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On Ideals of Minors Fixing a Submatrix

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Let $\mathscr{M}$ be a $n \times m$ matrix with entries in a noetherian ring R and let $\mathscr{M}'$ be the submatrix of $\mathscr{M}$ consisting of the first r columns $(r \leqslant n-1)$. Consider the ideal $J_r(\mathscr{M})$ of $n \times n$ minors of $\mathscr{M}$ involving the columns of $\mathscr{M}'$. We obtain the primary decomposition and the homological dimension of $J_r(\mathscr{M})$ in the generic case. The proofs rely heavily on the methods and the theory of weak d-sequences and straightening laws. As a byproduct we obtain exact conditions under which $J_r(\mathscr{M})$ is generated by a d-sequence and also a complete picture of the blowing-up algebras of $J_r(\mathscr{M})$ in that case. The latter proofs rely on recent methods developed by several authors such as those of sliding-depth, approximation complexes, Cohen–Macaulay residual intersections. To close the discussion we construct a free resolution of $J_r(\mathscr{M})$ when $m = n+1$ (the case $r = n-1$ had been treated before by the present authors). A side curiosity herein obtained is an example of a nonperfect radical 3-generated ideal of codimension 2 whose associated graded ring is a Cohen–Macaulay reduced ring that is *not* Gorenstein. Examples of this sort do not seem to abound. © 1986 Academic Press, Inc.

INTRODUCTION

Let R be a noetherian ring and let $f\colon F \to G$ be a map of free R-modules, where rank $F = m \geqslant n =$ rank G. As is well known the image $I_n(f)$ of the

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induced map $\Lambda^n f\colon \Lambda^n F \to \Lambda^n G \simeq R$—the so-called 0th Fitting of coker(f)—is identified with the ideal generated by the $n \times n$ minors of a matrix of f. Let there be given a decomposition $F = F' \oplus F''$, where F' and F'' are free modules with rank $F' = r \leqslant n-1$. Let $J_r(f)$ denote the ideal which is obtained as the image of the restriction of $\Lambda^n f$ to $\Lambda^r F' \otimes \Lambda^{n-r} F'' \subset \Lambda^n F$. Clearly, $J_r(f)$ is correspondingly identified with the ideal generated by the $n \times n$ minors fixing (say) the first r columns of a matrix of f.

This work grew out from an attempt to write an explicit generic free resolution of $J_r(f)$. We have succeeded in doing so for the case $m = n+1$ (cf. Theorem D for a precise statement). Thus, together with the main result of [A-S], we have the details of the resolution for the extremal cases $m = n+1$ (r arbitrary) and $r = n-1$ (m arbitrary).

Though failing to obtain an explicit free resolution of $J_r(f)$ for all values of r, n, and m, we were nevertheless able to compute its homological dimension in the generic case (Theorem B). The main tool employed here is a collection of results springing out of the work of Huneke on weak d-sequences and their related ideals. Coming to us as a surprise at first, it soon became clear that this was the natural frame to study the finer properties of the ideal $J_r(f)$. The proof of Theorem B resorts to ideal-theoretic results hinging on peripheral properties of weak d-sequences. An interesting byproduct we are able to give is the primary decomposition of the ideal $J_r(f)$ in the generic case; this result is labelled Theorem A.

The work is divided into three short sections. In the first section we deal with the results which have a flavour of weak d-sequences. It is in this section that we prove all the ideal-theoretic results on $J_r(f)$.

The second section contains information on the usual blowing-up algebras of $J_r(f)$, i.e., the symmetric and Rees algebras and the associated graded ring. First, we give necessary and sufficient conditions on the values of r and m in order that $J_r(f)$ be generated by a d-sequence. We then estimate certain depths thus giving what essentially amounts to be the proof that $J_r(f)$ is residually Cohen–Macaulay for those particular values of r and m. A byproduct is that the sliding-depth condition holds in these cases. This allows us to use the main result of the approximation complexes theory [H-S-V1], yielding the usual arithmetical properties for the associated blowing-up rings. Finally, the nature of the primary decomposition of $J_r(f)$ allows for the proof of the torsion freeness of the associated graded ring along with the traditional outcome of equality between ordinary and symbolic powers. The fact that the graded ring is not Gorenstein yields some insight into the question as to when the graded ring is reduced Cohen–Macaulay but not Gorenstein (cf. [H-S-V3]).

The third section describes a generic complex that, under suitable natural conditions, is a free resolution of $J_r(f)$. This should be confronted with the result of [A-S].

1. Ideal-Theoretic Results

This section is wholly concerned with the generic case. To be more precise, let $R=k[X_{ij}]$ where k is a field and X_{ij} $(1\leqslant i\leqslant n,\ 1\leqslant j\leqslant m)$ indeterminants over k. We think of (X_{ij}) as the matrix of a map $f\colon F\to G$ of free R-modules and, for fixed $1\leqslant r\leqslant n$, we think of the submatrix (X_{ij}), $1\leqslant i\leqslant n$, $1\leqslant j\leqslant r$, as the matrix corresponding to the restriction map $f'\colon F'\to G$, with $F=F'\oplus F''$ and rank $F'=r$.

We will need certain "reduced" Plücker relations. These are probably well known but, for lack of proper reference, we include a proof here.

