Congruence-semimodular and congruence-distributive pseudocomplemented semilattices

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Investigations into the structure of the congruence lattices of pseudocomplemented semilattices (PCS’s) were initiated in [10]. In this paper, we characterize the class of congruence-semimodular PCS’s (i.e. PCS’s with semimodular lattice of congruences) and the class of congruence-distributive PCS’s (i.e. with distributive congruence lattices). We give two characterizations of each class; one of these is a Dedekind-Birkhoff-type characterization which says that the exclusion in a certain sense of a single PCS $P_6$ determines the class of congruence-semimodular PCS’s, and the exclusion of the two PCS’s $P_6$ and $P_5$ (these are defined in the sequel) determines the class of congruence-distributive PCS’s. The other characterization shows that each of these classes is strictly elementary and gives explicitly the defining axiom for each class as a universal positive sentence (in the language of PCS’s). This paper is a continuation of [10] and borrows the notation and the results from it. For other information see the standard references [6] and [7].

§1. Basic definitions and lemmas

Recall that a pseudocomplemented semilattice (PCS) is an algebra $(S; \wedge, *, 0)$ where $(S; \wedge, 0)$ is a $\wedge$-semilattice with zero and $*$ is the pseudocomplementation and that the class of all PCS’s is an equational class. Let $S$ denote an arbitrary PCS and $B(S)$ and $N(S)$ denote respectively the set of closed (i.e. $a^{**} = a$) elements and that of non-closed (i.e. $a < a^{**}$) elements of $S$. It is well-known [5] that $B(S)$ is both a Boolean algebra and a PCS-subalgebra of $S$. Con $S$ denotes the congruence lattice of $S$ with $\Delta$, and $\nabla$ (or simply $\Delta$ and $\nabla$) as its least and greatest elements respectively; and the kernel of the homomorphism $**: S \to S$.

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$s \rightarrow s^{**}$, is denoted by $\Phi(S)$ (or simply $\Phi$). For $a, b \in S$ with $a < b$ \((a, b)\) and \((a, b)\) denote respectively $[a, b] - \{a\}$ and $[a, b] - \{a, b\}$.

Boolean algebras are here thought to be a subvariety of PCS's and the obvious fact that for a $BA$ $B$ the PCS-congruences on $B$ are precisely the $BA$-congruences on $B$ will be used repeatedly in the sequel.

The following sentences in the first-order language of PCS's are useful later:

\[(U) \ \forall x \forall y (x < y^{**} \rightarrow (x^* = y^* \text{ or } x \leq y)),\]
\[(U_p) \ \forall x \forall y ((x \wedge y^{**})^* = y^* \text{ or } x \wedge y^{**} \leq y),\]
\[(U') \ \forall x \forall y ((x = y^{**} \text{ and } x < y^{**}) \rightarrow x < y),\]
\[(D) \ \forall x \forall y (x < y^{**} \rightarrow (x \leq y \text{ or } y \leq x)),\]
\[(D_n) \ \forall x \forall y (x^* = y^* \rightarrow (x \leq y \text{ or } y \leq x)),\]
\[(D_p) \ \forall x \forall y (x \wedge y^{**} \leq y \text{ or } y \leq x \wedge y^{**}),\]
\[(D_{w,p}) \ \forall x \forall y (x \wedge y^{**} \leq y \wedge x^{**} \text{ or } y \wedge x^{**} \leq x \wedge y^{**}).\]

Note that, in the class of PCS's, $(U)$ is equivalent to $(U_p)$ and to $(U')$, $(D)$ is equivalent to the conjunction of $(D_n)$ and $(U')$, $(D)$ is equivalent to $(D_p)$ and $(D_{w,p})$ is equivalent to $(D_{w,p})$.

In Figure 1 the two PCS's $P_6$ and $P_5$ (mentioned earlier) are defined (their congruence lattices are also shown).

Observe that if $x \leq a^{**}$ then $x^* = (x \wedge a)^*$. Using this the reader can verify the following theorem which gives a characterization of certain principal congruences and is inspired by a result in Dean and Oehmke [3].
THEOREM 1.1. Let \( a, b \in S \) be such that \( a \leq b \) and \( a^* = b^* \). Then \( \theta(a, b) = \theta \) is characterized by

\[ x = y(\theta) \text{ iff either } x = y, \text{ or } x \leq b, y \leq b \text{ and } x \land a = y \land a. \]

Moreover, \( x = y(\theta) \) implies \( x^* = y^* \).

We say that an equivalence relation \( R \) on \( S \) is a semilattice-congruence on \( S \) iff \( R \) is a congruence on \( S \) when \( S \) is regarded as a semilattice, ignoring the operation \(*\). If \( a, b \in S \), we denote by \( \theta_{\text{semilat}}(a, b) \) the principal semilattice-congruence on \( S \) generated by \( \langle a, b \rangle \). For \( a, b \in S \) with \( a < b \) and \( a^* = b^* \), from Theorem 1.1 it follows that \( \theta_{\text{semilat}}(a, b) \) is indeed a PCS-congruence on \( S \).

