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Normal Rees Algebras

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INTRODUCTION

This paper is a case study of a problem in the boundary between commutative algebra and computer algebra. The question is that of deciding the completeness of all the powers of an ideal of a polynomial ring. The formulation of the problems and the setting up of the guideposts take place in the first area, while the actual navigation is done in the latter. We hope that the methodology that evolved may be used in other instances.

For an integral domain R of field of fractions K, the *integral closure* of a submodule I consists of all elements $z \in K$ satisfying an integral equation of the form

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0, \qquad a_i \in I^i.$$

This set, I_a, is a submodule of K and I is said to be *complete*—or *integrally closed*—if $I = I_a$. I will be called *normal* if all of its powers are complete.

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Key references are [ZS, Appendices 4 and 5] and [Li, Sect. II] for the wealth of related ideas and applications. In particular they prove that if R is a 2-dimensional regular local ring—and more generally in [Li], if R is a 2-dimensional rational singularity—then complete ideals are normal. This turns out to be no longer valid in higher dimensions and the extent to which it is violated provides one of the motivations here.

Besides the role that complete ideals play in Zariski's theory of desingularization of algebraic surfaces, two other reasons make its study appealing: (i) the large number of Cohen–Macaulay phenomena that is connected to normal ideals, and (ii) the possibility now open of looking at these ideals and constructions with the resources of computer algebra.

There is a convenient manner in which the powers of an ideal I and their integral closures can be coded: It suffices to use, respectively, the Rees algebras of the corresponding (multiplicative) filtrations, the subrings of $K[T]$ defined by

$$R(I) = \bigoplus I^n T^n = R[IT] \qquad \text{and} \qquad R_a(I) = \bigoplus (I^n)_a T^n.$$

$R_a(I)$ is then the integral closure of $R(I)$ in the ring $K[T]$. If $I \subset R$, it is convenient to redefine $R_a(I)$ as the integral closure of $R(I)$ in $R[T]$. Of course, if R is normal the two definitions agree. In any case, for an ideal I we shall refer to the family $\{(I^n)_a \cap R\}$, $n \geqslant 0$, as the I-integral filtration.

Another filtration that will play a role in our discussion is the symbolic algebra associated to a prime ideal P. Although it makes sense for non-domains, we shall restrict them to this case. $P^{(n)}$ will then be $R \cap P^n R_P$. In particular, for a Noetherian ring, $P^{(n)}$ is the P-primary component of P^n. The corresponding Rees algebra will be denoted by $R_s(P)$.

Our aim is the comparison between the algebras $R(I)$ and $R_a(I)$ for ideals of affine domains. This, it turns out, seems more direct than the comparison between an ideal and its integral closure. Indeed, except for very special cases, deciding whether a given ideal I, defined by a set of generators, is complete seems appreciably harder than asking the same of all powers of I. The latter has a simple formulation in a criterion that meshes the Jacobian condition with Serre's normality criterion. Several classes of ideals can then be examined for this property, particularly via a computer analysis.

The approach used here to these questions is syzygetic, that is, it depends on access to a presentation of $R(I) = B/J$, where B is a polynomial ring over R. Over the rationals, obtaining J is a straightforward matter, in view of several implementations of an algorithm that computes Gröbner bases of ideals. The analysis proceeds through the determination of the sizes of various determinantal ideals associated to J. To obviate system constraints, we push the calculations against some established facts to expedite the verification in several instances.

We shall now highlight the contents of this paper. Section 1 contains the abstract formulation of Serre's condition S_2 (Theorem 1.5) and other elements of the normality of $R(I)$. We use the occasion to provide another proof of a result of [Hu_1] (see also [Sc]) on the symbolic power algebra of a prime ideal.

In the next section we cast the normality of $R(I)$ in terms of a Jacobian condition on the ideal J and the unmixedness of (J, I). The main result here lies in the testing of this method against several contexts: hypersurfaces rings, almost complete intersections, two-dimensional rings, etc. An indication is given on how to extend these procedures to more general affine domains.

The last section describes a simple procedure to obtain J from a computer program able to generate the Gröbner basis of an ideal. Two versions of the Jacobian condition are available, and the translation of the question above on (I, J) in terms of the determinantal ideals of a Noether normalization is indicated. Carrying out these steps directly, however, was not always possible for the limitations mentioned; in turn this often led to alternative (but more restrictive) approaches.

Most of the quoted computer-analyzed examples could, *post facto*, be independently checked—or were otherwise insightful. Several questions, prompted by the copious lists obtained, are raised. Because of its variability, from one location to another, the often mentioned expression "system limitations/constraints" will be left ambiguous. Finally, one word about several of the "Remarks" and "Propositions"; they have the expressed aim of coding some facts in the literature into convenient steps to be tested by the computer.

1. Basic Theory

In this section we develop and review some facts about the ideal theory of Rees algebras, particularly those connected to divisoriality. The ground ring we have in mind is an affine domain, but we shall make several observations applicable to more general rings. For valuation theoretic aspects of normality we shall refer to [ZS]. Another point of interest will be the connection between normality and Cohen–Macaulayness in Rees algebras; [Ra] contains a detailed examination of this phenomenon for ideals of the principal class.

(1.1) *Serre's Condition.* We recall the following terminology [Mat]: A ring R is an S_1-ring if its zero ideal has no embedded prime. If, further, principal ideals generated by regular elements have no embedded primes, R will be called an S_2-ring.

Let A be an integral domain. Then $A = \bigcap A_{P'}$ where P runs over the prime ideals associated to principal ideals [Kap]. This representation has two immediate consequences:

(i) First, it follows that A will be integrally closed if each A_P is normal. When A is Noetherian, this observation along with its converse is Serre's normality criterion: A is normal $\Leftrightarrow A_P$ is a discrete valuation domain for each prime associated to a principal ideal. (It is usual to break this last formulation into two parts: (a) A satisfies S_2 and (b) A_p is a discrete valuation ring for each prime $\mathbf{p}$ of height one—the so-called R_1 condition.)

(ii) Let x be a nonzero element of A; then

$$A = A_x \cap \left(\bigcap A_p \right), \qquad \mathbf{p} \in \operatorname{Ass}(A/xA).$$

Assume that A is the extended Rees algebra of the ideal $I \subset R$:

$$A = R[It, u], \qquad u = t^{-1}.$$

The representation above may be written

$$A = A_u \cap \left(\bigcap A_P \right),$$

where P runs over the associated primes of uA. Since $A_u = R[t, t^{-1}]$, it follows therefore that

(1.2) COROLLARY. *If R is a normal domain then A is normal iff A_P is normal for each associated prime of uA.*

Because the normality of $R(I)$ and of A are equivalent notions, we have ([Ho], [Hu_1]):

(1.3) COROLLARY. *If R is normal and $\operatorname{gr}_I(R)$ is reduced, then $R(I)$ is normal.*

(1.4) *Remark.* Let R be a normal, affine, graded k-algebra

$$R = k + R_1 + R_2 + \cdots$$

and let $M = R_+ = R_1 + R_2 + \cdots$ be the irrelevant ideal of R. Suppose M is generated by R_1; then $\operatorname{gr}_M(R) \simeq R$, so that the Rees algebra $R(M)$ is normal.

