \lesssim \max\{n^{-1}, n^{1-\beta}\}\,.$$

Later on, we will briefly compare Theorem 3.1 for $\beta = 1$ with the convergence rate derived using the methodology introduced in Chapter 1. We shall see that faster rates of convergence can be established. However, this comes at the cost of working in a weaker framework in terms of the topology where the process takes place.

3.2 The 1-dimensional snapping-out Brownian motion

We will now use Proposition 3.1 to characterize the snapping-out Brownian motion via its generator. Once this is done, we will be able to apply the method presented in Chapter 1 to establish convergence and estimate its rate.

Lemma 3.1. The standard snapping-out Brownian motion over $\mathbb{G}$ has generator $\frac{1}{2}\Delta$ and its domain is given by

$$\mathfrak{D}\left(\mathsf{L}\right) := \left\{ f \in \mathcal{C}_0^2(\mathbb{G}) : f'' \in \mathcal{C}_0(\mathbb{G})\,,\ \text{and}\ \nabla f\left(0^+\right) = \nabla f\left(0^-\right) = \frac{\kappa}{2}\left[f\left(0^+\right) - f\left(0^-\right)\right] \right\} \tag{3.2.1}$$

where $\kappa = 2\alpha$.

Proof. The proof relies on the uniqueness of a solution of an ODE. Let G_α be given as in Proposition 3.1. Since $\{\mathsf{G}_\alpha : \alpha > 0\}$ is the resolvent family, for any α, the map $\mathsf{G}_\alpha : \mathcal{C}_0(\mathbb{G}) \to \mathfrak{D}\left(\mathsf{L}\right) \subset \mathcal{C}_0(\mathbb{G})$ is surjective.

In particular, let us fix $\alpha = 1$ and $g \in \mathcal{C}_0(\mathbb{G})$. From the surjectivity of G_1, it follows that there exists $f \in \mathfrak{D}\left(\mathsf{L}\right)$, such that $f(x) = \mathsf{G}_1 g(x)$ for every $x \in \mathbb{G}$. Since G_α satisfies the relations (3.1.3), one can see that f satisfies the desired boundary conditions. Indeed,

$$\begin{cases} f'(0^+) = (\mathsf{G}g)'(0^+) = (\mathsf{G}_1 g)'(0^-) = f'(0^-) \\ f'(0) = (\mathsf{G}_1 g)'(0) = \frac{\kappa}{2}\left[\mathsf{G}_1 g(0^+) - \mathsf{G}_1 g(0^-)\right] = \frac{\kappa}{2}\left[f(0^+) - f(0^-)\right] \end{cases} \tag{3.2.2}$$

It remains to characterize the generator. Observe now that, from equation (3.1.2),

$$\begin{aligned} g(x) &= \left(\mathsf{I}d - \tfrac{1}{2}\Delta\right) G_1 g(x) \\ &= f(x) - \tfrac{1}{2} f''(x) \qquad \forall x \in \mathbb{G}\,, \end{aligned}$$

and since f satisfies the boundary conditions (3.2.2), from standard ODE theory, it has unique solution. On the other hand, for every $x \in \mathbb{G}$, we have that

$$\begin{aligned} \tfrac{1}{2} f''(x) &= f(x) - g(x) \\ &= \mathsf{G}_1 g(x) - g(x) \\ &= [\mathsf{G}_1 - \mathsf{G}_1^{-1}\mathsf{G}_1] g(x) \\ &= [\mathsf{I}d - (\mathsf{I}d - \mathsf{L})]\mathsf{G}_1 g(x) \\ &= \mathsf{L} f(x)\,, \end{aligned}$$

concluding in this way that generator is $\frac{1}{2}\Delta$, fully characterizing the domain (3.2.1) □

In what follows, we modify the standard Snapping Out Brownian motion by exchanging the position of the process discontinuity. We exchanged the slow site, before living over the sites -1 and 0, now to some middle point over the bond $[-1, 0]$.

Consider $p \in \mathbb{R}$ fixed and define the set $\mathbb{J} := (-\infty, -p^-] \cup [p^+, \infty)$. We will define the toy model in this space.

Lemma 3.2. Let $p \in \mathbb{R}$. Then, for every $n \in \mathbb{N}$, there exists a linear operator $\tau_n : \mathbb{R} \to \mathbb{R}$ such that $\tau_n(p) \in \left(-\frac{1}{n}, \frac{0}{n}\right)$.

Proof. A countability argument ensures that we can always define a family of linear operators $\tau_n : \mathbb{R} \to \mathbb{R}$ indexed in n, which maps p to the usual lattice via $\tau_n(p) := -\frac{p}{n} \in \left(-\frac{1}{n}, 0\right)$ and acts linearly on the other points. □

Let then $\{\mathsf{P}^{\mathbb{J}}(t) : t \geq 0\}$ denote the Feller semigroup of the snapping-out Brownian motion $X_{\mathbb{J}}^{\text{snob}}$ defined over $\mathbb{J}$ whose generator $\mathsf{L}_{\mathbb{J}}$ is $\frac{1}{2}\Delta$ and whose domain is given by

$$\mathfrak{D}\left(\mathsf{L}_{\mathbb{J}}\right) := \left\{ f \in \mathcal{C}_0^2(\mathbb{J}) : f'\left(p^{\pm}\right) = \frac{\kappa}{2}\left(f\left(p^{+}\right) - f\left(p^{-}\right)\right) \right\}.$$

It remains to show that we can approximate this model using the same slow bond random walk: For each $n \in \mathbb{N}$, consider X_n^{slow} to be the simple random walk with a slow bond whose trajectories are defined on the scaled space $\mathbb{J}_n := \tau_n(\mathbb{R}) \cap \mathbb{Z}_n$ and whose generator is defined via (3.1.1) and speeded up by n^2. In addition, we restrict ourselves to the regime in which $\beta = 1$.