For \( c \in B(S) \) we define \( D_c(S) = \{ x \in S : x^{**} = c \} \) and refer to the elements of \( D_c(S) \) as \( c \)-dense; in particular we write \( D(S) \) for \( D_1(S) \) whose elements are called simply dense elements. It is easy to check that \( D_c(S) \), in particular \( D(S) \), is a subsemilattice of \( S \) (as a semilattice).

LEMMA 1.2. Let \( b \in B(S) \) and \( A \subseteq D_b(S) \). Suppose \( \langle x, y \rangle \in \theta(A \times A) \) with \( x \neq y \). Then there exist \( h \) and \( h' \) in \( A \) such that \( x \leq h \) and \( y \leq h' \).

Proof. If \( a_0 \) is an arbitrary fixed element of \( A \), we can write \( \theta(A \times A) = \bigvee_{a \in A} \theta(a_0, a) \). Then \( \langle x, y \rangle \in \theta(A \times A) \) implies that there exist \( x = x_0, x_1, \ldots, x_n = y \) in \( S \) and \( a_1, \ldots, a_n \) in \( A \) such that \( \langle x_i, x_j \rangle \in \theta(a_0, a_1), \langle x_1, x_2 \rangle \in \theta(a_0, a_2), \ldots, \langle x_{n-1}, y \rangle \in \theta(a_0, a_n) \). We may assume \( x \neq x_1 \) and \( x_n \neq y \). Since \( \theta(a_0, a_1) = \theta(a_1, a_0 \land a_1) \lor \theta(a_0 \land a_1, a_1) \), and \( a_0^* = a_1^* \) by hypothesis, we can see by Theorem 1.1 (using also the fact that \( a_0^* = a_0 \land a_1^* = a_1^* \)) that \( x \leq a_0 \) or \( x \leq a_1 \); similarly one sees that \( y \leq a_0 \) or \( y \leq a_n \). From this the lemma follows.

LEMMA 1.3. Let \( c \in B(S) \). Then

\[ \theta(D_c(S) \times D_c(S)) = \bigvee_{a \in D_c(S)} \theta(c, a) = \bigcup_{a \in D_c(S)} \theta(c, a) \]

Proof. It is clear that \( \theta(D_c(S) \times D_c(S)) = \bigvee_{a \in D_c(S)} \theta(c, a) \). Since \( \{ \theta(c, a) : a \in D_c(S) \} \) is directed upwards (a common upper bound for \( \theta(c, a_1) \) and \( \theta(c, a_2) \) is \( \theta(c, a_1 \land a_2) \) if \( a_1, a_2 \in D_c(S) \), the equality of the join and the union is immediate.

§2. Atoms in congruence lattices

To achieve our characterizations we need to examine first the atoms in the congruence lattices.
EXAMPLE 2.1.

Recall from [10] that \( (x, y) \in \hat{e} \) iff \( x \land e = y \land e \). Observe that, in the preceding example, \( \theta(h, g) \) is an atom in \( \text{Con } S \) while \( \hat{f} \) is not an atom in \( \text{Con } S' \). Observe also that \( \hat{e} \) is an atom in \( \text{Con } S \) and has an additional property, namely \( (x, y) \in \hat{e} \) and \( x \neq y \) imply \( x \leq e \) or \( y \leq e \), whereas \( \hat{f} \) fails to have this property. These considerations lead us to the following theorem. In the sequel we use \( \rightarrow \) and \( \Rightarrow \) for the covering relation and its dual respectively.

THEOREM 2.2. Let \( \alpha \in \text{Con } S \). Then \( \alpha \) is an atom in \( \text{Con } S \) iff \( \alpha \) is one or the other of the following two types:

1. (T1) \( \alpha = \{(a, b), (b, a)\} \cup \Delta \) for some \( a, b \in S \) such that \( a^* = b^* \) and \( a \rightarrow b \) (i.e. \( a \rightarrow b \) and \( \forall x (x < b \rightarrow x \leq a) \)),
2. (T2) \( \alpha = \hat{e} \) for some \( e \in B(S) \) with \( e \rightarrow 1 \) and \( \alpha \) has the following additional property:

\((*) \) If \( (x, y) \in \alpha \) and \( x \neq y \) then \( x \leq e \) or \( y \leq e \).

Proof. First note that if a relation is of type (T1) or of type (T2) then it is indeed an atom in \( \text{Con } S \).

To prove the reverse, suppose \( \alpha \) is an atom in \( \text{Con } S \). Then \( \alpha \) is \( \land \)-irreducible. From [10, Theorem 2.6] we have \( \alpha = ([1]_\land)^* \lor (\alpha \lor \Phi) \) and hence it follows that \( \alpha = ([1]_\land)^* \lor \alpha \subseteq \Phi \). We show that if the former holds then \( \alpha \) is an atom of type (T2), whereas the latter implies \( \alpha \) is an atom of type (T1).