The following result discusses more fully the condition S_2 for the Rees algebra $R(I)$.

(1.5) THEOREM. *Let R be a Noetherian ring satisfying S_2 and let I be an ideal containing a regular element. The following two conditions are equivalent*:

(1) *The Rees algebra $S = R(I)$ satisfies S_2.*

(2a) *The associated graded ring $G = \mathrm{gr}_I(R)$ satisfies S_1.*

(2b) *For each prime ideal $\mathfrak{p}$ of R, of height one, I_p is principal.*

Proof. (1) ⇒ (2): Let P^* be a prime ideal of G of height at least one, and denote by P its inverse image in S. Localize R at $\mathfrak{p} = R \cap P$ and denote still by R the resulting local ring. P is a prime ideal of S of height at least two. To prove (2a) consider the exact sequences [Hu_1]

$$0 \to (It) \to S \to R \to 0$$

and

$$0 \to IS \to S \to G \to 0.$$

Since P-depth$(R) > 0$ and P-depth$(S) > 1$, (It) has P-depth at least 2. But $(It) \simeq IS$ as S-modules, and therefore P-depth$(G) > 0$.

If $\dim(R) = 1$, the assumption is that S is a Cohen–Macaulay ring. Assume I is a proper ideal. We may extend R by a faithfully flat extension and thus assume that the residue field of R is infinite. Let x be a minimal reduction of I [NR]; that is, for some integer s, $I^{s+1} = xI^s$. We claim that $I = (x)$. It is easy to see that $\{x, xt\}$ form a system of parameters for P. As S is Cohen–Macaulay, these elements form a regular sequence. In such a case, for $r \in I$ we have the equation $r \cdot xt = x \cdot rt$, which shows that r must be a multiple of x. (See [NR] for a discussion of minimal reductions.)

(2) ⇒ (1): Let P be a prime ideal of S. As before put $\mathfrak{p} = R \cap P$, localize at $\mathfrak{p}$, but denote the localization still by R. If $I \not\subset \mathfrak{p}$, $S = R[t]$, which is a S_2-ring along with R. We may assume $I \subset \mathfrak{p}$ and that P has height at least two. If $It \not\subset P$, there exists—by the usual prime avoidance argument—a nonzero divisor $x \notin I$ such that $xt \notin P$. Localizing S at xt we get that x is a regular element of S and $(S/xS)_{xt} = \mathrm{gr}_I(R)_{xt}$, from which we get that depth S_P is at least two, since G is S_1. We may thus assume that $It \subset P$, so that P is the maximal irrelevant of S. First, if height$(\mathfrak{p}) = 1$, S is Cohen–Macaulay by (2b). Thus we take $\dim(R) > 1$. Let x be a regular element of $\mathfrak{p}$. Suppose that P is associated to S/xS, that is, assume that there exists a nonzero, homogeneous element h of S/xS with $Ph = O$. If degree$(h) = 0$, we would have $\mathfrak{p}h = 0$, and $\mathfrak{p}$ would be associated to Rx, which is a contradiction since height$(\mathfrak{p}) > 1$. Let then $h \in (S/xS)_s$; put $h = r^*$, $r \in I^s \backslash xI^s$. By assumption we have

$$(*)\ \text{(i)}\ \mathfrak{p}r \subset xI^{s+1}; \qquad \text{(ii)}\ rI^n \subset xI^{s+n}, \qquad n > 0.$$

The first condition implies that $\mathbf{p}(r/x) \subset R$, so that $r = ax$, $a \in R$, since grade $(\mathbf{p}) > 1$. We are going to show that $a \in I^s$. Consider the exact sequences of graded S-modules derived from G:

$$0 \to \bigoplus I^{n+1}/I^{n+2} \to \bigoplus I^n/I^{n+2} \to \bigoplus I^n/I^{n+1} \to 0.$$

The modules at the ends are submodules of G, so that they both have P-depth > 0, and thus the P-depth of the mid module is also strictly positive. Arguing inductively we get that for each integer q,

$$P\text{-depth}\ (T = \bigoplus I^n/I^{n+q}) > 0.$$

Suppose we have shown that $a \in I^q \backslash I^{q+1}$, $q < s$. Let b denote the class of a in $R/I^{q+1} \subset T$. But the equations $(*)$ imply that $Pb = 0$, which is a contradiction. ▮

The next section contains some criteria for the Rees algebra to be S_2. The survey article [HSV_1] discusses many instances of Cohen–Macaulay Rees algebras. Unfortunately it relies heavily on knowledge of the Koszul homology modules of the ideal that is not currently accessible through direct computation. On the other hand, because they have been treated elsewhere (see [EH] and its bibliography), we focus mainly on ideals outside the framework of invariant theory.

Finally, we take up a case of equality between the algebras $R_a(I)$ and $R_s(I)$, a situation that has been discussed in [Hu_1] and [Sc]. The approach here is perhaps simpler. Let R be a Noetherian ring and let P be a prime ideal. Suppose the localization of the algebra $G = \text{gr}_P(R)$ at the prime P is an integral domain (e.g., R_P = regular local ring). Let us see the meaning of this condition as reflected on the extended Rees algebras of P. Denote by A, B, and C the extended Rees algebras of P corresponding, respectively, to the P-adic, P-integral closure, and P-symbolic filtrations.

(1.6) *Remark.* If G is an integral domain, then $B = C$. Indeed, from (1.2) we have the equality

$$A = A_u \cap A_{(u)}.$$

But both A_u and $A_{(u)}$ contain B and C.

(1.7) THEOREM. *Let P be a prime ideal of the domain R be such that the localization G_P is an integral domain. The following are equivalent:*

(1) $B = C$.

(2) *G has a unique minimal prime.*

Proof. The hypothesis on G_P has the following immediate consequences: (i) $B \subset C$: we may localize at P to verify this; now appeal to (1.6); (ii) uC is a prime ideal: C/uC is a torsion-free R/P-module; localizing at P we obtain an embedding of C/uC into the integral domain G_P.

(1) $\Rightarrow$ (2): It is clear by the lying over theorem, since uB is a prime ideal.

(2) $\Rightarrow$ (1): Since uA has a unique minimal prime Q, and $A_P = B_P = C_P$, there is a unique minimal prime Q^* of B lying over uB. Denote by B' the integral closure of B in its field of quotients. B' is a Krull domain. The minimal primes $Q_1, ..., Q_s$ of uB' must each contract to Q^* in B. Localizing at P we conclude that uB' is a prime ideal. Let $q \in Q^*$; we have $q = u \cdot b' \in B'$. Since $q \in uC$, $b' \in R[t]$—that is, $b' \in B$ as desired. ∎

The final point of this section concerns the minimal prime ideals of $I \cdot R(I)$. We recall the notion of the analytic spread, $l(I)$, of an ideal I of the local ring $(R, \mathbf{m})$: set $l(I) =$ Krull dimension of $R(I) \otimes (R/\mathbf{m})$. Put otherwise, if the residue field of R is infinite—an innocuous hypothesis here—$l(I)$ is the number of generators of the smallest ideal J such that $I^{t+1} = J \cdot I^t$ for some integer t. As a consequence, $l(I) \leqslant \text{height}(\mathbf{m})$.