As a matter of notation, $\Delta^{i_1\cdots i_t}_{j_1\cdots j_t}$ will denote the $t\times t$ minor of (X_{ij}) involving rows $i_1,\dots,i_t$ and columns $j_1,\dots,j_t$. In the case of an $n\times n$ minor, we drop the superscripts. We then have

Lemma 1.1. *Given minors $\Delta^{i_1\cdots i_t}_{j_1\cdots j_t}$ and $\Delta_{j'_1\cdots j'_n}$ of (X_{ij}), the following relation holds*:

$$\Delta^{i_1\cdots i_t}_{j_1\cdots j_t}\cdot\Delta_{j'_1\cdots j'_n}=\sum_k \Delta^{i_1\cdots i_t}_{j_1\cdots j_{t-1}j'_k}\cdot\Delta_{j'_1\cdots j'_{k-1}j_tj'_{k+1}\cdots j'_n}.$$

Proof. For the sake of simplicity, assume $i_1=j_1=1,\dots,i_t=j_t=t$. We consider the enlarged $n\times(m+n-t)$ matrix

$$\left[\begin{array}{ccc|ccc|ccc}
X_{11} & \cdots & X_{1t} & X_{1t+1} & \cdots & X_{1m} & & & \\
\vdots & & \vdots & \vdots & & \vdots & & \bigcirc & \\
X_{t1} & \cdots & X_{tt} & X_{tt+1} & \cdots & X_{tm} & & & \\
\hline
X_{t+11} & \cdots & X_{t+1t} & X_{t+1t+1} & \cdots & X_{t+1m} & 1 & & \bigcirc \\
\vdots & & \vdots & \vdots & & \vdots & & \ddots & \\
X_{n1} & \cdots & X_{nt} & X_{nt+1} & \cdots & X_{nm} & \bigcirc & & 1
\end{array}\right]$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\underbrace{\qquad\qquad}_{n-t}$$

Clearly, we get an equality $\Delta^{1\cdots t}_{1\cdots t}=\Delta_{1\cdots t\,m+1\cdots m+n-t}$ between the $t\times t$ minor of the original matrix and an $n\times n$ minor of the enlarged matrix. By the same token, we see that $\Delta^{1\cdots t}_{1\cdots t-1j'_k}=\Delta_{1\cdots t-1j'_k m+1\cdots m+n-t}$. We now plug these equalities into the ordinary Plücker relation for $n\times n$ minors of the enlarged matrix in the following form:

$$\Delta_{1\cdots t\,m+1\cdots m-n+t}\cdot\Delta_{j'_1\cdots j'_n}=\sum_k \Delta_{1\cdots t-1j'_k m+1\cdots m+n-t}\cdot\Delta_{j'_1\cdots j'_{k-1}tj'_{k+1}\cdots j'_n}.\quad\blacksquare$$

Corollary 1.2. *Let $(f)=(X_{ij})$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$. Assume, moreover, that $m=n+1$. Then $J_r(f)=I_r(f')\cap I_n(f)$, where $I_r(f')$ is the ideal* *generated by the $r \times r$ minors of the submatrix* $(f') = (X_{ij})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$.

Proof. The nontrivial inclusion is $I_r(f') \cap I_n(f) \subseteq J_r(f)$. Denote $\Delta_{\hat{j}} = \Delta_{1 \cdots j-1 j+1 \cdots n+1}$ and set $\mathbf{\Delta}_j = (\Delta_{\hat{j}}, \ldots, \Delta_{\hat{r}})$, the ideal generated by all of $\Delta_{\hat{1}}, \ldots, \Delta_{\hat{r}}$ except $\Delta_{\hat{1}}, \ldots, \Delta_{\widehat{j-1}}$. Here $j = 1, \ldots, r$.

Clearly, $I_n(f) = J_r(f) + \mathbf{\Delta}_1$. Therefore, $I_r(f') \cap I_n(f) = J_r(f) + (I_r(f') \cap \mathbf{\Delta}_1)$ since $J_r(f) \subseteq I_r(f')$. Thus, it suffices to show that $I_r(f') \cap \mathbf{\Delta}_1 \subseteq J_r(f)$. We do this by recursion as follows.

First, observe that, as a particular case of Lemma 1.1, we obtain the following inclusions:

$$I_r(f') \cdot \Delta_{\hat{r}} \subseteq J_r(f),$$

$$I_j(f_j) \cdot \Delta_{\hat{j}} \subseteq \mathbf{\Delta}_{j+1} + J_r(f), \qquad j = 1, \ldots, r-1,$$

where (f_j) denotes the submatrix of f' formed with the first j columns. We then obtain

$$\begin{aligned} I_r(f') \cap \mathbf{\Delta}_1 = I_r(f') \cap ((\Delta_{\hat{1}}) + \mathbf{\Delta}_2) &\subseteq I_1(f_1) \cap ((\Delta_{\hat{1}}) + \mathbf{\Delta}_2) \\ &= (I_1(f_1) \cap (\Delta_{\hat{1}})) + \mathbf{\Delta}_2 \qquad \text{since } \mathbf{\Delta}_2 \subseteq I_1(f_1) \\ &= I_1(f_1) \cdot \Delta_{\hat{1}} + \mathbf{\Delta}_2 \qquad \text{since } \Delta_{\hat{1}} \notin I_1(f_1) \\ &\subseteq \mathbf{\Delta}_2 + J_r(f) \qquad \text{by the above inclusion } (j = 1). \end{aligned}$$

Thus, $I_r(f') \cap \mathbf{\Delta}_1 \subseteq I_r(f') \cap (\mathbf{\Delta}_2 + J_r(f)) = (I_r(f') \cap \mathbf{\Delta}_2) + J_r(f)$, and by recursion we are finished. ∎

To prove the main results of this section we recall a few facts from the theory of weak d-sequences and their related ideals as introduced in [HU1].