Lemma 3.3. Fix $u, t > 0$. Consider the 1-dimensional snapping-out Brownian motion $X_{\mathbb{J}}^{\text{snob}}$ over $\mathbb{J}$, by $\{\mathsf{P}^{\mathbb{J}}(t) : t \geq 0\}$ we mean its Feller semigroup and $\mathsf{L}_{\mathbb{J}}$ its respective generator. For each n, let the slow bond random walk X_n^{slow} speeded up by n^2 defined over $\mathbb{J}_n$, and consider $\{\mathsf{T}_{\mathsf{n}}(t) : t \geq 0\}$ to be its semigroup. Then, for each $t \geq 0$ inside compact intervals and for all $f \in \mathfrak{D}\left(\mathsf{L}_{\mathbb{J}}^2\right)$ it holds that

$$\left\|\pi_n \mathsf{P}^{\mathbb{J}}(t) f - \mathsf{T}_{\mathsf{n}}(t)\pi_n f\right\| = \left(\,\|f\| + \|\Delta f\|\,\right) \cdot O\left(\tfrac{1}{n}\right) + \left\|\Delta^2 f\right\| \cdot O\left(\tfrac{1}{n^2}\right).$$

Proof. For the sake of simplicity, define by $\pi_n : \mathcal{C}_0(\mathbb{J}) \to \mathcal{C}_0(\mathbb{Z}_n)$, the projection given by $\pi_n f = f|_{\mathbb{Z}_n}$. The proof consists of checking the convergence at -1 as well as at the origin 0 since at other sites the random walk is homogeneous, by verifying hypotheses (A1) to (A3) in order to apply Theorem 1.2.

Fix $f \in \mathfrak{D}\left(\mathsf{L}_{\mathbb{J}}^2\right)$. The boundary conditions of the process can be translated into the following relations

$$\begin{aligned}
&f'\left(-\tfrac{p}{n}^{+}\right) = f'\left(-\tfrac{p}{n}^{-}\right) = \alpha\left[f\left(-\tfrac{p}{n}^{+}\right) - f\left(-\tfrac{p}{n}^{-}\right)\right] \\
&\left(\tfrac{1}{2}\Delta f\right)'\left(-\tfrac{p}{n}^{+}\right) = \left(\tfrac{1}{2}\Delta f\right)'\left(-\tfrac{p}{n}^{-}\right) = \alpha\left[\tfrac{1}{2}\Delta f\left(-\tfrac{p}{n}^{+}\right) - \tfrac{1}{2}\Delta f\left(-\tfrac{p}{n}^{+}\right)\right].
\end{aligned}$$

Since the Snapping-Out Brownian motion has a discontinuity at $-\frac{p}{n}$, we have that

$$\begin{aligned}
n^2\mathsf{L}_n\pi_n f\left(\tfrac{0}{n}\right) &= n^2\tfrac{\alpha}{n}\nabla^n_{-1,0}\pi_n f\left(\tfrac{0}{n}\right) + n^2\nabla^n_{1,0}\pi_n f\left(\tfrac{0}{n}\right) \\
&= n\alpha\nabla^n_{-1,-p^-} f\left(-\tfrac{p}{n}^{-}\right) + n\nabla^n_{p^-,p^+} f\left(\tfrac{p}{n}^{+}\right) - n\alpha\nabla^n_{0,-p^+} f\left(-\tfrac{p}{n}^{+}\right) + n^2\nabla^n_{1,0} f\left(\tfrac{0}{n}\right) \\
&= n\alpha\nabla^n_{-1,-p^-} f\left(-\tfrac{p}{n}^{-}\right) - nf'\left(-\tfrac{p}{n}\right) - n\alpha\nabla^n_{0,-p^+} f\left(-\tfrac{p}{n}^{+}\right) \\
&\qquad + n^2\nabla^n_{1,-p^+} f\left(-\tfrac{p}{n}^{+}\right) - n^2\nabla_{0,-p^+} f\left(-\tfrac{p}{n}^{+}\right).
\end{aligned} \tag{3.2.3}$$

and

$$\begin{aligned}
n^2\mathsf{L}_n\pi_n f\left(-\tfrac{1}{n}\right) &= n\alpha\nabla^n_{0,-1} f\left(-\tfrac{1}{n}\right) + n^2\nabla^n_{-2,-1} f\left(-\tfrac{1}{n}\right) \\
&= n\alpha\nabla^n_{p^+,0} f\left(-\tfrac{p}{n}^{-}\right) + nf'\left(\tfrac{p}{n}\right) - n\alpha\nabla^n_{-1,p^+} f\left(-\tfrac{p}{n}^{+}\right) \\
&\quad + n^2\nabla_{-2,p^-} f\left(-\tfrac{p}{n}^{-}\right) - n^2\nabla^n_{-1,p^-} f\left(-\tfrac{p}{n}^{-}\right),
\end{aligned} \tag{3.2.4}$$

We begin by analyzing the first equation (3.2.3). The second equation (3.2.4) can be treated in a similar manner. By doing Taylor expansions around the discontinuities in (3.2.3), and by applying the boundary conditions, we obtain:

$$\begin{cases}
\nabla^n_{1,-p^+} f\left(-\tfrac{p}{n}^{+}\right) = \tfrac{p+1}{n} f'\left(-\tfrac{p}{n}\right) + \tfrac{1}{2}\left(\tfrac{p+1}{n}\right)^2 f''\left(-\tfrac{p}{n}^{-}\right) + \|\Delta f\| \cdot O\left(\tfrac{1}{n^3}\right) + \|\Delta^2 f\| \cdot O\left(\tfrac{1}{n^4}\right). \\
\nabla^n_{-1,p^-} f\left(-\tfrac{p}{n}^{+}\right) = \tfrac{p-1}{n} f'\left(-\tfrac{p}{n}\right) + \tfrac{1}{2}\left(\tfrac{p-1}{n}\right)^2 f''\left(-\tfrac{p}{n}^{-}\right) + \|\Delta f\| \cdot O\left(\tfrac{1}{n^3}\right) + \|\Delta^2 f\| \cdot O\left(\tfrac{1}{n^4}\right). \\
\nabla_{0,-p^+} f\left(-\tfrac{p}{n}^{+}\right) = \tfrac{p}{n} f'\left(-\tfrac{p}{n}\right) + \tfrac{1}{2}\left(\tfrac{p}{n}\right)^2 f''\left(-\tfrac{p}{n}^{-}\right) + \|\Delta f\| \cdot O\left(\tfrac{1}{n^3}\right) + \|\Delta^2 f\| \cdot O\left(\tfrac{1}{n^4}\right).
\end{cases} \tag{3.2.5}$$

The approximation arises from the boundary conditions imposed on the terms of orders 3 and 4, which introduce slight deviations.