(1.8) *Remark.* Let R be a universally catenarian integral domain and let I be an ideal. Let P be a prime ideal of $R(I)$ containing I, and put $\mathbf{p} = P \cap R$. Localizing at $\mathbf{p}$—and denoting by $\mathbf{m}$ the resulting maximal ideal—we get

$$\dim(R(I)/P) \leqslant \dim(R(I)/\mathbf{m}R(I)) = l(I_p),$$

with equality if P is a minimal prime of $IR(I)$.

The conditions of (1.7) can also be phrased in terms of analytic spreads (cf. [Hu_1]): $G = \text{gr}_I(R)$ has a unique minimal prime if and only if for each prime $\mathbf{p}$ of R properly containing I, then $\text{height}(\mathbf{p}) > l(I_p)$.

2. Normality Criteria

Let R be a Noetherian domain, and let $S = R(I)$ be the Rees algebra of the ideal $I = (f_1, ..., f_m)$. The natural presentation of S is a homomorphism

$$\phi\colon B = R[T_1, ..., T_m] \to S, \qquad \phi(T_j) = f_j T.$$

In this section we consider ways of describing the normality of S in terms of ϕ. Set $J = \text{kernel}(\phi)$. J is a graded ideal of B, $J = \bigoplus J_s$. J_1 is the R-module of all first-order syzygies of I, that is, all 1-forms in the variables T_j

$$a_1 T_1 + \cdots + a_m T_m$$

$a_j \in R$, such that

$$a_1 f_1 + \cdots + a_m f_m = 0.$$

Similarly, J_s consists of all first-order syzygies of I^s. (This indicates the information packed into J, so that access to it should enable one to rapidly decide properties of $R(I)$.)

If J is generated by J_1 we shall say that I is of *linear type*. In such a case $R(I)$ is the symmetric algebra, $\mathrm{Sym}(I)$, of I as an R-module.

To highlight the significance of J, we have the following formulation of Serre's normality criterion for Rees algebras.

(2.1) PROPOSITION. *Let R be a normal domain, and let $S = R(I)$ be the Rees algebra of the ideal I. $R(I)$ is normal if and only if the following conditions hold*:

(a) *The ideal (J, I) of B is unmixed (i.e., has no embedded prime).*

(b) *For each minimal prime P of (I, J), the image of J in the B/P-module P/P^2 has* $\mathrm{rank} = \mathrm{rank}(P/P^2) - 1$.

Proof. Part (a) is a recasting of Theorem 1.5, while (b) is requiring that localizations of $R(I)$ at essential height one primes be discrete valuation rings.

(2.2) *Remarks.* (i) Note that these conditions test for the completeness of I and of all of its powers. There exist few classes of ideals whose completion can be explicitly described; for monomial ideals, which have been repeatedly "discovered," see [KM].

(ii) There is also the question of when the completeness of all high powers of I implies the normality of I. Using a Veronese subring of $R(I)$, it is easy to see that it will be so if and only if $R(I)$ satisfies S_2.

(iii) An earlier version of this proposition was used in [Va] to classify the defining prime ideals of monomial curves of equations $x = t^a$, $y = t^b$, and $z = t^c$ into normal and non-normal ideals.

(2.3) *Standard Jacobian Criterion.* Given the ideal J of the presentation of $R(I)$, the verification of the conditions of (2.1) is far from straightforward. Condition (b), in case R is a polynomial ring over a field of characteristic zero, can be tested through that part of the usual Jacobian criterion that pertains to normality. That is, let $h_1, ..., h_s$ be a set of generators of J and consider the Jacobian matrix

$$M = \frac{\partial(h_1, ..., h_s)}{\partial(x_1, ..., x_n, T_1, ..., T_m)}.$$

Let N be the ideal generated by all $(m-1)\times(m-1)$ minors of M; then (b) is equivalent to height $(J, N)\geq m+1$ (see [Mat, Sect. 29]). The determination of the size of this ideal may take longer than the direct verification of (b) through the identification of the minimal primes of (I, J). ▮

To indicate the usefulness of (2.1) we shal now discuss several classes of examples. We refer to Section 3 for the method used to obtain the presentation ideal J.

(2.4) EXAMPLE: HYPERSURFACES. Let $R=A/(F)$ be a hypersurface ring. Here $A=K[X_1, ..., X_n]$, $F=F_r+F_{r+1}+\cdots\in A$, with F_i homogeneous of degree i and $F_r\neq 0$. Assume R is a normal domain and $r\geq 2$. Set $\mathbf{m}=(x_1, ..., x_n)=(X_1, ..., X_n)/(F)$. We will test the normality of $R(\mathbf{m})$ against the conditions of Proposition 2.1.

First, one reads the presentation ideal J of $R(\mathbf{m})$. For this purpose it is convenient to use the upgrading operator of the Rees algebra $R_A(X_1, ..., X_n)=A[T_1, ..., T_n]/(X_iT_j-X_jT_i)$, $1\leq i, j\leq n$ (cf. [HSV_1], where its inverse—the so-called downgrading operator—was considered). This is an additive map that acts on $R_A(X_1, ..., X_n)$ as a $K[T_1, ..., T_n]$-homomorphism by means of $X_i\to T_i$. (Warning: The map is defined only on the Rees algebra, not on $A[T_1, ..., T_n]$.) Now for $G\in A[T_1, ..., T_n]$, let $\lambda(G)\in A[T_1, ..., T_n]$ denote an arbitrarily chosen lifting of the image of G under the upgrading operator. Also set $\lambda^i(G)=\lambda(\lambda^{i-1}(G))$, for any integer $i\geq 1$.

We claim that the presentation ideal L of $R(\mathbf{m})$ as an $A[T_1 ..., T_n]$-algebra is given by

$$L=(X_iT_j-X_jT_i, F, \lambda(F), ..., \lambda^r(F)).$$

Indeed, suppose $G(X, \mathbf{T})$ is a polynomial, homogeneous of degree s in the $\mathbf{T}$-variables, such that $G(x, Tx)=T^s\cdot G(x, x)=0$. This means that $G(X, X)$ is a multiple of $F(X)$, $G(X, X)=F(X)\cdot H(X)$. Apply the upgrading operator s times on $G(X, X)$ to obtain $G(X, T)$ back. If $s\leq r$, on the product $F\cdot H$ apply it to the F factor only; if $s\geq r$, after applying λ r times to F, the remaining $s-r$ times apply it to H. It follows that the difference $G(X, T)-\lambda^s(F)\cdot H$ (and correspondingly $G(X, T)-\lambda^r(F)\cdot\lambda^{s-r}(H)$ in the other case) lies in the ideal generated by the Koszul polynomials $X_iT_j-X_jT_i$.