CASE 1. Let \( \alpha = ([1]_\land)^* \); then from \( \alpha \neq \Delta \) we easily see that there exists an element \( e \in [1]_\land \cap B(S) \) with \( e \neq 1 \). Since \( \Delta \subseteq \hat{e} \subseteq \alpha \) and \( \alpha \) is an atom, we get \( \alpha = \hat{e} \). Clearly \( e \rightarrow 1 \) in \( S \) and \( \alpha \cap \Phi = \Delta \). We now show that \( \alpha \) has the property \((*)\). Let \( (x, y) \in \alpha \) and \( x \neq y \). Thus \( x^* \neq y^* \) as \( \alpha \land \Phi = \Delta \), and \( (x^*, y^*) \in (\hat{e})_\lor \). Then it can be easily seen by an argument for Boolean algebras that \( x^{**} \leq e \) or \( y^{**} \leq e \) from which it follows that \( \alpha \) is an atom of type (T2).
CASE 2. Let $\alpha \leq \Phi$. Now $\alpha \neq \Delta$ implies that there exist $a$, $b$ in $S$ such that $(a, b) \in \alpha$ and $\alpha \neq b$. Then $a^* = b^*$, and since $\alpha$ is an atom, it follows that $\theta(a, b) = \alpha$. Now $(a, a \wedge b) \in \alpha$ and since $\alpha \neq b$, we may assume that $a \neq a \wedge b$ and hence $\theta(a, a \wedge b) = \alpha$ which implies that $(a, b) \in \theta(a, a \wedge b)$. Then from Theorem 1.1 we see that $b < a$. If $b \neq a$ then there exists an element $u$ in $S$ such that $u < a$ and $u \neq b$, so $(u \wedge b, u) \in \theta(a, b)$ and hence $\theta(u \wedge b, u) \subseteq \theta(a, b)$. However $\theta(a, b) \notin \theta(u \wedge b, u)$. Thus $b \leftarrow a$ and so $\alpha$ is an atom of type (T1).

Remark 2.3. Let $\alpha = \bar{e}$ be an atom of type (T2) and let $(x, y) \in \alpha$ with $x \neq y$. Then (i) $x^* \neq y^*$, (ii) $x \leftarrow y$ or $y \leftarrow x$, and (iii) $x = x^{**} \land y$ or $y = y^{**} \land x$.

§3. Semimodular congruence lattices

In this section we shall be concerned with the characterization of the PCS's whose congruence lattices are semimodular. It is easily verified that all PCS's with at most 5 elements have semimodular congruence lattices, while Con $P_6$ is not semimodular. The special role played by $P_6$ in this section is implicit in the following lemma.

Lemma 3.1. Suppose there are two incomparable elements $b$ and $d$ in $S$ such that $b \in B(S)$ and $d$ is dense in $S$. Then there exists a quotient algebra $S'$ of $S$ with the following properties:

1. $[b] \in B(S')$ and $[d]$ is dense in $S'$,
2. $[b]$ and $[d]$ are incomparable,
3. $[b] \leftarrow [1]$, $[b \land d] \leftarrow [b]$, $[d] \leftarrow [1]$ and $[b \land d] \leftarrow [d]$,
4. $N(S') = \{[b \land d], [d]\}$.

Proof. First we note that property (1) holds in any quotient algebra of $S$. We construct a sequence of PCS's $S = S_0, S_1, S_2, \ldots, S_5$ such that $S_i = S_{i-1} / \psi_i$ where $\psi_i$ is a suitably chosen congruence on $S_{i-1} = S_{i-1}$, for $i = 1, \ldots, 5$, and $S'$ will be the quotient $S_5$. For $s \in S$ we define $[s] = [s] \psi_1$ and $[s] = [s] \psi_1 \psi_2$ and we write $[1]$ for $[1]$, $i = 1, \ldots, 5$. The congruences $\psi_i$ will be so chosen that:

(i) $[b]$, and $[d]$, are incomparable, for $i = 1, \ldots, 5$,
(ii) $[b] \leftarrow [1]$ in $S_i$, $i \geq 2$,
(iii) $[b \land d] \leftarrow [b]$ in $S_i$, $i \geq 3$,
(iv) $[b \land d] \leftarrow [d] \leftarrow [1]$ in $S_i$, $i \geq 4$, and
(v) $N(S_5) = \{[b \land d], [d]\}$.

STEP 1. Let $[b, 1]_b = [b, 1] \cap B(S)$. Then $[b, 1]_b$ is a Boolean lattice. Let $U$ be an ultrafilter in $[b, 1]_b$. Define $\psi_i = (U)^n$ where $(U)$ is the filter in $S$ generated by
U and let $S_1 = S/\psi_1$. Then it is clear that $[[b]]_1 \vee [1]_0 = ([b], [1])$ and hence if $[x]_1 > [b]$, then $[x]_1$ is dense in $S_1$. We claim that $[d]_1 \neq [1]$, for, if $(d, 1) \in (U)$ then $d \land f = f$ for some $f \in U$. Then $f \equiv u$ for some $u \in U$ and hence $d \land u = u$, implying $d \equiv u \equiv b$ which is a contradiction. Thus $[d]_1 \neq [1]$, and similarly $[b \land d]_1 \neq [b]_1$. If $(b \land d, d) \in \psi_1$ then $(b \land d)_2 \equiv [b], d \equiv [1]$ which is impossible – thus $[b]_1$, $[d]_1$, $[b \land d]_1$ and $[1]$ are distinct from each other.