Now, by substituting the relations from (3.2.5) into (3.2.3), and applying the same procedure to equation (3.2.4), we obtain the following results

$$\begin{cases} n^2\mathsf{L}_n\pi_n f\left(-\frac{1}{n}\right) = -\alpha f'\left(-\frac{p}{n}\right) + pf''\left(-\frac{p}{n}^+\right) \\ \qquad\qquad\qquad +\frac{1}{2}f''(0) + \|\Delta f\|\cdot O\big(\frac{1}{n}\big) + \|\Delta^2 f\|\cdot O\big(\frac{1}{n^2}\big)\,, \\ n^2\mathsf{L}_n\pi_n f\left(\frac{0}{n}\right) = \alpha f'\left(-\frac{p}{n}\right) + (1-p)f''\left(-\frac{p}{n}^-\right) \\ \qquad\qquad\qquad +\frac{1}{2}f''\left(-\frac{p}{n}^+\right) + \|\Delta f\|\cdot O\big(\frac{1}{n}\big) + \|\Delta^2 f\|\cdot O\big(\frac{1}{n^2}\big)\,. \end{cases}$$

It is worth highlighting that the relation above holds as an equality, except for terms that converge to zero depending on $\|\Delta f\|$ and $\|\Delta^2 f\|$. As we shall see later, we will introduce operators designed to correct these constant terms, ensuring that only $\frac{1}{2}\Delta$ at the point remains, alongside terms that vanish as they converge to zero.

Let us in the direction of obtaining the family of correction operators. Firstly, we define auxiliary straight lines g_f^1 and g_f^2 as follows: For any fixed $f \in \mathfrak{D}\left(\mathsf{L}^2\right)$ put

$$\begin{aligned} g_f^1\left(\tfrac{x}{n}\right) &:= -\Big[f'\left(-\tfrac{1}{n}\right) - f'\left(-\tfrac{p}{n}\right)\Big]x + f'\left(-\tfrac{p}{n}\right)\,, \\ g_f^2\left(\tfrac{x}{n}\right) &:= \Big[f'\left(\tfrac{p}{n}\right) - f'\left(\tfrac{0}{n}\right)\Big]x - f'\left(-\tfrac{p}{n}\right)\,. \end{aligned}$$

As we shall see, these straight lines play a crucial role in ensuring convergence near the slow bonds by effectively “gluing" the behavior in these region.

For each $n \in \mathbb{N}$ fixed, let $\mathrm{H} : \mathbb{R} \to \mathbb{R}$ be a function satisfying that $\mathrm{H}(0) = 0$ and it is Lipschitz. Additionally, for any polynomial p of degree 1, it holds

$$\lim_{|x|/n\to\infty} \frac{p\left(\frac{x}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x}{n}\right)} = 0, \qquad \forall n \in \mathbb{N}\,. \tag{3.2.6}$$

For example, we could simple take $\mathrm{H}(x) = x^2$. Let then consider the family of linear operators $\{\Xi_n^{\mathrm{I}} : n \geq 0\}$, acting over local functions $\Xi_n : \mathcal{C}_0(\mathbb{J}) \to \mathcal{C}_0(\mathbb{Z}_\mathsf{n})$ via

$$\Xi_n^{\mathrm{I}} f\left(\tfrac{x}{n}\right) := \begin{cases} \frac{1}{n}\frac{g_f^1\left(\frac{x+1}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x+3}{n}\right)} & , \textit{ if } \frac{x}{n} \leq -\frac{3}{n}\,, \\ \frac{1}{n}f'\left(-\frac{1}{n}\right) & , \textit{ if } \frac{x}{n} = -\frac{2}{n}\,, \\ \frac{1}{n}f'\left(-\frac{p}{n}\right) & , \textit{ if } \frac{x}{n} = -\frac{1}{n}\,, \\ -\frac{1}{n}f'\left(-\frac{p}{n}\right) & , \textit{ if } \frac{x}{n} = \frac{0}{n}\,, \\ -\frac{1}{n}f'\left(\frac{0}{n}\right) & , \textit{ if } \frac{x}{n} = \frac{1}{n}\,, \\ \frac{1}{n}\frac{g_f^2\left(\frac{x}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x-2}{n}\right)} & , \textit{ if } \frac{x}{n} \geq \frac{2}{n}\,. \end{cases}$$

As previously mentioned, the operator is well-defined, that is, the process is defined over the space of continuous functions vanishing at infinity mapping $\mathbb{Z}_n$ into $\mathbb{R}$. Thus, the hypothesis (A1) holds.

The boundary conditions in addition to $f \in \mathfrak{D}\left(\mathsf{L}^2\right)$, yields that

$$\begin{aligned} n^2\mathsf{L}_n\Xi_n f\left(\tfrac{0}{n}\right) &= n\alpha\left[\Xi^{\mathrm{I}} f\left(-\tfrac{1}{n}\right) - \Xi^{\mathrm{I}} f\left(\tfrac{0}{n}\right)\right] + n^2\left[\Xi^{\mathrm{I}} f\left(\tfrac{1}{n}\right) - \Xi^{\mathrm{I}} f\left(\tfrac{0}{n}\right)\right] \\ &= \alpha\left[f'\left(-\tfrac{p}{n}\right) + f'\left(-\tfrac{p}{n}\right)\right] - n\left[f'\left(\tfrac{0}{n}\right) - f'\left(-\tfrac{p}{n}\right)\right] \\ &= 2\alpha f'\left(-\tfrac{p}{n}\right) + n\left[\tfrac{p}{n}f''\left(-\tfrac{p}{n}^+\right) + \tfrac{p^2}{2n^2}f'''\left(-\tfrac{p}{n}^+\right) + \tfrac{p^3}{3!n^3}f''''(\eta)\right] \\ &= 2\alpha f'\left(-\tfrac{p}{n}\right) + pf''\left(-\tfrac{p}{n}^+\right) + \|\Delta f\|\cdot O\big(\tfrac{1}{n}\big) + \left\|\Delta^2 f\right\|\cdot O\big(\tfrac{1}{n^2}\big)\,. \end{aligned} \tag{3.2.7}$$