The presentation ideal of $R(\mathbf{m})$ as an $R[T_1, ..., T_n]$-algebra is obtained by making, in the list above, the substitution $X_i\to x_i$. Further, the minimal primes of $(\mathbf{m}, J)\subset R[T_1, ..., T_n]$ are easily seen to be of the form $P_k=(\mathbf{m}, H_k)$, where $F_r=\prod_1^q H_k^{a_k}$ is the prime factorization of $F_r(\mathbf{T})=\lambda^r(F_r)$.

Condition (a) of Proposition 2.1 is trivially satisfied in this situation since $G = \mathrm{gr}_m(R) = K[T_1, \ldots, T_n]/(F_r(T))$ is even Cohen–Macaulay.

Consider condition (b) of that proposition. We first claim that the image of J in any P_k/P_k^2 is generated by $(x_i T_j - x_j T_i, \lambda^{r-1}(F_r), \lambda^r(F_r) + \lambda^r(F_{r+1}))$. Indeed, $\lambda^i(F) \in \mathbf{m}^2 \cdot R[T]$ for $0 \leqslant i \leqslant r-2$ and $i \geqslant r+2$.

We separate into two cases.

(1) $a_k = 1$: In this case it is clear that the localization of $R(\mathbf{m})$ at P_k is a discrete valuation ring.

(2) $a_k \geqslant 2$: For simplicity set $a = a_k$, $P = P_k$, and $H = H_k(T)$. We have $\lambda^r(F_r) = F_r(T) = H^a \cdot \prod_1^q H_j(T)^{a_j} \in P^2$, $j \neq k$.

We also have that $\lambda^{r-1}(F_r) \in P^2$. Indeed, let $d = \deg(H)$; then, by the definition of the upgrading operator, we may write

$$\begin{aligned}\lambda^{r-1}(F_r) &= \lambda^{d-1}(H) \cdot \lambda^{r-d}(H^{a-1} \cdot \textstyle\prod H_j(T)^{a_j}) \\ &= \lambda^{d-1}(H) \cdot H(T)^{a-1} \cdot \textstyle\prod H_j(T)^{a_j},\end{aligned}$$

$j \neq k$, where $\lambda^{d-1}(H) \in \mathbf{m}R[T]$. Since $a \geqslant 2$, we conclude that $\lambda^{r-1}(F_r) \in \mathbf{m} \cdot H(T)\, R[T] \subset P^2$.

The upshot is that the image of J in P/P^2 is generated by the Koszul relations $x_i T_j - x_j T_i$, plus the single element $\lambda^r(F_{r+1}) \in \mathbf{m}R[T]$, which can be written as $\sum_1^n G_i(T)\, x_i$, for suitable $G_i(T) \in K[T]$. Now $(P/P^2)_P$ is a vector space over $(R[T]/P)_P$ with basis defined by $\{x_1, \ldots, x_n, H(T)\}$. With respect to this basis, the image of J is described by the matrix

$$\begin{bmatrix} & G_1(T) \\ \mathbf{K} & \vdots \\ & G_n(T) \end{bmatrix},$$

where $\mathbf{K}$ is the matrix of the Koszul relations for the T_i's.

If H divides F_{r+1} (in A) then $H(T)$ divides $G_i(T)$ for each i. In this case, the above matrix has the same rank as $\mathbf{K}$, which is $n-1$. We thus see that condition (b) of Proposition 2.1 would be violated. Conversely, if H does not divide F_{r+1}, consider the following $n \times n$ minor:

$$\begin{bmatrix} T_2 & \cdots & T_n & G_1(T) \\ -T_1 & \cdots & 0 & G_2(T) \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ 0 & \cdots & -T_1 & G_n(T) \end{bmatrix},$$

where we assumed that T_1 does not divide H. This minor is just $(-T_1)^{n-2} \cdot F_{r+1}(T)$, which does not lie in P.

Summing up, we have shown: $R(\mathbf{m})$ is normal if and only if H_k does not divide F_{r+1} for every k such that $a_k \geqslant 2$.

Remarks. (i) It follows from [GS] that $R(\mathbf{m})$ is Cohen–Macaulay if and only if $r < n$.

(ii) If R is no longer a hypersurface, the presentation of $R(\mathbf{m})$ could—by using the argument above—still be found once a Gröbner basis of the defining ideal of R is available. To address this point, see Section 3.

(2.5) *Almost Complete Intersections.* Let R be a Cohen–Macaulay and let I be an ideal of height g, generated by $g+1$ elements. The following cuts the burden of verifying the conditions of Proposition 2.1 in several cases. (H_1 stands for the 1-dimensional Koszul homology module of I, $S = B/IB$.)

(2.6) PROPOSITION. *If, moreover, I is generically a complete intersection, then its Rees algebra satisfies S_2.*

Proof. The hypothesis implies that I is unmixed and that the approximation complex associated to I is exact [HSV_1]. There exists then an exact sequence

$$0 \to H_1 \otimes R/I[x_1, ..., x_{g+1}] \to R/I[x_1, ..., x_{g+1}] \to G \to 0.$$

By Theorem 1.5 it suffices to verify that G satisfies S_1. Let P be an associated prime of G and set $\mathbf{p}$ = inverse image of P in R. We may assume that $\mathbf{p}$ is the maximal ideal of R.

In all cases, by [Ao], the depth of H_1 is at least $\inf\{\dim(R/I), \operatorname{depth}(R/I) + 2\}$. If I is not Cohen–Macaulay, $\operatorname{depth}(H_1) \geqslant 2$, and the complex says that $\mathbf{p}G$ has grade at least 1, which is a contradiction. When I is Cohen–Macaulay, G will be Cohen–Macaulay as well (cf. [HSV_1]). ∎

Note that if I is as in (2.6) it is of linear type, $J = (J_1)$, and the verification of (2.1)(b) can now proceed in one of two ways: (i) If R is, say, a polynomial ring over a field of characteristic 0, one can use the standard Jacobian criterion; cf. (2.3). (ii) Determining the minimal primes of (I, J) may be made easier by the fact that it suffices to look for primes P with $\operatorname{ht}(P \cap R = \mathbf{p}) = 1 + \operatorname{ht}(I)$, and such that the minimal number of generators $v(I_p) = g + 1$.

(2.7) EXAMPLE: MONOMIAL PRIMES. Let I be the prime ideal of $k[x, y, z]$ defining the monomial curve (t^a, t^b, t^c), $a < b < c$, $\gcd(a, b, c) = 1$. I is given by the 2×2 minors of the matrix

$$\begin{bmatrix} z^{a_1} & x^{a_2} & y^{a_3} \\ y^{b_1} & z^{b_2} & x^{b_3} \end{bmatrix}.$$

The exponents can be found by the algorithm described originally in [He], or through the Gröbner basis algorithm of [Bu_1]. Although these are obviously related here, the former is preferable if one wants to study the distribution of normal primes.