**STEP 2.** Consider the set $E = ([b]_1, [1])$ in $S_1$; from step 1 every element of $E$ is dense in $S_1$, so $E$ is a filter in $S_1$. Define $\psi_2 = \bar{E}$ and $S_2 = S_1/\psi_2$. If $[[x]]_1 \in \psi_2$ then $[x]_1 \land [e]_1 = [e]_1$ for some $[e]_1 \in E$ and hence $[e]_1 \equiv [x]_1$ which implies that $[x]_1 \in E$. From this it is immediate that $[b]_2 \neq [1]$ and $[d]_2 \neq [1]$. Thus we get $[b]_2 - [1]$ and therefore the closed element $[b]_2$ and the dense element $[d]_2$ are incomparable giving $[b]_2 \neq [b \land d]_2$ and $[d]_2 \neq [b \land d]_2$.

**STEP 3.** Let $A = D_{[b]_3}([b]_2)$ where $D_{[b]_3}([b]_2) = \{ [x]_1 : [x]_1 \equiv [b]_2 \}$, and note that $[b \land d]_2 \in A$. Consider $\psi_3 = \theta(A \times A)$ and define $S_3 = S_2/\psi_3$. From Lemma 1.2 it follows that if $[[x]]_2, [[y]]_2 \in \psi_3$ and $[x]_2 \neq [y]_2$ then $[x]_2 \leq [b]_2$ and $[y]_2 \leq [b]_2$, which leads us to conclude that $[b \land d]_3 \neq [b]_3$, $[b \land d]_3 \neq [d]_3$, $[b]_3 \neq [1]$ and $[d]_3 \neq [1]$ – thus showing that $[b]_3$ and $[d]_3$ are incomparable. We also have that $[b \land d]_3 \equiv [b]_3$ and $D_{[b]_3}([b]_2)$ which implies that if $[x]_3$ is dense in $S_3$ then $[x]_3 \equiv [b \land d]_3$ since $[x \land b \land d]_3 \equiv [b]_3$.

**STEP 4.** Let $M = ([b \land d]_3, [1]) - ([b]_2)$ and let $S_4 = S_3/\psi_4$ where $\psi_4 = \theta(M \times M)$. We note that if $[[m]]_4 \in M$ then $[[m]]_4 \equiv ([b \land d])_3 \equiv ([b]_3)$ and hence $[m]_4$ is dense in $S_4$ since $[m]_4 \equiv [b \land d]_3$ – thus $M = D_{[b]_3}([1])$. Then, since $\psi_4 \subseteq \Phi(S_4)$, we have $[b]_4 \neq [1]$; from Lemma 1.2 it follows that $[d]_4 \neq [1]$ and hence $[b]_4$ and $[d]_4$ are incomparable. In $S_4$ we then have $D_{[b]_4} = ([d]_4, [1])$ and $[b \land d]_4 \equiv [d]_4 \equiv [1]$.

**STEP 5.** Letting $K = B(S_4) - ([b]_4, [1])$, define $S_5 = S_4/\psi_5$ where $\psi_5 = \bigvee_{[k]_5 \in K} \theta(D_{[k]_5}([b]_5) \times D_{[k]_5}([d]_5))$. By Lemma 1.3 we get $\psi_5 = \bigcup_{[a]_4 \in D_{[b]_4}} \theta([a]_4, [k]_4)$. Then $[b]_5 \neq [1]$ and $[d]_5 \neq [1]$, hence $[b]_5$ and $[d]_5$ are incomparable and $N(S_5) = ([b \land d]_5, [d]_5)$. Of course, we have $[b \land d]_5 \equiv [b]_5 \equiv [1]$ and $[b \land d]_5 \equiv [d]_5 \equiv [1]$. Thus $S' = S_5$ is a quotient of $S$ having all the desired properties (1)–(4).

**THEOREM 3.2.** Suppose there are two incomparable elements $b$ and $d$ in $S$ such that $b \in B(S)$ and $d$ is dense in $S$. Then there exists in $\text{Con} S$ an interval $[\alpha, \beta]$ isomorphic with $N_5$. 
Proof. Using Lemma 3.1 we obtain a quotient algebra $S'$ of $S$ with the properties (1)–(4) of the lemma. Let us now define four congruences $\theta_i$, $i = 1, \ldots, 4$ on $S'$ as $\theta_1 = \{(b)\}^\circ$, $\theta_2 = \theta([b \land d], [b])$, $\theta_3 = ([d])^\circ$ and $\theta_4 = ([b \land d])^\circ$. By Theorem 2.2 $\theta_1$ and $\theta_2$ are atoms; it is easily seen then that the congruences $\Delta_{S'}$, $\theta_1$, $\theta_2$, $\theta_3$, $\theta_4$ form the interval $[\Delta_{S'}, \theta_4]$ in Con $S'$ isomorphic with $N_4$. If $\alpha$ and $\beta$ are congruences on $S$ corresponding to $\Delta_{S'}$ and $\theta_4$, then it follows that $[\alpha, \beta] \equiv N_2$ – proving the theorem.

We are now ready to give a necessary condition on $S$ in order for Con $S$ to be upper-semimodular.