Let H be a finite partially ordered set (poset). Let $\{\Delta_\alpha\}_{\alpha \in H}$ be a set of elements in a ring R indexed by H. Let $\Lambda \subseteq H$ be an ideal (i.e., $\alpha \in \Lambda$ and $\beta \leqslant \alpha$ imply $\beta \in \Lambda$). An ideal J of R is an *H-ideal* if J is generated by elements indexed by an ideal $\Lambda \subseteq H$. In particular, the ideal $I = (\Delta_\alpha)_{\alpha \in H}$ is an H-ideal. For any given $\alpha \in H$, we have the ideal $\Lambda_\alpha = \{\beta \in H \mid \beta < \alpha\}$; the corresponding H-ideal will be denoted I_α. Finally, if J is any ideal of R, we denote by J^* the subideal of J generated by all elements Δ_β, $\Delta_\beta \in J$.

The main concept in the theory is this: we say that $\{\Delta_\alpha\}_{\alpha \in H}$ form a *weak d-sequence* if for any H-ideal J and any $\alpha \in H$ such that $I_\alpha \subseteq J$, $\Delta_\alpha \notin J$, the following hold:

(1) $(J : \Delta_\alpha)^*$ is an H-ideal,

(2) $(J : \Delta_\alpha) \cap I = (J : \Delta_\alpha)^*$,

(3) $\Delta_\beta \in (J: \Delta_\alpha) \Rightarrow \Delta_\alpha \Delta_\beta \in JI$,

(4) $\Delta_\alpha \notin (J: \Delta_\alpha) \Rightarrow (J: \Delta_\alpha) = (J: \Delta_\alpha^2)$.

An ideal of R of the form $(J: \Delta_\alpha) + I$, where J is an H-ideal such that $I_\alpha \subseteq J$, $\Delta_\alpha \notin J$, is called a *related ideal* to the weak d-sequence $\{\Delta_\alpha\}_{\alpha \in H}$.

Now, we specialize the above notions to our present context of determinantal ideals. Thus, H is the set of all arrays $(j_1 \cdots j_n)$, where $1 \leqslant j_1 < \cdots < j_n \leqslant m$. One decrees

$$(j_1 \cdots j_n) \leqslant (j'_1 \cdots j'_n) \Leftrightarrow j_1 \leqslant j'_1, ..., j_n \leqslant j'_n.$$

This turns H into a poset and the $n \times n$ minors $\{\Delta_{j_1 \cdots j_n}\}$ of the matrix (X_{ij}) form a weak d-sequence [HU1, 1.19]. Clearly, in our old notation, $I = I_n(f)$. Moreover, every related ideal to this weak d-sequence is a perfect ideal [HU1, Proof of Theorem 3.1]. Incidentally, it goes without saying that both these results rely heavily on nontrivial facts from the theory of algebras with straightening laws (cf. [HU1, Propositions 3.A and 3.B]).

After these preliminaries, we are ready for our next lemmata.

LEMMA 1.3. *Let (X_{ij}) be an $n \times m$ generic matrix and let $I_r(f')$ denote the ideal of $r \times r$ minors of the submatrix (X_{ij}), $1 \leqslant j \leqslant r \leqslant n$. Let $\Delta = \Delta_{1 \cdots r-1 r+1 \cdots n+1}$ and denote I_Δ the H-ideal $I_{(1 \cdots r-1 r+1 \cdots n+1)}$. Then $I_r(f') = I_\Delta: \Delta$.*

Proof. We consider the $n \times (n+1)$ submatrix of (X_{ij}) formed with the first $n+1$ columns, which will be denoted (g). We think of g as a map of free A-modules, with $A = k[X_{ij}]$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$ and, accordingly, g' will denote its restriction to an A-direct summand of rank r. Now, by a special case of Lemma 1.1 (as used in the last stage of the proof of Corollary 1.2), we see that the inclusion $I_r(g') \subseteq J_r(g): \Delta$ holds in A (note $\Delta \in A$). However, by Corollary 1.2, $J_r(f)$ has no embedded primes and as $\Delta \notin J_r(g)$, we must have $I_r(g') = J_r(g): \Delta$.

To conclude, note that $R = A[X_{ij}]$, $1 \leqslant i \leqslant n$, $n+1 \leqslant j \leqslant m$, a polynomial ring over A. Therefore, one has

$$I_r(f') = I_r(g')\, R = (J_r(g): \Delta)\, R = J_r(g)\, R: \Delta = I_\Delta: \Delta,$$

where the equality $I_\Delta = (\Delta_{1 \cdots r r+2 \cdots n+1}, ..., \Delta_{1 \cdots r r+1 \cdots n})$ $(= J_r(g)\, R)$ follows straightforwardly from the definitions. ∎

COROLLARY 1.4. *For the generic matrix $(f) = (X_{ij})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and the submatrix $(f') = (X_{ij})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant r$, the ideal $I_r(f') + I_n(f) \subset k[X_{ij}]$ is a perfect ideal.*

Proof. This follows immediately from our general remarks preceding Lemma 1.3. ▌

One of our main results now follows.