The only prime that matters, in verifying (2.1), is $P=(x, y, z)B$. Looking at the rank of the image of J in P/P^2, it follows that the Rees algebra $R(I)$ is not normal in exactly one of the following cases:

(i) $\inf\{a_1, a_2, a_3\} > 1$ or $\inf\{b_1, b_2, b_3\} > 1$;

(ii) $a_1 = b_2 = 1$, and the other exponents are >1.

Examples of these cases are, respectively, (7, 8, 10) and (6, 7, 16), while (3, 4, 5) corresponds to a normal prime. (These are the minimal examples of each kind.)

(2.8) EXAMPLE: TWO-DIMENSIONAL RINGS. Let I be an ideal of $k[x, y]$, and assume that I is (x, y)-primary. (In order not to clutter the text we only discuss "short" ideals; the presentation ideal J was found by the method discussed in Section 3.)

(a) $I=(y^5, x^2y^2, x^3+y^4)$: The ideal J is minimally generated by

$$(y^3T_2 - x^2T_1,\ -y^2T_3 + xT_2 + yT_1,\ -xyT_1T_3 + y^2T_2^2 + xT_1^2,$$
$$(-xT_1T_2 + yT_1^2)\,T_3 + yT_2^3 - T_1^3).$$

B/J satisfies R_1, but it follows from [EG] that it does not satisfy S_2. More precisely, if an ideal J of B, of height two, and generated by $2+r$ elements satisfies S_r, then it must be Cohen–Macaulay. (For another decision method, see (3.9).)

(b) $I=(x^{n+1}, y^n, x^n + y^{n-1} + xy)$, $n>3$: The ideal of relations J is generated by

$$h_1 = -x^{n-2}y^{n-2}T_1 + (y^{n-2} + x - x^{n-1}y^{n-3})\,T_2 + (x^{n-1}y^{n-2} - y^{n-1})\,T_3$$
$$h_2 = (y + x^{n-1} + x^{2n-3}y^{n-3})\,T_1 + (x^{2n-2}y^{n-4} - x^{n-1}y^{n-3})\,T_2$$
$$+ (x^{n-1}y^{n-2} - x^n - x^{2n-2}y^{n-3})\,T_3$$
$$h_3 = -x^nT_3^2 + (x^{n-1}y^{n-3}T_2 + (x^{n-2}y^{n-2} + y + x^{n-1})\,T_1)\,T_3$$
$$-x^{n-1}y^{n-4}T_2^2 + (-2x^{n-2}y^{n-3} + 1)\,T_1T_2 - x^{n-3}y^{n-2}T_1^2.$$

These generators were abstracted from several numerical examples. J is obviously Cohen–Macaulay and a direct check shows that (2.1)(b) is satisfied as well. The ideals corresponding to $n=2, 3$ are also complete.

(2.9) *Cohen–Macaulay Algebras.* There are a number of ways that an ideal such as J could be ascertained to be Cohen–Macaulay. For instance, according to [GS], the reduction exponent of I must be two, that is, there must be a quadratic polynomial in J, whose ideal of coefficients of the T-variables is R. More generally, if I is generated by $\{f_1, ..., f_m\}$, then J is Cohen–Macaulay iff the following equality holds:

$$(T_1, ..., T_m)^2 = (f, g) \cdot (T_1, ..., T_m) + J_2^*.$$

Here J_2^* is the component J_2 of J read mod $\mathbf{m}$, and f and g are generic linear polynomials in the T's, with coefficients in k. In particular, this says that the dimension of J_2^* must be at least $\binom{m-1}{2}$.

It was observed, but not explained, that for a three-generated ideal I, a generating set for J—in the examples above and in several other cases—could be obtained as follows. Write the generators of J_1 as

$$f = xa + yb, \qquad g = xc + yd.$$

Then the element $h = ad - bc$ is clearly in J. If h has unit content, then $J = (f, g, h)$, which is not difficult to show. Otherwise repeat the construction on the pairs (f, h) and (g, h), and so forth. If B/J satisfies R_1, then this procedure yielded the full ideal J.

An exhaustive search for normal ideals of $k[x, y]$ suggested the following problem.

(2.10) *Question*: Are the Rees algebras of complete ideals of $k[x, y]$ always Cohen–Macaulay?[1]

One would expect the answer to this to be read off Zariski's description of the complete ideals of $k[x, y]$ (cf. [ZS, Appendix 5]).

(2.11) *Symbolic Powers.* We now consider deciding whether for a given prime ideal I, its ordinary and symbolic powers coincide. (R will be a polynomial ring over a field of characteristic zero.)

(2.12) PROPOSITION. *Let $f \in R \backslash I$ be such that the ordinary and symbolic powers of the localization I_f coincide. Then*

(a) *If f is regular modulo (I, J), that is, if $(J, I): f = (J, I)$, then (I, J) is a prime ideal—and the ordinary and symbolic powers of I coincide.*

(b) *$L = \bigcup_{n \geq 1} (I, J): f^n$ is a minimal prime ideal of (I, J). If L is the radical of (I, J) then the integral closure of the ordinary powers of I coincides with its symbolic powers.*

Proof. Part (a) is clear. That L is prime follows by localizing at R_f; we then appeal to Theorem 1.7.

[1] Craig Huneke has answered this question affirmatively.

Remarks. (i) Since in case (a) the Rees algebra $R(I)$ would be normal, (2.1)(b) is an early obstruction.

(ii) Picking f may be accomplished in the following manner: Determine the Jacobian ideal D of I; any element $f \in D \setminus I$—which always exists—will do. We observe that if R is an arbitrary Noetherian ring and I is a prime ideal, it follows from [Br], cf. also [EM], that the set of prime ideals $\mathbf{p}$ for which $(I_p)^n = (I_p)^{(n)}$, for all n, defines an open set of Spec(R).

(iii) Note that if $(I, J) \neq (I, J){:}f$, we may determine where the symbolic and ordinary primes of I first differ.

(iv) Both ideals $(I, J){:}f$ and $\bigcup_{n \geqslant 1}(I, J){:}f^n$ can be determined from any program able to generate Gröbner bases (see Section 3). As a matter of fact, the second ideal, in some systems, is computed rather more easily!

An example is that considered in [Ho], where P is the prime defining the surface $k[u^2, u^3, uv, v]$, which was also studied in [EH]. We consider the next monomial ring not covered by their analyses: $R = k[u^3, u^4, uv, v]$. $I \subset k[x, y, z, w]$ is minimally generated by $(-xw^3 + z^3,\ x^2w^2 - yz^2,\ x^3w - y^2z,\ yw - xz,\ y^3 - x^4)$. If $f = x$ we have the situation described above.