By an analogous computation, one can check that

$$n^2\mathsf{L}_n\Xi^{\mathrm{I}}_n f\left(-\tfrac{1}{n}\right) = -2\alpha f'\left(-\tfrac{p}{n}\right) - (1-p)f''\left(-\tfrac{p}{n}^-\right) + \|\Delta f\|\cdot O\big(\tfrac{1}{n}\big) + \left\|\Delta^2 f\right\|\cdot O\big(\tfrac{1}{n^2}\big)\,, \tag{3.2.8}$$

and therefore, we partially correct the constant terms. We now aim to prove that the $n^2\mathsf{L}_n\Xi_n f$ decay to zero as $n\to\infty$ when evalute outside $\left\{-\tfrac{1}{n},\tfrac{0}{n}\right\}$. Observe that

$$n^2\mathsf{L}_n\Xi^{\mathrm{I}} f\left(-\tfrac{2}{n}\right) = n^2\mathsf{L}_n\Xi^{\mathrm{I}} f\left(\tfrac{1}{n}\right) = 0 \leq \|f\|\cdot O\big(\tfrac{1}{n}\big)$$

Consider now, $\frac{x}{n}$ outside $\{-\frac{2}{n},-\frac{1}{n},\frac{0}{n},\frac{1}{n}\}$, and without loss of generality, suppose $\frac{x}{n}\geq\frac{0}{n}$. Thus

$$\begin{aligned}
n^2\mathsf{L}_n\Xi^{\mathrm{I}}_n f\left(\tfrac{x}{n}\right) &= n^2\Big[\Xi^{\mathrm{I}} f\left(\tfrac{x+1}{n}\right) + \Xi^{\mathrm{I}} f\left(\tfrac{x-1}{n}\right) - 2\Xi^{\mathrm{I}}_n f\left(\tfrac{x}{n}\right)\Big]\\
&= n\left[\frac{g^1_f\left(\frac{x+1}{n}\right)}{1+\frac{1}{n}\mathbf{H}\left(\frac{x-1}{n}\right)} + \frac{g^1_f\left(\frac{x-1}{n}\right)}{1+\frac{1}{n}\mathbf{H}\left(\frac{x-3}{n}\right)} - 2\frac{g^1_f\left(\frac{x}{n}\right)}{1+\frac{1}{n}\mathbf{H}\left(\frac{x-2}{n}\right)}\right]\\
&= g^2_f\left(\tfrac{x+1}{n}\right)\left[\frac{\mathrm{H}\left(\frac{x-2}{n}\right)-\mathrm{H}\left(\frac{x-1}{n}\right)}{\left(1+\frac{1}{n}\mathrm{H}\left(\frac{x-1}{n}\right)\right)\left(1+\frac{1}{n}\mathrm{H}\left(\frac{x-2}{n}\right)\right)}\right] + \frac{n}{1+\frac{1}{n}\mathbf{H}\left(\frac{x-2}{n}\right)}\Big[f'\left(-\tfrac{p}{n}\right) - f'\left(\tfrac{0}{n}\right)\Big]\\
&\quad + g^2_f\left(\tfrac{x-1}{n}\right)\left[\frac{\mathrm{H}\left(\frac{x-2}{n}\right)-\mathrm{H}\left(\frac{x-3}{n}\right)}{\left(1+\frac{1}{n}\mathrm{H}\left(\frac{x-3}{n}\right)\right)\left(1+\frac{1}{n}\mathrm{H}\left(\frac{x-2}{n}\right)\right)}\right] - \frac{n}{1+\frac{1}{n}\mathbf{H}\left(\frac{x-2}{n}\right)}\Big[f'\left(-\tfrac{p}{n}\right) - f'\left(\tfrac{0}{n}\right)\Big]\\
&= \|f\|\cdot O\big(\tfrac{1}{n}\big)\,,
\end{aligned}$$

and the last inequality comes from the Lipschitz property of H as well the boundary condition imposed to f. The same computations ensures the same rate for $\frac{x}{n}\leq\frac{0}{n}$. Therefore, it follows the following estimate

$$n^2\mathsf{L}_n\Xi^{\mathrm{I}}_n f\left(\tfrac{x}{n}\right) = \|f\|\cdot O\big(\tfrac{1}{n}\big)\,.$$

It is important to highlight that the corrections (3.2.7) and (3.2.8) partially guarantee the desired convergence. Specifically, with these adjustments, the terms involving the first derivative do not vanish; more than that, their intensity is amplified. Consequently, it becomes necessary to introduce a second operator to address this convergence issue. Following a similar approach to our earlier considerations, let us define the following straight lines

$$\begin{aligned}
h^1_f\left(\tfrac{x}{n}\right) &:= -\alpha f'\left(-\tfrac{p}{n}\right)x\,,\\
h^2_f\left(\tfrac{x}{n}\right) &:= \alpha f'\left(-\tfrac{p}{n}\right)x\,.
\end{aligned}$$

as well the operator $\Xi^{\mathrm{II}}_n:\mathcal{C}_0(\mathbb{J})\to\mathcal{C}_0(\mathbb{Z}_n)$ given by

$$\Xi^{\mathrm{II}}_n f\left(\tfrac{x}{n}\right) := \begin{cases}
\frac{1}{n^2}\frac{h^1_f\left(\frac{x-2}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x-2}{n}\right)} & ,\ \text{if } \frac{x}{n}\geq\frac{2}{n}\\
\frac{1}{n^2}\frac{\alpha f'\left(-\frac{p}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x}{n}\right)} & ,\ \text{if } \frac{x}{n}=\frac{1}{n}\\
\frac{1}{n^2}2\alpha f'\left(-\frac{p}{n}\right) & ,\ \text{if } \frac{x}{n}=\frac{0}{n}\\
-\frac{1}{n^2}2\alpha f'\left(-\frac{p}{n}\right) & ,\ \text{if } \frac{x}{n}=-\frac{1}{n}\\
-\frac{1}{n^2}\frac{\alpha f'\left(-\frac{p}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x+1}{n}\right)} & ,\ \text{if } \frac{x}{n}=-\frac{2}{n}\\
\frac{1}{n^2}\frac{h^2_f\left(\frac{x+3}{n}\right)}{1+\frac{1}{n}\mathrm{H}\left(\frac{x+3}{n}\right)} & ,\ \text{if } \frac{x}{n}\leq-\frac{3}{n}
\end{cases}$$