Recall that a lattice $L$ is upper-semimodular iff $a \rightarrow a \land b$, $b \rightarrow a \land b$ and $a \neq b$ implies $a \lor b \rightarrow a$ and $a \lor b \rightarrow b$, $a, b \in L$. (Lower-semimodularity is defined dually.)

THEOREM 3.3. If Con $S$ is upper-semimodular then $S$ satisfies:

\[(U) \forall x \forall y(x < y \Rightarrow (x^* = y^* \text{ or } x \preceq y)).\]

Proof. Assume that $(U)$ fails in $S$. Then there are $x$ and $y$ in $S$ such that $x < y^*$, $x^* \neq y^*$ and $x \neq y$. The first two of these relations imply $x^{**} < y^{**}$ and hence $y \not\preceq x^{**}$, while it follows from the last that $x^{**} \neq y$ – thus we see that $x^{**}$ and $y$ are incomparable, hence that $y < x^{**}$. Consider the quotient $T = S/(y^{**})^\circ$. Clearly $[y]$ is dense in $T$ and $[x^{**}]$ is closed in $T$. If $[y] \subseteq [x^{**}]$ then $[y, y \land x^{**}] \in (y^{**})^\circ$, i.e. $y \land x^{**} \in y^{**}$, so $y \preceq x^{**}$ which is false, proving that $[y] \not\subseteq [x^{**}]$. Similarly it can be shown that $[x^{**}] \not\subseteq [y]$ and so we obtain that $[x^{**}]$ and $[y]$ are incomparable. With $b = [x^{**}]$ and $d = [y]$ in Theorem 3.2 we see that Con $T$, and hence Con $S$, contains an interval isomorphic with $N_4$. Since $N_4$ is not upper-semimodular, Con $S$ is not upper-semimodular, proving the theorem.

The following lemma is implicit in Lemma 3.2.

LEMMA 3.4. $(U)$ holds in $S$ iff $P_4 \not\in SH(S)$.

Proof. Suppose $(U)$ fails in $S$, then as in the proof of Theorem 3.3 we let $T = S/(y^{**})^\circ$, $b = [x^{**}]$ and $d = [y]$. Therefore there is a quotient $T'$ of $T$ with the properties (1)–(4) of the Lemma 3.1 and hence $[b]^* < [d]$ in $T'$. Then the elements $[1]$, $[b]$, $[d]$, $[b \land d]$, $[b]^*$ and $[0]$ form a subalgebra of $T'$ isomorphic to $P_4$. Since $T' \in H(S)$, $P_4 \in SH(S)$. Conversely, assume $(U)$ holds in $S$ and $P_4 \in SH(S)$. Now $(U)$ is equivalent to the positive universal sentence $(U_p)$ for PCS's and hence it follows from well-known preservation properties of first-order sentences that $(U)$ is true in $P_4$. But with $x = b$ and $y = d$ $(U)$ fails in $P_4$ and we have a contradiction.
We shall now study the PCS's with the property (U).

**LEMMA 3.5.** Let $S$ satisfy (U) and let $\alpha$ be an atom of type (T2) in Con $S$. Then $\alpha = (\alpha)_S \cup \Delta_{S - B(S)}$.

**Proof.** Let $(x, y) \in \alpha$ with $x \neq y$. Then using Remark 2.3 one has that $x^* \neq y^*$ and $x \not\rightarrow y$, say. Now applying (U) for $x^{**}$ and $y$ we get $x^{**} \prec y$ and thus $x = x^{**}$, whence $y = y^{**}$.

We shall now give a "strong" converse of Theorem 3.4. Recall that a lattice $L$ is semimodular iff for $a, b \in L, a \rightarrow a \wedge b \rightarrow a \vee b \rightarrow b$.