The ideal J is nonminimally generated by the polynomials

$$(((-xyw - x^2z)\,T_2 - x^2yT_1)\,T_4^2 + T_1T_3^2 + T_2^2T_3,$$
$$T_1T_5 + (xyw + x^2z)\,T_4^2 - T_2T_3,\ xzwT_4^2 - T_1T_3 - T_2^2,\ -T_2T_5 + x^2yT_4^2 - T_3^2,$$
$$(-y^2wT_2 - xy^2T_1)\,T_4 - w^2T_3^2 + 2xwT_2T_3 - x^2T_2^2,\ -x^3y^2T_1T_4 - yz^2T_3^2$$
$$+ (x^3w + 2y^2z)\,T_2T_3 + (-y^3 - x^4)\,T_2^2,\ (-x^3zT_2 - x^3yT_1)\,T_4 - z^2T_3^2$$
$$+ 2yzT_2T_3 - y^2T_2^2,\ (xw^3 - z^3)\,T_2 + (x^2w^2 - yz^2)\,T_1,\ xw^3T_3$$
$$+ (-x^2w^2 - yz^2)\,T_2 - y^2zT_1,\ xw^2T_4 + zT_2 + yT_1,\ -z^3T_3 + (x^2w^2 + yz^2)\,T_2$$
$$+ x^3wT_1,\ zwT_3 + (-yw - xz)\,T_2 - xyT_1,\ z^2T_4 + wT_2 + xT_1,$$
$$(x^2w^2 - yz^2)\,T_3 + (-x^3w + y^2z)\,T_2,\ x^2wT_4 + zT_3 - yT_2,\ yzT_4$$
$$+ wT_3 - xT_2,\ wT_5 - y^2T_4 + xT_3,\ zT_5 - x^3T_4 + yT_3).$$

It was verified that $(*)$ $(J, I){:}\,x = (J, I)$, that is, x is regular modulo (J, I). It follows therefore that the symbolic powers of I are its ordinary powers. Moreover the sequence $\{x, w, T_3, T_1 - T_5 - z, T_4 - y - z\}$ turns out to be a regular system of parameters for B/J; this suffices to show that J is a Cohen–Macaulay ideal.

The verification $(*)$ was conducted in the following manner. First the ideal $(J, I){:}\,x$ was computed. It was not possible, however, to find a Gröbner basis of the ideal (J, I), so that a comparison of the two ideals could be effected. Instead we argued as follows: Each homogeneous generator h (in the T_i-variables) of (J, I): x, of degree, say, r, was mapped into R via $T_i \to$ (ith generator of I) and shown to belong to I^{r+1}—and this is clearly sufficient to show h belonged in (J, I). (It was only necessary to consider $r \leqslant 3$.) Although these powers of I may have a large number of generators, they contain far fewer indeterminates.

There are also many examples among the prime ideals of $k[x, y, z]$ defined by the equations $x = t^a + t^b$, $y = t^c$, and $z = t^d$. The simplest corresponds to $(a, b, c, d) = (2, 4, 3, 5)$. It would be interesting to find out when the ideals of [Moh] have this property.

(2.13) *Question*: Are the Rees algebras of prime ideals of polynomial rings, whose symbolic powers coincide with the ordinary powers, always Cohen–Macaulay?

The final topic of this section is the comparison of the algebras $R_a(I)$ and $R_s(I)$ of (1.7). Although, in principle, this is dealt with in Proposition 2.12(b). the ideal (I, J) may be unwieldy. We consider one case when the approach of (1.8) is preferable. Let I be a prime ideal of $R = k[x_1, ..., x_{n+1}]$, defining a projective curve of P^n. Denote by M the irrelevant maximal ideal of R. If I is generated by m elements, suppose we have available the presentation ideal J of the Rees algebra $R(I)$. In particular we have a presentation

$$\phi: R^s \to R^m \to I \to 0.$$

(2.14) PROPOSITION. *The integral closure of $R(I)$ coincides with the symbolic algebra of I if and only if*:

(a) *The analytic spread $l_M(I) \leqslant n$.*

(b) *The homogeneous ideal L generated by the $(m-n+1)$-sized minors of ϕ is M-primary. (In other words, I is a complete intersection in codimension one.)*

Proof. It is a rephrasing of the requirements of (*I*.7) (cf. [Hu_1]). For the necessity, asking that the analytic spread of I at each prime Q of height n be less than n is, according to [CN], equivalent to demanding that I_Q be a complete intersection. For the converse, the condition on L makes I_P a complete intersection for each maximal ideal P distinct from M. ∎

Both conditions are readily tested if m is small. It leaves unanswered the question of when $R(I)$ is actually normal.

As an example we consider the smooth curve of P^3 given by the equations

$$x = u^d$$
$$y = u^{d-1}v$$
$$z = uv^{d-1}$$
$$w = v^d.$$

The defining ideal I of $R=k[x, y, z, w]$ is minimally generated by $\{yw^{d-2}-z^{d-1},\ y^2w^{d-3}-xz^{d-2}, \ldots, y^{d-1}-x^{d-2}z,\ yz-xw\}$. The presentation ideal J contains the polynomials

$$T_iT_{j+1}-T_jT_{i+1}-a_{ij}T_d^2, \qquad 1\leqslant i<j<d-1,$$

where $a_{ij}\in M=(x, y, z, w)$.

It follows that $J(0)=J$ (evaluated at M) contains the ideal N generated by the 2×2 minors of the matrix

$$\begin{bmatrix} T_1 & T_2 & \cdots & T_{d-2} \\ T_2 & T_3 & \cdots & T_{d-1} \end{bmatrix}.$$

Because this determinantal ideal is prime of height $d-3$ and $l_M(I)\geqslant 3$ by [CN], $J(0)=N$ and $l_M(I)=3$. Since I is a complete intersection outside of M, it follows from (2.14) that the integral closure of $R(I)$ is the symbolic power algebra. For $d\leqslant 5$, we verified the normality of $R(I)$; possibly the algebras coincide for all values of d.

3. Presentation of Rees Algebras

We shall consider in this section the question of deciding the normality of a Rees algebra from the point of view of computer algebra. We assume from now on that R is the polynomial ring $k[x_1, \ldots, x_n]$.

(3.1) *Gröbner Bases.* A notion that allows for explicit computations in several situations in commutative algebra is that of the Gröbner basis of an ideal. We briefly recall some of its pertinent properties and refer to the growing literature on this topic (e.g., [Ba], [Bu_1], [Bu_2], [MM], [Ro]).

One aim is to provide a simulacrum of a division algorithm for the ring of polynomials in several variables. This is accomplished, for instance, in the following manner. Set $R=k[x_1, \ldots, x_n]$ and identify the set of monomials of R with the additive semigroup $\mathbf{N}^n$. Pick a total order for $\mathbf{N}^n$ which is compatible with its semigroup structure. Finally, define the "degree" of a monomial as the corresponding vector of exponents in $\mathbf{N}^n$, and the degree $d(f)$ of a nonzero polynomial f of R as the *supremum* of the degrees of the monomials that occur with nonzero coefficients in f.