THEOREM A. *Let* $(f)=(X_{ij})$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant m$ *and* $(f')=(X_{ij})$, $1\leqslant i\leqslant n$, $1\leqslant j\leqslant r$. *Then* $J_r(f)=I_r(f')\cap I_n(f)$.

Proof. On clearly sees that $I_r(f')^*=J_r(f)$ (in particular, $J_r(f)$ is an H-ideal on the poset H of maximal minors!). Then, by Lemma 1.3 and property (2) of a weak d-sequence, we get the desired result. ▌

Our second main result depends on knowing the grade of the perfect ideal $I_r(f')+I_n(f)$. We isolate this as a lemma.

LEMMA 1.5. $\operatorname{grade}(I_r(f')+I_n(f))=m-r+1$.

Proof. The result holds in fact over $A[X_{ij}]$, for any ring A which is a localization of a polynomial ring over k.

For in this case $I_r(f')+I_n(f)$ is a prime ideal as well [HU1, Proof of Theorem 3.1]. We then use the inductive device of Northcott [N]. Thus, we consider the ring of fractions $A[X_{ij}]_{X_{11}}=A[X_{i1},X_{1j}]_{X_{11}}[X_{i'j'}]$, where $i=1,\ldots,n$; $j=1,\ldots,m$; $2\leqslant i'\leqslant n$; $2\leqslant j'\leqslant m$. In the latter, $(I_r(f')+I_n(f))_{X_{11}}=I_r(g')+I_n(g)$, where g is an $(n-1)\times(m-1)$ matrix of indeterminates over $A[X_{i1},X_{1j}]_{X_{11}}$ and g' the initial $(n-1)\times(r-1)$ submatrix. Since $I_r(f')+I_n(f)$ is prime, grade $(I_r(f')+I_n(f))=\operatorname{grade}(I_r(g')+I_n(g))$.

Therefore, by induction, we are reduced to the case $r=1$. Here $I_r(f')+I_n(f)=(X_{11},\ldots,X_{n1})+I_n(f'')$, where $(f'')=(X_{ij})$, $1\leqslant i\leqslant n$, $2\leqslant j\leqslant m$. But $X_{11},\ldots,X_{n1}$ is an $A[X_{ij}]/I_n(f'')$-sequence. Therefore, grade $(I_r(f')+I_n(f))=\operatorname{grade} I_n(f'')+n=m$, as was to be shown. ▌

THEOREM B. *If* $m\geqslant n+1$ *and* $n\geqslant r+1$ *then the homological dimension of* $J_r(f)$ *is* $m-r-1$.

Proof. Using Theorem A we get the exact sequence

$$0\to R/J_r(f)\to R/I_r(f')\oplus R/I_n(f)\to R/(I_r(f')+I_n(f))\to 0.$$

By Theorem B, $R/(I_r(f')+I_n(f))$ has homological dimension $m-r+1$. The middle term has homological dimension (h.d.) equal to $\max\{m-n+1,\ n-r+1\}$. We thus separate the discussion into two cases:

(1) $m-n+1\geqslant n-r+1$.

We have $m-r+1\gneqq m-n+1$ as we are assuming $r\leqslant n-1$. Therefore, a standard argument shows that the homological dimension of $R/J_r(f)$ is

$\geqslant m-n+1$. If strict inequality takes place, then it must be the case that h.d. $R/J_r(f)=m-r+1-1=m-r$. In the event of equality we must have $m-r+1 \leqslant m-n+1+1$ and since we are given $m-n+1 \lneqq m-r+1$, then $m-r+1=m-n+2$ must hold, or in other words, $r=n-1$. But, in this case we know already that h.d. $R/J_r(f)=m-n+1$ [A-S]. Therefore, in any case we have h.d. $R/J_r(f)=m-r$.

(2) $n-r+1 \geqslant m-n+1$.

The discussion is entirely similar except when the alternative h.d. $R/J_r(f)=n-r+1$ arises. By analogy with the preceding case, we have here $m-r+1=n-r+2$, i.e., $m=n+1$. Thus, in any case, h.d. $R/J_r(f)=m-r$. ∎

2. Blowing-Up Rings and the d-Sequence Property for $J_r(f)$

In this paragraph we are still concerned with the generic case.

For simplicity, we will denote $J_r=J_r(f)$. For an R-module E, $S(E)$ will stand for its symmetric algebra; for a given ideal $I \subset R$, $R(I)$ and $\mathrm{gr}_I(R)$ will denote, respectively, its Rees algebra and associated graded ring.

The following is the main result of this section. It completely clarifies the difference between the extremal cases $r=n-1$ or $m=n+1$ and all other cases.

Theorem C. *Let (X_{ij}), $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, be a generic matrix over a field k and let J_r denote as before the ideal of $n \times n$ minors fixing an $n \times r$ submatrix. Then*:

(i) *$S(J_r)=R(J_r)$ if and only if $r=n-1$ or $m=n+1$.*

(ii) *If $r=n-1$ or $m=n+1$, the following hold*:

(ii$_1$) *$S(J_r)=R(J_r)$ is a Cohen–Macaulay integrally closed domain,*

(ii$_2$) *$S(J_r/J_r^2)=\mathrm{gr}_{J_r}(R)$ is a Cohen–Macaulay reduced ring,*

(ii$_3$) *$gr_{J_r}(R)$ is a torsion-free R/J_r-module. In particular, we obtain equality $J_r^s=J_r^{(s)}$ of ordinary and symbolic powers throughout.*

Remark. The theorem provides us with generic examples of *non*perfect ideals of codimension 2 whose blowing-up rings behave in best possible way. Moreover, these ideals have an arbitrary number of generators $\geqslant 3$. Such systematic examples do not seem to appear in the literature.