Similarly, $\Xi_n^{\mathrm{II}} f$ satisfies (A1), and therefore, defining $\Phi_n := \pi_n + \Xi_n^{\mathrm{I}} + \Xi_n^{\mathrm{II}}$ it is immediate this new operator also satisfies (A1). A similar computations guarantees that

$$n^2 \mathsf{L}_n \Xi_n^{\mathrm{II}} f\left(-\tfrac{1}{n}\right) = -\alpha f'\left(-\tfrac{p}{n}\right) + \|f\| \cdot O\big(\tfrac{1}{n}\big)$$
$$n^2 \mathsf{L}_n \Xi_n^{\mathrm{II}} f\left(\tfrac{0}{n}\right) = \alpha f'\left(-\tfrac{p}{n}\right) + \|f\| \cdot O\big(\tfrac{1}{n}\big).$$

and, outside these sites, we also have that

$$n^2 \mathsf{L}_n \Xi_n^{\mathrm{II}} f\left(\tfrac{x}{n}\right) = \|f\| \cdot O\big(\tfrac{1}{n}\big).$$

Thus, we are able to apply the Theorem 1.2. Then, for any $t > 0$ in a compact interval, one can check that

$$\left\|\pi_n \mathsf{P}^{\mathbb{J}}(t) f - \mathsf{T}_\mathsf{n}(t)\pi_n f\right\| = (\|f\| + \|\Delta f\|) \cdot O\big(\tfrac{1}{n}\big) + \left\|\Delta^2 f\right\| \cdot O\big(\tfrac{1}{n^2}\big).$$

□

In order to establish convergence over the Skorohod space $\mathsf{D}_{\mathbb{J}}[0,\infty)$ as well to derive the *weak Berry-Esseen* estimate, it remains to construct the basis for the topology under consideration. Subsequently, with this basis, we will be able to compare the results obtained through this approach via Lemma 3.3 against the ones presented in Theorem 3.1.

Lemma 3.4. There exists sequences $\{f_{k,j} : k, j \geq 0\} \subset \mathfrak{D}\left(\mathsf{L}_{\mathbb{J}}^2\right)$ satisfying the hypotheses (B2) and (B3).

The proof of Lemma 3.4 presented above closely mirrors the construction detailed in Theorem 2.2.

Proof. Firstly, note that Lemma 2.1 ensures that $\mathrm{span}(A)$ is dense in $\mathcal{C}_0(\mathbb{R})$. Consider then f_k precisely as in (2.3.4). Observe, in this particular case, that $f_k \in \mathfrak{D}\left(\mathsf{L}_{\mathbb{J}}^2\right)$ for $k \geq 4$, and hence, in this range of k, define $f_{j,k} := f_k$.

To finish the proof, for $k \in \{0, 1, 2, 3\}$, construct $f_{j,k}$ following the same procedure detailed in the proof of Theorem 2.2. □

The hypothesis (B1) is straightforwardly verified for this model. It remains to check (B4).

Lemma 3.5. The semigroup $\mathsf{P}_{\mathbb{J}}(t)$ of the snapping-out Brownian motion is Lipschitz.

Proof. For sake of simplicity, let us suppose that $p = 0$. From the Proposition 3.2, the semigroup $\mathsf{P}_{\mathbb{J}}(t)$ has an explicit form given by (3.1.4) and (3.1.5). We show by straightforward computation that the hypothesis (B4) holds. Initially, let us suppose that $v > u > 0$. For sake of simplicity, define

$$A_{z,y} := e^{-\kappa z} \int_0^\infty \left[\left(\tfrac{z-y+\kappa t}{2t}\right) e^{-\frac{(z-y)^2}{2t}} + \left(\tfrac{z+y+\kappa t}{2t}\right) e^{-\frac{(z+y)^2}{2t}}\right] f_{\mathsf{odd}}(y)\mathbf{d}y\,,$$

and $M(t) := |\sqrt{2\pi t}|^{-1}$. Observe that $\|f_{\mathsf{even}}\|, \|f_{\mathsf{odd}}\| \leq \|f\|$. From the triangle inequality, for every $t \geq 0$ we have that

$$\begin{aligned}
|\mathsf{P}_{\mathbb{J}}(t)f(u) - \mathsf{P}_{\mathbb{J}}(t)f(v)| &\leq M(t)\left|\int_{\mathbb{R}} e^{-\frac{(u-y)^2}{2t}} f_{\mathsf{even}}(y)\mathbf{d}y - \int_{\mathbb{R}} e^{-\frac{(v-y)^2}{2t}} f_{\mathsf{even}}(y)\mathbf{d}y\right| \\
&\quad + M(t)\left|e^{\kappa u}\int_u^\infty A_{z,y}\mathbf{d}z - e^{\kappa v}\int_v^\infty A_{z,y}\mathbf{d}z\right| \\
&\leq M(t)\,\|f_{\mathsf{even}}\| \int_{\mathbb{R}} \left|e^{-\frac{(u-y)^2}{2t}} - e^{-\frac{(v-y)^2}{2t}}\right|\mathbf{d}y + M(t)\left|(e^{\kappa u} - e^{\kappa v})\int_u^\infty A_{z,y}\mathbf{d}z + e^{\kappa v}\int_u^v A_{z,y}\mathbf{d}z\right| \\
&\leq M(t)\left[\|f\| \int_{\mathbb{R}} \left|e^{-\frac{(u-y)^2}{2t}} - e^{-\frac{(v-y)^2}{2t}}\right|\mathbf{d}y + |e^{\kappa u} - e^{\kappa v}| \cdot \int_0^\infty |A_{z,y}|\,\mathbf{d}z + e^{\kappa z}\int_u^v |A_{z,y}|\,\mathbf{d}z\right].
\end{aligned} \tag{3.2.9}$$