For an ideal I, define $d(I)$ as the union of the degrees of the nonzero elements of I. $d(I)$ is a sub-semigroup of $\mathbf{N}^n$, satisfying $d(I)+\mathbf{N}^n\subset d(I)$. By the Hilbert basis theorem it follows that

$$d(I)=\bigcup\,(d(f_i)+\mathbf{N}^n), \qquad 1\leqslant i\leqslant r.$$

One can then verify that the images of the monomials g, $d(g) \notin d(I)$, form a vector space basis for R/I over k. The detailed study of this basis goes back to Macaulay.

(3.2) DEFINITION. $(f_1, ..., f_r)$ is a *Gröbner basis* of I.

We pass over the discussion of the various effectivity issues, but point out some of the properties of this construction that are relevant to our questions.

(i) A Gröbner basis of I is a generating set for the ideal and may be readily used to test membership in I. Indeed, the condition on the (f_i) is equivalent to: Every element f of I can be written as $f = \sum a_i f_i$, with $d(f) \geq d(a_i f_i)$.

Different orderings of $\mathbf{N}^n$ give rise to Gröbner bases with distinctive properties. For instance, the so-called graded lexicographic ordering

$$(a_1, ..., a_n) < (b_1, ..., b_n) \Leftrightarrow$$
$$\text{the first nonzero entry of } (\textstyle\sum b_i - \sum a_i, a_1 - b_1, ..., a_n - b_n) \text{ is positive}$$

embodies several additional computational efficiencies (cf. [MM]).

Because of the computer algebra system used, we focus here on Gröbner bases derived from the strict lexicographic ordering of the variables. These allow the following constructions:

(ii) Given an ideal I of $k[x_1, ..., x_n]$, find the contraction of I to the subring of R defined by a subset of the indeterminates. It suffices to find a Gröbner basis of I corresponding to an ordering of the variables that lists the variables of the subring first. The desired intersection is generated by the basis elements contained in the subring.

In turn this property may be used to compute (T is an indeterminate over R):

(a) The intersection of two ideals A and B: $A \cap B = (A \cdot T, B(T-1)) \cap R$.

(b) Decide whether an element f of R is regular modulo an ideal A: That is, compute $A: f$ $(= ((A \cdot T, (1-T) f) \cap R)/f)$ and compare it to A. Similarly, one can compute $\bigcup_{n \geq 1} A: f^n$ as $(A \cdot T, (1 - Tf)) \cap R = (A, (1 - Tf)) \cap R$. In fact, because of the last equality, we have found that determining the ideal $\bigcup_{n \geq 1} A: f^n$ is a convenient procedure to compute depths of cyclic modules.

(c) Carry out Noether normalization and compute the height of an ideal and systems of parameters of affine algebras. (This is discussed more explicitly later in this section.)

Presently there exist several implementations of an algorithm of Buchberger [Bu_1] that constructs Gröbner bases. The particular implementation we used was written by G. Zacharias, and runs in Macsyma.

One approach to access J is based on the following elementary fact.

(3.3) PROPOSITION. *The ideal of relations J of the presentation of $R(I)$ may be obtained in the following manner. In the ring*

$$A = Q[x_1, ..., x_n, T_1, ..., T_m, T]$$

consider the ideal L generated by the polynomials $T_j - Tf_j$, $j = 1, ..., m$. J is then the intersection of L and the subring $B = Q[x_1, ..., x_n, T_1, ..., T_m]$.

Proof. It is clear that $J \supset L \cap B$. Conversely, if $f(T_1, ..., T_m)$ is an element of J, we write

$$f(T_1, ..., T_m) = f(Tf_1 + (T_1 - Tf_1), ..., Tf_m + (T_m - Tf_m))$$

and use the Taylor expansion to show $f \in L$. ▮

The expected complexity of computing J is rather high. The underlying reason is that the theoretical complexity of computing a Gröbner basis has a doubly exponential cost as a function of the number of variables, while obtaining J as above makes it, likely, depend so heavily on the number of generators of I as well. Note that the payoff is extremely high as we are obtaining the first-order syzygies of all the powers of I.

(3.4) *Remark.* Given a submodule E of a free module $R^n = \mathrm{Re}_1 \oplus \cdots \oplus \mathrm{Re}_n$, the presentation of the symmetric algebra of E modulo its torsion can be computed in the same manner: For each generator $f_j = \sum a_{ij} e_i$ of E, define the corresponding element of L as $T_j - \sum a_{ij} U_i$, a variable U_i for each basis element e_i. Although, in principle, a free presentation of E could also be obtained by this procedure, it is obviously quite wasteful. Instead, the much leaner one of [MM] is recommended.

(3.5) EXAMPLE. Let I be the ideal generated by the 2×2 minors of the generic symmetric matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

Finding J by the method outlined required the Gröbner basis of an ideal involving 13 variables. We found that $J = (J_1)$, in agreement with [HVV], where one had to appeal to various aspects of the rich structure of I. (Incidentally, the Rees algebra of I is normal.) On the other hand, we ran into system difficulties while attempting to prove the similar equality for the ideal of 2×2 minors of a generic matrix of the same size (which is known to be of linear type, cf. [Hu_2], and shown to satisfy (2.1)(b) by R. Villarreal, although the normality of $R(I)$ has not been established).

We shall discuss how the question of the unmixedness of (I, J) can be approached. It is much less direct than the testing of the Jacobian condition.

Given a nonzero ideal $P=(h_1, ..., h_s)$ of the polynomial ring $B=Q[x_1, ..., x_n]$, a Noether normalization of the affine algebra B/P can be obtained as follows. Effect a linear change of variables—done on-line, usually—so that one of the elements of P, f_1, is monic in one of the variables, say x_1. Consider $P_1=P\cap B_1$, $B_1=Q[x_1, ..., x_n]$. Note that B/P is integral over B_1/P_1, so that in particular $\mathrm{ht}(P)=\mathrm{ht}(P_1)+1$. In this manner, heights, analytic spreads, and systems of parameters can be computed. Note that when all the changes of variables are taken into account we have a sequence of elements $f_i(y_i, ..., y_n)\in B=Q[y_1, ..., y_n]$, $1\leqslant i\leqslant \mathrm{height}(I)$, monic in y_i, lying in the ideal I.

This procedure is to be applied to the ideal $P=(I, J)$ of $B=Q[x_1, ..., x_n, T_1, ..., T_m]$. First, note that all of the minimal primes of (I, J) have the same height. This follows because the associated graded ring $G=B/(I, J)$ can be expressed as the quotient of the affine domain $R[IT, T^{-1}]$, modulo the ideal (T^{-1}), so that the equal chain condition applies (cf. [Mat, Sect. 14]). The normalization process is to have m of the steps above, as $\dim(R)=\dim(G)$. We will be left with a subring $C=Q[y_1, ..., y_n]$. Since the minimal primes of G must, by the preceding, contract to the zero ideal of C, the following formulation is immediate.