**THEOREM 3.6.** If $S$ satisfies (U) then Con $S$ is semimodular.

**Proof.** Let $\beta, \delta \in$ Con $S$ such that $\delta \rightarrow \beta \wedge \delta$. Consider the quotient $S_i = S/\beta \wedge \delta$ and let $\beta_i, \delta_i$ be the congruences on $S_i$ corresponding to $\beta, \delta$ respectively. Then $\beta_i \wedge \delta_i = \Delta_{S_i}$, and $\delta_i$ is an atom in Con $S_i$. In view of [10, Theorem 4.9] we can assume that $\delta_i$ is of type (T2). Let us write $\Phi_i$ for $\Phi(S_i)$. From [10, Theorem 2.6] it follows that $\beta_i \vee \Phi_i = \left((1)_{\beta_i}\right)_{\Phi_i} \left((1)_{\beta_i}\right)$, and Lemma 3.5 gives $\delta_i \wedge (\beta_i \vee \Phi_i) = \Delta_{S_i}$, and we have by [10, Theorem 4.7] that $[\Delta_{S_i}, \delta_i] = [\beta_i \vee \Phi_i, \beta_i \vee \Phi_i \vee \delta_i]$. Now we claim that $(\beta_i \vee \delta_i) \wedge (\beta_i \vee \Phi_i) = \beta_i$. In fact, from [10, Theorem 2.8] we know $\beta_i \wedge \delta_i = \delta_i \circ \beta_i \circ \delta_i$, and as noted earlier $\beta_i \vee \Phi_i = \left((1)_{\beta_i}\right)_{\Phi_i} \left((1)_{\beta_i}\right)$. Thus we only need to show that $(\delta_i \circ \beta_i \circ \delta_i) \wedge \left((1)_{\beta_i}\right)_{\Phi_i} \left((1)_{\beta_i}\right) \subseteq \beta_i$. Let $(x, y) \in \delta_i \circ \beta_i \circ \delta_i$ and $(x, y) \in \left((1)_{\beta_i}\right)_{\Phi_i} \left((1)_{\beta_i}\right)$. Thus $\delta_i \circ \beta_i \circ \delta_i \wedge \left((1)_{\beta_i}\right)_{\Phi_i} \left((1)_{\beta_i}\right) \subseteq \beta_i$. Next suppose $x$ and $y$ are non-closed. In this case we get from $x \delta_i \beta_i \delta_i y$ that $x = x$ and $y = y$, implying that $(x, y) \in \beta_i$. Finally let us assume that $x$ is closed and $y$ is not closed. Then $x \delta_i \beta_i \delta_i = x \delta_i \beta_i$, i.e., $x \delta_i \beta_i \delta_i$, and $x \delta_i \beta_i y$ must be closed. Thus we get $x \beta_i y^{**}$. On the other hand, from the other relations we get $x \left((1)_{\beta_i}\right)_{\Phi_i} y^{**} = x y^{**} \left((1)_{\beta_i}\right)_{\Phi_i} y^{**}$ which simplifies to $x \left((1)_{\beta_i}\right)_{\Phi_i} y^{**}$ and hence $x \beta_i y^{**}$. Thus we have $x \beta_i y^{**} \beta_i y$, i.e., $x \beta_i$, proving the claim. Then from [10, Theorem 4.7] we get $[\beta_i, \beta_i \vee \delta_i] = [\beta_i \vee \Phi_i, \beta_i \vee \Phi_i \vee \delta_i]$ and the latter, as was noted earlier in the proof, is isomorphic to $[\Delta_{S_i}, \delta_i]$. Thus $[\Delta_{S_i}, \delta_i] = [\beta_i, \beta_i \vee \delta_i]$, yielding $\beta_i \vee \delta_i \rightarrow \beta_i$ from which it is immediate that $\beta \vee \delta \rightarrow \beta$ in $S$, proving the theorem.

We have thus proved the following theorem, which includes the two characterizations mentioned in the introduction.

**THEOREM 3.7.** For any PCS $S$ the following are equivalent.

1. Con $S$ is upper-semimodular,
2. $S$ satisfies (U),
(3) Con $S$ is semimodular,
(4) $P_o \not\in SH(S)$,
(5) $S$ satisfies $(U_o)$.

Moreover, if Con $S$ is of finite length, each of the above conditions is equivalent to

(6) Con $S$ is a graded lattice.

Proof. (1) implies (2) by Theorem 3.3, (2) implies (3) by Theorem 3.6, that (3) implies (1) is obvious, (4) and (2) are equivalent by Lemma 3.4 and (2)$\iff$(5) was noted earlier. If Con $S$ is of finite length then from the proof of Theorem 3.3 it follows that (6) implies (2), while (1)$\iff$(6) is well-known.

In view of the fact that the sentence $(U_o)$ is both universal and positive and that universal sentences are preserved under the formation of substructures while positive sentences are preserved under homomorphic images, the following corollary is immediate.

COROLLARY 3.8. The class of PCS's $S$ such that Con $S$ is semimodular is closed under formation of subalgebras and quotients.

We would like to point out that even if $S$ is of finite length, Con $S$ need not be of finite length as is shown by the following example.

EXAMPLE 3.9.

If $S$ has, as shown in Figure 3 as infinite antichain consisting of $a_i$ ($i \in \mathbb{Z}$), then Con $S$ contains an infinite Boolean sublattice generated by the congruences $\theta(0', a_i)$, $i \in \mathbb{Z}$ and hence is not of finite length.

In this connection, therefore, the following theorem is of some interest.

![Figure 3](image)
THEOREM 3.10. If Con $S$ if of finite length $l$ then $S$ is finite.

Proof. The hypothesis implies that $[\Phi, V]$ is of finite length. Hence $[\Phi, V]$ is finite because it is isomorphic with Con $B(S)$ which, it is known, is finite iff it is of finite length. This implies that $B(S)$ is finite. Let $c \in B(S)$ be arbitrarily fixed. From the hypothesis it can be easily argued that every chain in $D_c$ is finite and every antichain in $D_c$ has width at most $l$. Then the fact that $D_c$ is finite follows from Dilworth's result: If $P$ is a poset of width $m$, then $P = \bigcup_{i=1}^{m} C_i$, where $C_i$ is a chain in $P$, $1 \leq i \leq m$. Now $S = \bigcup\{D_c : c \in B(S)\}$. Since each $D_c$ is finite and $B(S)$ is finite, it follows that $S$ is finite.

Using an argument similar to the one in the above theorem, it can also be shown that if Con $S$ satisfies the ascending chain condition or descending chain condition then Con $S$ is finite. Thus we have the following.

THEOREM 3.11. T.f.a.e.