Invoking the mean value theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}} \left| e^{-\frac{(u-y)^2}{2t}} - e^{-\frac{(v-y)^2}{2t}} \right| \mathbf{d}y &\leq |v-u| \int_{\mathbb{R}} \sup_{\xi\in[u,v]} \left| \tfrac{d}{d\xi} e^{-\frac{(\xi-y)^2}{2t}} \right| \mathbf{d}y \\ &= |v-u| \int_{\mathbb{R}} \sup_{\xi\in[u,v]} \left| \tfrac{(\xi-y)}{t} e^{-\frac{(\xi-y)^2}{2t}} \right| \mathbf{d}y \\ &\leq |v-u| \int_{\mathbb{R}} \sup_{\xi\in[u,v]} \tfrac{|\xi-y|}{t} e^{-\frac{(\xi-y)^2}{2t}} \mathbf{d}y \\ &\leq K|v-u| \end{aligned} \tag{3.2.10}$$

for some constant $K > 0$, and

$$|e^{\kappa u} - e^{\kappa v}| \leq \sup_{\xi\in[u,v]} \left| \tfrac{d}{d\xi} e^{\kappa\xi} \right| \cdot \left| v-u \right| = \sup_{\xi\in[u,v]} \left| \kappa e^{\kappa\xi} \right| \cdot \left| v-u \right| \leq \kappa e^{\kappa v} |v-u| \,. \tag{3.2.11}$$

By replacing (3.2.10) and (3.2.11) into (3.2.9) we achieve that

$$\begin{aligned} |\mathsf{P}_{\mathbb{J}}(t)f(u) - \mathsf{P}_{\mathbb{J}}(t)f(v)| \leq 2M(t)\,\|f\| \cdot |u-v| + M(t)\kappa e^{\kappa v}|v-u| \cdot \int_0^\infty |A_{z,y}|\, \mathbf{d}z \\ + e^{\kappa v} \int_u^v |A_{z,y}|\, \mathbf{d}\, z \,. \end{aligned} \tag{3.2.12}$$

We claim now that

$$\int_0^\infty |A_{z,y}|\, \mathbf{d}z \leq \|f\| \left[\tfrac{1}{\kappa} + \sqrt{\tfrac{\pi t}{2}} \right] . \tag{3.2.13}$$

Denote by

$$I_1 := \int_0^\infty \left| \tfrac{z-y+\kappa t}{2t} \right| e^{-\frac{(z-y)^2}{2t}} \mathbf{d}\, y \,, \quad \text{and} \quad I_2 := \int_0^\infty \left| \tfrac{z+y+\kappa t}{2t} \right| e^{-\frac{(z+y)^2}{2t}} \mathbf{d}\, y \,,$$

Therefore, one can obtain the following upper bound

$$\begin{aligned} I_1 &\leq \int_0^\infty \left[\tfrac{|z-y|}{2t} + \tfrac{\kappa}{2} \right] e^{-\frac{(z-y)^2}{2t}} \mathbf{d}\, y \\ &= \int_0^\infty \tfrac{|z-y|}{2t} e^{-\frac{(z-y)^2}{2t}} \mathbf{d}\, y + \int_0^\infty \tfrac{\kappa}{2} e^{-\frac{(z-y)^2}{2t}} \mathbf{d}\, y \\ &= \tfrac{1}{2} + \tfrac{\kappa}{2} \sqrt{\tfrac{\pi t}{2}} \end{aligned}$$

and similarly, the same upper bound holds for I_2. From these upper bounds, it follows

$$\begin{aligned} \int_0^\infty |A_{z,y}|\, \mathbf{d}\, z &\leq \int_0^\infty \left| e^{-\kappa z} \int_0^\infty \left[\left(\tfrac{z-y+\kappa t}{2t} \right) e^{-\frac{(z-y)^2}{2t}} + \left(\tfrac{z+y+\kappa t}{2t} \right) e^{-\frac{(z+y)^2}{2t}} \right] f_{\mathsf{odd}}(y) \mathbf{d}y \right| \mathbf{d}\, z \\ &\leq \|f_{\mathsf{odd}}\| \left| 1 + \kappa \sqrt{\tfrac{\pi t}{2}} \right| \int_0^\infty e^{-\kappa z} \mathbf{d}\, z \\ &\leq \|f\| \left[1 + \kappa \sqrt{\tfrac{\pi t}{2}} \right] \int_0^\infty e^{-\kappa z} \mathbf{d}\, z \end{aligned}$$

Finally, using the bound obtained, one can check that

$$\begin{aligned} e^{\kappa v} \int_u^v |A_{z,y}|\, \mathbf{d}\, z &\leq e^{\kappa v}\, \|f\| \left[1 + \kappa \sqrt{\frac{\pi t}{2}} \right] \int_u^v e^{-\kappa z} \mathbf{d}\, z \\ &\leq e^{\kappa(v-u)}\, \|f\| \left[1 + \kappa \sqrt{\frac{\pi t}{2}} \right] (v-u) \end{aligned} \tag{3.2.14}$$

which permits us to conclude the case $v > u > 0$. Observe now that the same argument holds for $v < u < 0$. Therefore, replacing equations (3.2.13) and (3.2.14) in (3.2.12), one can conclude that the semigroup $\mathsf{P}_{\mathbb{J}}(t)$ is Lipschitz. □

Observe that, as previously noted, by the right choice of the scale and starting points, we can avoid the Lemma 3.5.

Theorem 3.2. Fix $t > 0$ and denote by μ_t^{snob} and $\mu_{tn^2}^{\text{slow}}$ the probability distributions at time t on $\mathbb{J}$ induced by $X_{\mathbb{J}}^{\text{snob}}$ and $n^{-1}X_{tn^2}^{\text{slow}}$ respectively, starting from the points $\frac{\lfloor un \rfloor}{n} \in \mathbb{Z}_n$ and $u \in \mathbb{Z}$. Then

$$\mathbf{d}(\mu_t^{\text{snob}}, \mu_{tn^2}^{\text{slow}}) \lesssim n^{-1}$$

and $n^{-1}X_{tn^2}^{\text{slow}} \Rightarrow X_{\mathbb{J}}^{\text{snob}}$ in the J_1-Skorohod topology of $\mathsf{D}_{\mathbb{J}}[0, \infty)$.