We will prove the above theorem by enhancing a few technical results which may be of some interest on their own.

LEMMA 2.1. *Let the notation be as before and assume $r = n-1$. Then the generators of the ideal J_{n-1}, considered as a subset of the weak d-sequence of all $n \times n$ minors, form themselves a weak d-sequence.*

Proof. Note the generators of J_{n-1} are indexed by the set $H' = \{(1 \cdots n-1n), \ldots, (1 \cdots n-1m)\}$. It is easy to check that this set is an ideal in the poset H of all arrays $(j_1 \cdots j_n)$ indexing the $n \times n$ minors. On the other hand, it is clear that H' is linearly ordered in the induced order from H. Therefore, by [HU1, Proposition 1.3], the generators $\Delta_{1 \cdots n-1n}, \ldots, \Delta_{1 \cdots n-1m}$ form a weak d-sequence. ∎

PROPOSITION 2.2. *Keeping the same notation, if $r = n-1$ or $m = n+1$ then the generators of J_r form a d-sequence.*

Proof. Precisely, we will show the generators of J_r form a d-sequence for the (linear) order induced from that of the poset of all $n \times n$ minors.

For $m = n+1$ the result follows from the work of Huneke (cf., e.g., [HU2]). Thus, we assume $r = n-1$. In this case $J := J_r = (\Delta_{1 \cdots n-1n}, \ldots, \Delta_{1 \cdots n-1m})$. Denote $J_l = (\Delta_{1 \cdots n-1n}, \ldots, \Delta_{1 \cdots n-1n+l})$, $0 \leqslant l \leqslant m-n$. By [H-S-V1, Lemma 12.1], we are to show that $(J_l : \Delta_{n+l+1}) \cap J = J_l$, $0 \leqslant l \leqslant m-n-1$.

For this we proceed as follows. First, observe that J_l is an H'-ideal, where H' is the (linearly) ordered set of generators of J. Moreover, it is also clear that $J_l = I_{\Delta_{1 \cdots n-1n+l+1}}$, in the notation explained in the preceding section, and that $\Delta_{1 \cdots n-1n+l+1} \notin J_l$. Since the generators of J form a weak d-sequence on H' (Lemma 2.1), we have, by definition, that $(J_l : \Delta_{n+l+1})^*$ is an H'-ideal and $(J_l : \Delta_{n+l+1}) \cap J = (J_l : \Delta_{n+l+1})^*$. By the only H'-ideals are of the form J_t, for some $0 \leqslant t \leqslant m-n$. Therefore, $(J_l : \Delta_{n+l+1}) \cap J = J_t$, for some $t \geqslant l$. On the other hand, the generating relations of the ideal J [A–S] are Plücker relations. In particular, $J_l : \Delta_{n+l+1}$ is generated by the generators of J_l and in addition by other $n \times n$ minors *not* belonging to J. We are then forced to conclude that $(J_l : \Delta_{n+l+1}) \cap J = J_l$. ∎

For the next lemma we recall a concept that has been introduced in [H-S-V2, Sect. 6]. Thus, let $I \subset R$ be an ideal in a local ring and let H_l be the lth Koszul homology module on a set of generators of I. The ideal I is said to satisfy the *sliding-depth condition* if

$$\operatorname{depth} H_l \geqslant \dim R - v(I) + l, \quad l \geqslant 0,$$

where $v(I)$ stands for the minimal number of generators of I. This property is known to be independent of the set of generators of I and to localize whenever R is Cohen–Macaulay [H-S-V2, Sect. 6].

LEMMA 2.3. *Let as before J_r denote the ideal of $n \times n$ minors involving the first r columns of (X_{ij}). If $r = n-1$ or $m = n+1$ then J_r satisfies the sliding-depth property.*

Proof. In both cases we will show that if $x_1, ..., x_s$ are the canonical generators of J_r, canonically ordered so that they form a d-sequence, then depth $R/(x_1, ..., x_l) \geqslant \dim R - l$ for $1 \leqslant l \leqslant s$. This is all we need to prove the sliding depth condition in this case (cf. [H-S-V1, Proposition 7.1]; also [H-S-V; Lemma 3.7]).

The argument for the two cases are slightly different, so we will consider them separately.

(1) $r = n-1$.

Here $J := J_r = (\Delta_{1 \cdots n-1n}, ..., \Delta_{1 \cdots n-1m})$ and $J_l = (\Delta_{1 \cdots n-1n}, ..., \Delta_{1 \cdots n-1n+l})$. We consider the initial $n \times (n+l)$ submatrix of (X_{ij}) and set $\tilde{R} = k[X_{ij}]$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+l$, so that $R = \tilde{R}[X_{ij}]$, $1 \leqslant i \leqslant n$, $n+l+1 \leqslant j \leqslant m$. Then J_l is the extension to R of the ideal $\tilde{J}_l = (\Delta_{1 \cdots n-1n}, ..., \Delta_{1 \cdots n-1n+l})\tilde{R}$. From our present Theorem B (or [A-S]) $\tilde{R}/\tilde{J}_l$ has homological dimension $n+l-n+1 = l+1$, hence R/J_l has this same homological dimension over R. It follows that depth $R/J_l = \dim R - (l+1)$ as intended.