(1) Con $S$ is of finite length,

(2) Con $S$ satisfies A.C.C.,

(3) Con $S$ satisfies D.C.C.,

(4) Con $S$ is finite,

(5) $S$ is finite.

§4. Congruence-distributive PCS's

In this section we shall give the characterizations of the class of congruence-distributive PCS's. The general program to arrive at these characterizations follows closely the one carried out in section 3. Accordingly the proofs of the results in this section are similar to those of the corresponding theorems of section 3 and hence the presentation in this section will be sketchy, leaving most of the details to the reader (or see [9]). Use Lemma 1.3 to prove the following.

LEMMA 4.1. Let $K \subseteq B(S)$ have the property: If $k_1 \leq k_2$ and $k_2 \in K$ then $k_1 \in K$. Then

$$\bigvee_{k \in K} \theta(D_k(S) \times D_k(S)) = \bigcup_{k \in K, a \in D_k(S)} \theta(k, a).$$

COROLLARY 4.2.

$$\Phi = \bigcup_{b \in B(S)} \theta(b, a).$$
The role of $P_s$ will be clear from the following lemma which in many respects is similar to Lemma 3.1.

**Lemma 4.3.** Suppose there are two incomparable elements $d$ and $h$ both dense in $S$. Then there exists a quotient algebra $S''$ of $S$ with the following properties:

1. $[d]$ and $[h]$ are dense in $S'$.
2. $[d]$ and $[h]$ are incomparable.
3. $[d \wedge h] \preceq [d]$, $[d \wedge h] \preceq [h]$, $[d] \preceq 1$ and $[h] \preceq 1$.
4. $N(S'') = \{[d], [h], [d \wedge h]\}$.

**Proof.** We shall briefly indicate how to construct a sequence of PCS's $S = S_0, S_1, \ldots, S_5$ such that $S_i = S_{i-1}/\xi_i$ where $\xi_i$ is a suitably chosen congruence on $S_{i-1}$, $i = 1, \ldots, 5$, and $S_5$ will have the properties (1)–(4) and hence will qualify to be $S''$. Let $[s]_n$ denote the congruence class of $[s]_{n-1}$ modulo $\xi_n$, with $[s]_0 = s$ for $s \in S$, and $[1]$ denotes $[1]_5$, $i = 1, \ldots, 5$.

**Step 1.** The set $M = \{x \in D(S) : x \succ d \text{ and } x \succ h\} = ([d], [h])$ is a filter in $S$. With $\xi_1 = M$ and $S_1 = S/M$ one verifies that $[d]_1$ and $[h]_1$ are incomparable and $[1]$ is the only common upper bound of $[d]_1$ and $[h]_1$.

**Step 2.** Let $K = D(S_1) - \{[d]_1, [1]\} \cup \{[h]_1, [1]\}$, $\xi_2 = \bigvee_{[k]_1 \in K} \theta([k]_1, [d \wedge h]_1)$ and $S_2 = S_1/\xi_2$. Then using Lemma 1.2 it follows that $[d]_2$ and $[h]_2$ are incomparable, $[d \wedge h]_2 \preceq [d]_2$, $[d \wedge h]_2 \preceq [h]_2$ and $D(S_2) = \{[d]_2, [1]\} \cup \{[h]_2, [1]\} \cup \{[d \wedge h]_2\}$.

**Step 3.** Let $H = \{[d]_2, [1]\}$ and define $\xi_3 = \bigvee_{[x]_2 \in H} \theta([x]_2, [d]_3)$ and $S_3 = S_2/\xi_3$. Then again using Lemma 1.2 one sees that $[d]_3$ and $[h]_3$ are incomparable, $[d]_3 \preceq [1]$ and $D(S_3) = \{[h]_3, [1]\} \cup \{[d]_3, [d \wedge h]_3\}$.

**Step 4.** Let $G = \{[h]_3, [1]\}$, let $\xi_4 = \bigvee_{[x]_2 \in G} \theta([x]_3, [h]_3)$ and let $S_4 = S_3/\xi_4$. Then $[d]_4$ and $[h]_4$ are incomparable, $[h]_4 \preceq [1]$ and $D(S_4) = \{[1], [d]_4, [h]_4, [d \wedge h]_4\}$.

**Step 5.** Let $T = \{[t]_4 \in B(S_4) : [t]_4 \neq [1]\}$. Define $S_5 = S_4/\xi_5$ where $\xi_5 = \bigvee_{[t]_4 \in T} \theta(D_{[t]_4}(S_4))$. Then using Lemma 4.1 it is easy to see that $[d]_5$ and $[h]_5$ are incomparable, and $N(S_5) = \{[d]_5, [h]_5, [d \wedge h]_5\}$. Thus $S'' = S_5$ is the quotient algebra that was sought, proving the lemma.