In Theorem 3.1, a significantly slower rate of convergence was obtained, despite our employing a finer topology. This highlights an important topological distinction: convergence in the bounded Lipschitz metric implies convergence in the weaker metric d, but the converse does not hold. Nevertheless, establishing convergence under the metric d ensures convergence in the Skorohod topology, thereby fully characterizing the associated Feller processes.

In contrast, the model presented here, although structurally simpler than the framework in [Erhard et al., 2021], achieves a substantially faster convergence rate. It is noteworthy that we do not require a stronger topology or explicit semigroup constructions if we choose the starting point as well the scale in which the discrete processes lie in the right way. The result emphasizes the simplicity and elegance of the method here employed, which, while not robust in a broader semigroup context, proves sufficient to guarantee convergence in the Skorohod topology.

Chapter 4

Open Problem

Throughout this chapter, we aim to outline the future milestones. We strive to present a comprehensive roadmap about the conjecture and the possible key challenges. By delving into these future objectives, we aim to provide a clear vision of the steps required to prove the conjectures.

The conjecture here presented was inspired by the papers [Franco et al., 2011] and [Franco and Tavares, 2019] as well from the stochastic process studied by Lejay in his work [Lejay, 2016].

4.1 The d-dimensional Snapping-out Brownian motion

Let Ω be a simply connected region inside $\mathbb{R}^d$ of codimension 0, and consider $\Omega \subset \mathbb{R}^d$ having smooth boundary $\partial\Omega$. By $\{e_1, \dots, e_d\}$ we mean the canonical basis of $\mathbb{R}^d$, and we consider the natural partition of $\mathbb{R}^d$ via Ω, that is, the partition given by the disjoint sets $\overline{\text{int}\,(\Omega)}$, $\overline{\text{ext}\,(\Omega)}$ and $\partial\Omega$ that are, respectively, the interior of the region Ω, the exterior, and the boundary of that region Ω. Let then

$$\mathbb{G} := \overline{\text{int}\,(\Omega)} \cup \partial\Omega \cup \overline{\text{ext}\,(\Omega)}\,.$$

For each $n \in \mathbb{N}$, consider $\mathbb{Z}_n^d := \mathbb{Z}_n \times \cdots \times \mathbb{Z}_n$ representing d copies of the lattice $\mathbb{Z}_n := \frac{1}{n}\mathbb{Z}$. Fix a random point $x/n \in \mathbb{R}_n^d$ and observe that, if the bond $[x/n, (x+e_j)/n]$ has vertices in each of regions $\text{int}\,(\Omega)$ and $\text{ext}\,(\Omega)$, say for example $(x+e_j)/n \in \text{int}\,(\Omega)$ and $x/n \in \text{ext}\,(\Omega)$, then there exists

$$\tfrac{u}{n} \in [x/n, (x+e_j)/n] \cap \partial\Omega\,.$$

Denote by $\overrightarrow{\zeta}(u)$ the unitary exterior normal vector to the smooth surface $\partial\Omega$ in this point $\frac{u}{n} \in \partial\Omega$.

Let X^n be a random walk over $\mathbb{Z}_n^d$ and equipp it with the following dynamic: Whenever the process X attempts to cross the boundary $\partial\Omega$, it will be constrained to remain within its current region until a specific condition is met: For instance, consider the bond $[\frac{x}{n}, \frac{x+e_j}{n}]$ where j denotes some direction on $\mathbb{R}^d$, and one vertex, for example $\frac{x}{n}$, lies inside the region Ω while the other one $\frac{x+e_j}{n}$ lies outside Ω. For such bonds, the exchange rate of the random walk X is given by n^{-1} times the absolute value of the inner product between the vector field $\overrightarrow{\zeta}(u)$ and e_j. For all other edges of the lattice– Specifically, those where both bonds are either entirely inside or outside the region Ω– the transition rate is equal to 1. We refer to X^n as the n-th rescaling of the *slow bond random walk*, and the boundary $\partial\Omega$ can be understood as a permeable membrane. See figure 4.1.

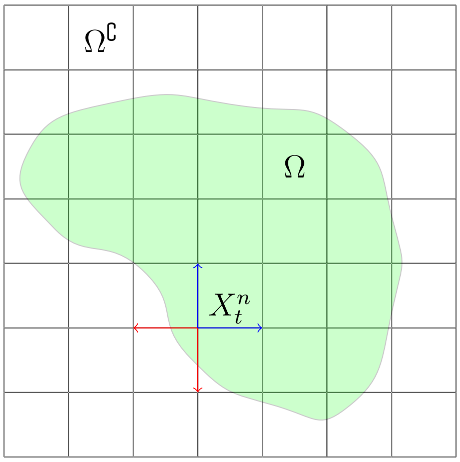

Figure 4.1: The region in green represents the smooth surface Ω over $\mathbb{R}^2$, and in the region white, its complement $\Omega^{\complement}$. In red we represent the slow rate of the SRW and in blue the standard rate.

The continuous counterpart, which we conjecture to be the SNOB in higher dimensions, should appear to exhibit dynamics closely resembling those of the 1-dimensional snapping-out Brownian motion.

A possible dynamic would be the following: Consider $\{B_t : t \geq 0\}$, a Brownian motion over int$(\Omega) \cup$ ext(Ω), and, without loss of generality, let us assume the process begins inside the membrane, in ext(Ω). We sample an exponential random vector $Z = \prod_{u \in \partial\Omega} Z^u$ independent of B. Each time the Brownian motion B hits the slow membrane in $u \in \partial\Omega$ the particle is reflected in a direction determined by the vector $\overrightarrow{\zeta}(u)$, and its local time, here defined by L_t^u increases. Denote by

$$\mathsf{L}_t = \prod_{u \in \partial\Omega} \mathsf{L}_t^u$$

the local time of the boundary $\partial\Omega$. When any coordinate, say $u \in \partial\Omega$ reache the value Z^u, the Brownian motion is killed. Then, the particle is reborn: upon rebirth, a fair coin is tossed to determine in which region the particle restarts its motion. It is noteworthy that this dynamic is close to the one 1-dimensional, and it generalizes the 1-dimensional case.