(2) $m = n+1$.

Here $J := J_r(f) = (\Delta_{1 \cdots r \cdots \widehat{n+1}}, ..., \Delta_{1 \cdots r\widehat{r+1} \cdots n+1})$ and the initial ideal $J_l = (\Delta_{1 \cdots r \cdots \widehat{n+1}}, ..., \Delta_{1 \cdots r+l\widehat{r+l+1} \cdots n+1})$. Only note that now J_l has $n+1-(r+l)$ generators! In fact, J_l is precisely the ideal of $n \times n$ minors fixing the initial $n \times (r+l)$ submatrix (g), i.e., $J_l = J_{r+l}(g)$ in our standard notation. Thus, by Theorem B, we have depth $R/J_l =$ depth $R/J_{r+l}(g) = \dim R - (n+1-(r+l))$, as was to be shown. ∎

Remark. Because of Proposition 2.2, we have $\nu_P(J_r) \leqslant ht(P)$ for every prime $P \supseteq J_r$. Together with Lemma 2.3, it has just be shown that J_r is a *residually Cohen–Macaulay* ideal in the terminology of [HU3] (cf. also [H-V-V, Sect. 3). In fact, the canonical generators of J_r as above form a typical sequence of elements with all the desired properties that intervene in the latter concept [H-V-V, Definition 3.1].

We now switch to the

Proof of Theorem C. (i) If $r = n-1$ or $m = n+1$ then, by Proposition 2.2, the generators of J_r form a d-sequence. Under this condition, one knows that $S(J_r) = R(J_r)$ [HU2].

Conversely, suppose both $m \geqslant n+2$ and $r \leqslant n-2$. Consider the two minors $\Delta_{1 \cdots rr+1 \cdots n}$ and $\Delta_{1 \cdots rr+1 \cdots n-2n+1n+2}$, both belonging to J_r. We have the Plüker relation:

$$\begin{aligned}\Delta_{1 \cdots rr+1 \cdots n}\Delta_{1 \cdots rr+1 \cdots n-2n+1n+2}\\ &= \Delta_{1 \cdots rr+1 \cdots n-1n+1}\Delta_{1 \cdots rr+1 \cdots n-2n}\\ &\quad - \Delta_{1 \cdots rr+1 \cdots n-1n+2}\Delta_{1 \cdots rr+1 \cdots n-2nn+1},\end{aligned}$$

where all involved minors belong to J_r. This relation thus shows that $S_2(J_r) \neq J_r^2$, as required.

(ii) Since $v_P(J_r) \leqslant ht(P)$ for any prime $P \supseteq J_r$ and since J_r satisfies the sliding depth condition, we have by [H-S-V1, Theorem 9.1] that both $S(J_r) = R(J_r)$ and $S(J_r/J_r^2) = \mathrm{gr}_{J_r}(R)$ are Cohen–Macaulay rings. As is well known, $R(J_r)$ is integrally closed provided $\mathrm{gr}_{J_r}(R)$ is reduced. Now, since R/J_r is reduced, $\mathrm{gr}_{J_r}(R)$ is reduced whenever it is a torsion-free R/J_r-module. Thus, we have to show this fact alone. For this we claim that it suffices to prove the stronger estimates $v_P(J_r) \leqslant ht(P) - 1$, for every prime $P \supseteq J_r$ containing properly a minimal prime of J_r. Indeed, once these estimates are granted, we depth-chase along the approximation complex of [H-S-V1]:

$$\begin{aligned}0 \to H_t \to H_{t-1} \otimes S_1(G) \to \cdots \to H_1 \otimes S_{t-1}(G)\\ \to S_t(G) \to J_r^t/J_r^{t+1} \to 0,\end{aligned}$$

where G is a free R-module of rank $= v(J_r)$, using the sliding depth estimates for the H_i's, thereby obtaining torsion freeness of J_r^t/J_r^{t+1}.

Therefore, we are indeed reduced to proving the estimates $v_P(J_r) \leqslant ht(P) - 1$, whenever P contains properly some minimal prime of J_r. For this, consider first the case where $P \supseteq J_r = I_r(f') \cap I_n(f)$ but $P \not\supseteq I_r(f')$ (the case $P \not\supseteq I_n(f)$ is entirely similar). Then we are assuming $P \supsetneqq I_n(f)$. Since $S(J_r) = R(J_r)$, we know that $v_P(J_r) = l_P(J_r)$, where l_P stands for the analytic spread at P. But $l_P(J_r) = l_P(I_n(f))$ as $P \not\supseteq I_r(f')$, and for the ideal of maximal minors we have $l_P(I_n(f)) \leqslant ht(P) - 1$ provided $P \supsetneqq I_n(f)$ (cf., e.g., [HU4, Theorem 3.5]). We are thus done in this case. Otherwise, we have $P \supseteq I_r(f') + I_n(f)$. But $ht(I_r(f') + I_n(f)) = m - r + 1$ by Lemma 1.5, which even exceeds $v(J_r)$ by 1. This completes the proof. ∎