**Theorem 4.4.** Suppose there are two incomparable elements $d$ and $h$ both dense in $S$. Then there exists an interval in $\text{Con} S$ which is isomorphic with the following lattice.
Proof. By Lemma 4.3 there exists a quotient $S''$ of $S$ with the properties (1)–(4) mentioned in the lemma. We now define four congruences $\psi_\alpha$, $\psi_\beta$, $\psi_\omega$, $\psi_\rho$ by $\psi_\alpha = \theta([d] \land [h], [d])$, $\psi_\beta = \theta([d \land h], [h])$, $\psi_\omega = ([d])^\ast$ and $\psi_\rho = ([h])^\ast$. Then it is clear, in view of Theorem 2.2, that the interval $[\Delta_\alpha, \psi_\omega \lor \psi_\rho]$ of $\text{Con } S''$ is isomorphic with the lattice of Figure 4 and so there is an interval in $\text{Con } S$ isomorphic with the lattice of Figure 4—this proves the theorem.

THEOREM 4.5. If $\text{Con } S$ is lower-semimodular then $S$ satisfies $(D)$.

Proof. From the proof of Theorem 3.3 $S$ satisfies $(U)$ and therefore, it is sufficient to show that $S$ satisfies $(D_n)$.

Let us suppose $S$ fails to satisfy $(D_n)$. Then there is a closed element $c$ in $S$ such that $D_c(S)$ is not totally ordered and hence there are elements $s$, $t$ in $D_c(S)$ which are incomparable. Now consider $S_1 = S/c$ and let $d = [s]$ and $h = [t]$. Then using Theorem 4.4 one can conclude that $\text{Con } S_1$ is not lower-semimodular and hence $\text{Con } S$ also is not lower-semimodular.

We shall now give a model-theoretic condition which is equivalent to $(D)$. $\text{SH}(S)$ denotes the class of all subalgebras of homomorphic images of $S$.

THEOREM 4.6. $(D)$ holds in $S$ iff $P_c \notin \text{SH}(S)$ and $P_s \notin \text{SH}(S)$.

Proof. $(D)$ holds in $S$ iff $(U)$ and $(D_n)$ hold in $S$. If $(U)$ holds in $S$ then by Theorem 3.4 $P_c \notin \text{SH}(S)$. If $(D_n)$ is true in $S$ and if $P_s \in \text{SH}(S)$ then $(D_{n,p})$ would be true in $P_s$ which is impossible, thus $P_s \notin \text{SH}(S)$. For the converse observe that if $(D_n)$ fails in $S$ then $0,1,d,h,d \land h$ form a subalgebra of $S_1$ (see the proof of the preceding theorem) isomorphic to $P_s$ and then apply Theorem 3.4.

THEOREM 4.7. If $S$ satisfies $(D)$ then $\text{Con } S$ is distributive.
Proof. Let \( S \) satisfy \((D)\) and let \( x = y(\theta \land (\theta_1 \lor \theta_2)) \) for \( \theta, \theta_1, \theta_2 \in \text{Con} \ S \). It suffices to show that \( x = y((\theta \land \theta_1) \lor (\theta \land \theta_2)) \). Now there are \( u_1, \ldots, u_n \) in \( S \) such that \( u_i < x, i = 1, \ldots, n \) and \( x\theta_1 u_i \theta_2 u_2 \theta_1 \cdots \theta_2 y \). If \( y \) is closed, we may suppose that \( x \) is closed, but then the result follows since \( \text{Con} \ B(S) \) is distributive. So assume \( y \) is not closed. If \( x^* = y^* \) then use an argument similar to [8], and if \( x^* \neq y^* \) then use the arguments in the above two cases to conclude the result.

Thus we have the following theorem which includes the two characterizations mentioned in the introduction (see Varlet [14] for \((4) \implies (5)\)).

**THEOREM 4.8. T.f.a.e.:**

(1) \( \text{Con} \ S \) is lower-semimodular,
(2) \( S \) satisfies \((D)\),
(3) \( \text{Con} \ S \) is distributive,
(4) \( \text{Con} \ S \) is modular,
(5) \( \text{Con} \ S \) is a Heyting lattice,
(6) \( P_0 \notin \text{SH}(S) \) and \( P_5 \notin \text{SH}(S) \).

Note that the class of all \( \text{PCS}\)'s whose congruence lattices are distributive is closed under formation of subalgebras and quotients.

We have seen in [10] that the interval \([\Phi, \lor] \) is distributive and also that the interval \([\Delta, \Phi] \) satisfies the condition \((B)\). Also observe that although \( \text{Con} \ P_0 \) is not distributive, the interval \([\Delta, \Phi] \) is distributive, and that \( P_0 \) satisfies \((D_0)\). This motivates the following theorem whose proof is left to the reader.

**THEOREM 4.9. T.f.a.e.**

(1) \([\Delta, \Phi] \) is lower-semimodular,
(2) \( S \) satisfies \((D_0)\),
(3) \([\Delta, \Phi] \) is distributive,
(4) \([\Delta, \Phi] \) is modular,
(5) \( P_5 \notin \text{SH}(S) \).

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REFERENCES


Added in proof: Methods of this paper also yield the equivalence of \( \mu \)-semidistributivity and distributivity on \( \text{Con} S \). For further related results see Mathematica Slovaca 29 (1977), 381–395; Proceedings of the Third Brazilian Conference on Mathematical Logic, Sociedade Brasileira de Logica, São Paulo, 1980, pp. 281–307 and Mathematica Japonica 25 (1980), 519–521.