Let us first observe that, once considered the lattice $\mathbb{Z}_n^d$, we can homogenize the type of the points of discontinuity in the sense of letting every discontinuity over bonds instead sites, justifying this way the toy model.

Main Goals:

- Existence: It is necessary to ensure that there exists a Feller semigroup associated to the desired stochastic process .
- Uniqueness: It is necessary to guarantee that the stochastic process is unique and it is rigorously well-defined .
- Rate of convergence: Estimate a *weak Berry-Esseen* estimate for the convergence that comes from uniqueness.

In the first condition, somehow inspired by works of Franco et al., we conjecture that the higher dimensional SNOB is a diffusion having as generator $\frac{1}{2}\Delta$ subjected to the following conditions: For every f in the domain of the generator, it should hold that

$$\begin{aligned}\nabla f\left(u^{+}\right)&=\nabla f\left(u^{-}\right),\\ \nabla f\left(u\right)&=\left[f\left(u^{+}\right)-f\left(u^{-}\right)\right]\cdot\overrightarrow{\zeta}\left(u\right),\end{aligned}$$

for any $u\in\partial\Omega$. Observe the second condition, which generalizes the problem to higher dimensions, building upon the case in one dimension. This generalization comes from the fact that the exterior unit normal vector encodes all the necessary geometric information about the boundary's orientation.

The primary goal, therefore, is to establish a Hille-Yosida type theorem for this generator, which will ensure the existence of the strongly continuous contraction semigroup for the higher-dimensional SNOB. To address this, we consider the operator $\mathsf{L}_\Omega:=\frac{1}{2}\Delta$ which domain is described via

$$\mathfrak{D}\left(\mathsf{L}_\Omega\right):=\left\{f\in\mathcal{C}_0^2(\mathbb{G}):\nabla^2 f\in\mathcal{C}^0(\mathbb{G})\ \text{and}\ \nabla f\left(u\right)=\left[f\left(u^{+}\right)-f\left(u^{-}\right)\right]\cdot\overrightarrow{\zeta}\left(u\right)\right\}\tag{4.1.1}$$

Conjecture 4.1. Consider the linear operator $\mathsf{L}_\Omega:\mathfrak{D}\left(\mathsf{L}_\Omega\right)\to\mathcal{C}_2(\mathbb{G})$. Then

1. $\mathfrak{D}\left(\mathsf{L}_\Omega\right)$ is dense in $\mathcal{C}_2(\mathbb{G})$,
2. L_Ω is dissipative,
3. For some $\lambda_0>0$, we have that $\mathsf{L}_\Omega-\lambda_0 I$ is surjective, where by I we mean the identity operator.

It is noteworthy that the model above is a highly dependent model on the connected region Ω, and from this dependence several challenges may arise.

The second objective lies in showing uniqueness, what we believe is possible by using the method presented in Chapter 1. To address it, we consider then the slow bond random walk whose generator L_n acts on local functions $f:\mathbb{Z}_n^d\to\mathbb{Z}_n^d$ as follows

$$\mathsf{L}_n f\left(\tfrac{x}{n}\right):=\sum_{i=1}^{d}\left[\theta^n_{x,x+e_i}\nabla^n_{x,x+e_i}f\left(\tfrac{x}{n}\right)+\theta^n_{x,x-e_i}\nabla^n_{x,x-e_i}f\left(\tfrac{x}{n}\right)\right]\tag{4.1.2}$$

where

$$\theta^n_{x,x+e_i}:=\begin{cases}\frac{|\overrightarrow{\zeta_{x,i}}\cdot e_i|}{n}, & \textit{if}\ \frac{x}{n}\in\text{int}\left(\Omega\right)\ \text{and}\ \frac{x+e_i}{n}\in\text{ext}\left(\Omega\right)\\ & \text{or}\ \frac{x+e_i}{n}\in\text{int}\left(\Omega\right)\ \text{and}\ \frac{x}{n}\in\text{ext}\left(\Omega\right)\\ 1, & \text{otherwise.}\end{cases}$$

and $\overrightarrow{\zeta_{x,i}}$ denotes the vector $\overrightarrow{\zeta}(u)$, where $u\in\left[\frac{x}{n},\frac{x+e_i}{n}\right]\cap\partial\Omega$. This dynamic is the same as in [Franco and Tavares, 2019].

From the random walk above, we expect the following

Conjecture 4.2. For each $n\in\mathbb{N}$, consider the operators L_Ω whose domain is given by (4.1.1) and L_n defined as in (4.1.2).

- The operator L_Ω can be approximated by L_n in the sense of (A2) and (A3).
- There exist functions $\{f_{k,j}:k,j\geq 0\}\in\mathfrak{D}\left(\mathsf{L}^2_\Omega\right)$ satisfying (B2) and (B3).

Key challenges:

As previously mentioned, the process given by L_Ω is a highly dependent model on the region Ω. Thus, the first approach that one can consider is choosing Ω to be a hyperplane or a $(d-1)$-dimensional sphere. In the second case, the primary challenge lies in identifying a function f that solves the following harmonic boundary value problem:

1. The Laplace operator applied to f safisfies ,

$$\Delta f\,(x) = 0, \qquad \forall x \in \mathbb{R}^d \setminus \mathbb{S}^{d-1}\,.$$

2. On the bounday, $f(x)$ matches a given continuous function $\varphi(x)$ vanishing at infinity ,

$$f\,(x) = \varphi(x), \qquad \forall x \in \mathbb{S}^{d-1}\,.$$

3. The functions φ belongs to the domain $\mathfrak{D}\,(\mathsf{L})$,

$$\nabla\varphi(x) = \big[\varphi(x^+) - \varphi(x^-)\big] \cdot \overrightarrow{\zeta}\,(x)\,,\, \forall x \in \mathbb{S}^{d-1}\,.$$

It is noteworthy that f will play the role as the correction operator for the conditions (A2) and (A3) ensuring the convergence of semigroups.

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