Remark. In the above setting, the ring $\mathrm{gr}_{J_r}(R)$ is not Gorenstein in general. Thus, let $m = 4$, $n = 2$, $r = n - 1 = 1$. Here $v(J_r) = 3$ and $\mathrm{gr}_{J_r}(R)$ is a factor ring of $R[T_1, T_2, T_3]$ by an ideal generated by 4 elements; as the homological dimensions of $\mathrm{gr}_{J_r}(R)$ over $R[T_1, T_2, T_3]$ is 3, it is well known that $\mathrm{gr}_{J_r}(R)$ cannot be Gorenstein [B-E, Theorem 1.1]. An explicit resolution of $\mathrm{gr}_{J_r}(R)$ over $S=R[T_1, T_2, T_3]$ is

$$0 \to S^3 \xrightarrow{\begin{bmatrix} X_{11} & X_{21} & 0 \\ -X_{12} & -X_{22} & T_1 \\ X_{13} & X_{23} & -T_2 \\ -X_{14} & -X_{24} & T_3 \\ 0 & 0 & X_{21} \\ 0 & 0 & -X_{11} \end{bmatrix}} S^6 \xrightarrow{\begin{bmatrix} \Delta_{34} & 0 & -\Delta_{14} & -\Delta_{13} & -T_3X_{13}+T_2X_{14} & -T_3X_{23}+T_2X_{24} \\ -\Delta_{24} & -\Delta_{14} & 0 & \Delta_{12} & T_3X_{12}-T_1X_{14} & T_3X_{22}-T_1X_{24} \\ \Delta_{23} & \Delta_{13} & \Delta_{12} & 0 & T_2X_{12}-T_1X_{13} & T_2X_{22}-T_1X_{23} \\ 0 & 0 & 0 & 0 & -X_{11} & -X_{21} \end{bmatrix}} S^4$$

$$\xrightarrow{[\Delta_{12}\Delta_{13}\Delta_{14}\Delta_{34}T_1 - \Delta_{24}T_2 + \Delta_{23}T_3]} S.$$

COROLLARY 2.4. *There is a local ring R and an ideal $I \subset R$ such that:*

(i) *R is regular and R/I is reduced.*

(ii) *I is a codimension* 2 (*generically complete intersection*) *ideal generated by* 3 *elements.*

(iii) $\mathrm{gr}_I(R)$ *is R/I-torsion free* (*hence reduced by* (i)).

(iv) $\mathrm{gr}_I(R)$ *is Cohen–Macaulay.*

But

(v) $\mathrm{gr}_I(R)$ *is not Gorenstein.*

As far as we can say no such examples were known before. As a matter of fact, if one requires a bit more about I, namely, that it be moreover height-unmixed (e.g., prime) then $\mathrm{gr}_I(R)$ must indeed be a Gorenstein ring [H-S-V4]. At the other end of the spectrum, one can give examples of a Gorenstein local ring R of embedding dimension 4 and dimension 2 for which the graded ring $\mathrm{gr}_m(R)$ is a reduced Cohen–Macaulay ring, but not a Gorenstein ring [H-S-V3, Example (1.2)].

3. AN EXPLICIT FREE RESOLUTION

We now come to grips with the question of constructing an explicit free resolution of J_r. As explained in the Introduction, we are only able to do accomplish this for the "extremal" cases where $r=n-1$ [A-S] or $m=n+1$ (Theorem D).

We briefly review the set-up. R is a noetherian ring and $f\colon F \to G$ is a map of free R-modules, where rank $F=n+1$ and rank $G=n$. We are given moreover a direct sum decomposition $F=F' \oplus F''$, where F' and F'' are free modules and rank $F' = r \leqslant n-1$. $J_r = J_r(f)$ is then the image of the restriction of $\Lambda^n f$ to $\Lambda^r F' \otimes \Lambda^{n-r} F'' \subset \Lambda^n F$.

Now, associated to the dual f'^* of the restriction $f' = f_{|F'}: F' \to G$ there is the well-known complex [B-E]

$$\mathbb{G}(f'^*):\quad 0 \to S_{n-r-1}(F') \otimes \Lambda^n G^* \to S_{n-r-2}(F') \otimes \Lambda^{n-1} G^*$$

$$\to \cdots \to S_0(F') \otimes \Lambda^{r+1} G^* \xrightarrow{\varepsilon_{f'^*}} G^* \xrightarrow{f'^*} F'^*.$$

Next, we will need the dual f''^* of the restriction $f'' = f_{|F''}: F'' \to G$. If we replace the tail of $\mathbb{G}(f'^*)$,

$$S_0(F') \otimes \Lambda^{r+1} G^* \xrightarrow{\varepsilon_{f'^*}} G^* \xrightarrow{f'^*} F'^*,$$

by the composite

$$S_0(F') \otimes \Lambda^{r+1} G^* \xrightarrow{\varepsilon_{f'^*}} G^* \xrightarrow[f''^*]{f'^*} F''^* \xrightarrow{\sim} \Lambda^r F' \otimes \Lambda^{n-r} F'',$$

we will obtain the following modified complex

$$\mathbb{G}(f'^*, f''^*): 0 \to S_{n-r-1}(F') \otimes \Lambda^n G^* \xrightarrow{\partial_{n-r+1}} S_{n-r-2}(F') \otimes \Lambda^{n-1} G^*$$

$$\to \cdots \to S_0(F') \otimes \Lambda^{r+1} G^* \xrightarrow[\partial_2]{G_2} \Lambda^r F' \otimes \Lambda^{n-r} F''.$$

It is well known [B-E] that the complex $\mathbb{G}(f'^*)$ is acyclic if and only if grade $I_r(f'^*)$ ($=$grade $I_r(f')) \geqslant n-r+1$. The main result in the section reads in turn as follows.

THEOREM D. *The following conditions are equivalent*:

(i) $\mathbb{G}(f'^*, f''^*)$ *is a free resolution of the ideal* J_r.

(ii) grade $I_r(f') \geqslant n-r+1$ *and grade* $I_n(f) \geqslant 2$.

Proof. (ii)$\Rightarrow$(i). By the remark on the exactness of $\mathbb{G}(f'^*)$ we need only show that $\mathbb{G}(f'^*, f''^*)$ is exact at the terms $S_0(F') \otimes \Lambda^{r+1} G^*$ and $\Lambda^r F' \otimes \Lambda^{n-r} F''$.

First, we check exactness at $S_0(F') \otimes \Lambda^{r+1} G^*$. Thus, let $a \in \ker(\partial_2)$. Then $\varepsilon_{f'^*}(a) \in \ker(f''^*)$, by construction, and $\varepsilon_{f'^*}(a) \in \ker(f'^*)$ since $\mathbb{G}(f'^*)$ is a complex. Therefore $\varepsilon_{f'^*}(a) \in \ker(f^*)$. But $\ker(f^*) = (0)$ is more than granted by the assumption grade $I_n(f) \geqslant 2$. Again, by the exactness of $\mathbb{G}(f'^*)$ at $S_0(F') \otimes \Lambda^{r+1} G^*$, we are through.

Next, we verify exactness at $\Lambda^r F' \otimes \Lambda^{n-r} F'' \simeq F''^*$. Under the full hypothesis that grade $I_n(f) \geqslant 2$, we have the well-known short acyclic complex

$$0 \to G^* \xrightarrow{f^*} F^* \simeq \Lambda^n F \xrightarrow{\Lambda^n f} R,$$

which embeds into the commutative diagram

$$\begin{tikzcd}
 & & F'^* & & \\
0 \arrow[r] & G^* \arrow[ur, "f'^*"] \arrow[r, "f^*"] \arrow[dr, "f''^*"] & F^* \arrow[u, "i_1'^*"'] \arrow[d, "i''^*"] \arrow[r, phantom, "\simeq"] & \Lambda^n F \arrow[r, "\Lambda^n f"] & R \\
 & S_0(F') \otimes \Lambda^{r+1} G^* \arrow[u, "\varepsilon_{f'^*}"] \arrow[r, "\partial_2"] & F''^* \arrow[r, phantom, "\simeq"] & \Lambda^r F' \otimes \Lambda^{n-r} F'' \arrow[u, hook] \arrow[r, "\partial_1"] & R,
\end{tikzcd}$$

where $i'\colon F' \to F$ and $i''\colon F'' \to F$ are the natural inclusion maps. From this diagram one easily sees, by using exactness of $\mathbb{G}(f'^*)$ at G^*, that the bottom sequence is exact at $\Lambda^r F \otimes \Lambda^{n-r} F''$.

(i) $\Rightarrow$ (ii) First, we show that $\mathbb{G}(f'^*)$ is acyclic. As remarked before this implies grade $I_r(f') \geq n-r+1$.

Clearly, it suffices to check exactness at $S_0(F') \otimes \Lambda^{r+1} G^*$ and G^*. The first is trivial as $\ker(\varepsilon_{f'^*}) \subset \ker(\partial_2)$. Thus, we turn to the term G^*. Let $b \in \ker(f'^*)$. Then, an easy consequence of Laplace rule shows that $f''^*(b)$ belongs to the kernel of the structural map

$$F''^* \simeq \Lambda^r F' \otimes \Lambda^{n-r} F'' \xrightarrow{\partial_1} J_r.$$

By assumption, we can find $a \in S_0(F') \otimes \Lambda^{r+1} G^*$ such that $\partial_2(a) = f''^*(b)$. We now claim that $\varepsilon_{f'^*}(a) = b$.

At any rate, we have $f'^*(\varepsilon_{f'^*}(a) - b) = 0$ and also $f''^*(\varepsilon_{f'^*}(a) - b) = \partial_2(a) - f''^*(b) = 0$. Hence, $f^*(\varepsilon_{f'^*}(a) - b) = 0$. However, f^* is injective as grade $I_n(f) \geq$ grade $J_r \geq 1$, J_r being an ideal with a free resolution. Therefore, $\varepsilon_{f'^*}(a) = b$, thereby showing exactness of $\mathbb{G}(f'^*)$ at G^*.

Finally, to show that grade $I_n(f) \geq 2$ it suffices to show that grade $J_r \geq 2$. But this follows by an standard argument of reduction to the generic case [A-S]. This concludes the proof of the theorem. ∎

Note added in proof. Corollary 1.4 is contained in J. Eagon and M. Hochster: Cohen–Macaulay rings, invariant theory and the generic perfection of determinantal loci, *Amer. J. Math.* **53** (1971), 1020–1058. Theorem A can be obtained in a quicker manner without resorting to properties of weak d-sequences. We thank W. Bruns for calling our attention to these facts.

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