(3.6) PROPOSITION. *(I, J) is unmixed if and only if G is a torsion-free C-module.*

The question could, in principle, be resolved if the primary decomposition of (I, J) was available: The height of each component would be checked for unmixedness. Short of this we suggest two partial approaches.

(3.7) *Free Resolutions.* The steps in the normalization process may often be carried out in a manner so as to keep track of the generators and relations of G as a C-module. Indeed, if we denote by I_0 the ideal generated by the monic polynomials f_i in the normalization proces, we have the exact sequence

$$0\to (I, J)/I_0\to B/I_0\to B/(I, J)\to 0$$

in which B/I_0 is a free C-module of rank s equal to the product of $\deg_{x_i}(f_i)$, $i\leqslant \mathrm{height}(I)$.

One then obtains an explicit presentation of G as a C-module

$$\phi: C^r \to C^s$$

with r bounded by s times the number of generators for (I, J). Keeping r and s small is facilitated by the large number of linear polynomials in the T_i-variables and drastically helped if the analytic spread of the ideal I is small (actually the *supremum* of the analytic spreads of I at the maximal ideals).

The determinantal ideals associated to ϕ provide a (incomplete) measure of information concerning the torsion-freeness of G. Comprehensive details would require a fuller free presentation of the module G—which is computable through the use of a Gröbner basis algorithm (cf. [MM]). To make use of this, let us recall some facts. Given a matrix ϕ with entries in a ring R, denote by $I_t(\phi)$ the ideal of R generated by the t-sized minors of ϕ.

All that one would is an application of a theorem of [BE]: Start with a free presentation of G

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0.$$

Denote by $I_j(G)$ the ideal $I_{r_j}(\phi_j)$, $r_j =$ rank of $\phi_j: F_j \to F_{j-1}$. Then G is torsion-free if and only if height$(I_j(G)) \geqslant j+1$, $1 \leqslant n$.

In one particular case the task above can be eased. Assume that I is an $(x_1, ..., x_n)$-primary ideal. We can replace the condition $\text{ht}(I_n(G)) \geqslant n+1$, that is, $I_n(G) = R$, with the testing of the equality $(I, J)\colon (y_1, ..., y_n) = (I, J)$. Indeed, the height condition tests whether maximal ideals are associated to (I, J)—reduced here to $(x_1, ..., x_n, T_1, ..., T_m)$. Since $(I, J, (y_1, ..., y_n))$ is primary with respect to that maximal ideal, we can cast the condition in the residual form above.

Because of the difficulty in implementing fully the scheme above—that is, checking (2.1)(b) through a free resolution of G—it is desirable to have more direct means for testing that condition. We point out two simple situations derived from the theory of the approximations complexes [HSV_1].

Let I be an ideal satisfying the following two conditions: (i) I is syzygetic, that is, the symmetric square of I, $S_2(I)$, and I^2 coincide; (ii) for each prime ideal $\mathbf{p} \supset I$, $v(I_p) \leqslant \text{height}(\mathbf{p})$.

In terms of the presentation ideal J, the meaning of these conditions is:

(i) is equivalent to $J_2 = B_1 \cdot J_1$. As for the other condition, one gets from J_1 a presentation of I:

$$\phi: R^r \to R^m \to I \to 0.$$

(ii) is then equivalent to (cf. [HSV_2])

$$\text{height}(I_t(\phi)) \geqslant m - t + 1, \qquad 1 \leqslant t \leqslant m - 1.$$

(3.8) PROPOSITION. *Let I be an ideal as above, of height g, satisfying one of the two following conditions*:

(a) *R/I satis fies S_2, $v(I) \leqslant g + 2$, and I^2 has no associated prime of height $> g + 2$.*

(b) *I is Cohen–Macaulay, $v(I) \leqslant g + 3$, and I^2 has no associated prime of height $> g + 3$.*

Then $R(I)$ satisfies S_2.

Remarks. These conditions are rarely independent. Thus if in (a) I is Cohen–Macaulay, then both (i) and the condition on I^2 follow from the others. On the other hand, (b) underlies the verification of Example 3.5. It should be pointed out that for an ideal I that is generically a complete intersection, the determination of whether a power I^t has embedded primes of height greater than an integer k is harder than asking whether it has embedded primes of height less than k.

Proof. We give a proof of (b); the other part is similar. As in Proposition (2.6), we consider the approximation complex of I (H_i denotes the Koszul homology modules of I, and $S = B/IB$):

$$0 \to H_3 \otimes S[-3] \to H_2 \otimes S[-2] \to H_1 \otimes S[-1] \to S \to G \to 0.$$

The hypotheses imply that: H_3 is Cohen–Macaulay, H_2 is an S_2-module, and H_1 is a torsion-free R/I-module (cf. [HSV_2]). Since I satisfies (ii), it will follow from [HSV_1] that the complex above is exact.

Let P be an associated prime of G; we must show that P is a minimal prime. Denote by $\mathbf{p}$ the inverse image of P in R; localizing at $\mathbf{p}$ we may assume that $\mathbf{p}$ is the unique maximal ideal of R. If $v(I) < g + 3$, G is Cohen–Macaulay by [HSV_1]. On the other hand, if $v(I) = g + 3$, I satisfies sliding depth (cf. [HSV_2]) and again G will be Cohen–Macaulay. So we may assume that height$(\mathbf{p}) > g + 3$. In this case $\mathbf{p}$ is not associated to I^2 and it will follow that depth$(H_1) \geqslant 2$; appealing to [HSV_2, Example 4.7] we get depth$(H_2) \geqslant 3$. Since depth$(H_3) =$ dimension$(R/I) \geqslant 4$, we get that the ideal $\mathbf{p}G$ has depth at least 1, which is a contradiction as $\mathbf{p}G \subset P$. ∎

(3.9) *Subrings.* One can seek to detect the C-torsion of G in one of its C-subalgebras, particularly those in the normalization process above. At the penultimate step, for instance, B_{m-1}/P_{m-1} is torsion-free over C if and only if P_{m-1} is principal. (This weeded out several cases of the class of examples considered in (2.8).) Obviously, the normalizing subrings B_i/P_i

will not necessarily inherit torsion if that is present in G—unless, possibly, the changes of variables are sufficiently generic. The sensitivity of the Gröbner basis algorithm to the number of indeterminates militates against this approach.

EXAMPLE. Let $I = (x^4 + y^4, x^3y^2, x^5 + x^4y + y^5, x^2y^3)$. The ideal J is generated by

$$(T_4^2 + T_2T_3 + (-y-x)\,T_1T_2,\ T_3^2 + (-2y-x)\,T_1T_3 + T_2^2 + (y^2+xy)\,T_1^2,$$
$$y^2T_3 - x^2T_2 - y^3T_1,\ yT_4 + xT_3 + (-xy-x^2)\,T_1,\ -xT_4 + yT_2).$$

It is easy to verify that (2.1)(b) is satisfied. Since the subspace of 2-forms J_2^* has dimension two, J is not Cohen–Macaulay (cf. (2.9)). A Noether normalization of $G = B/(I, J)$ is the subring $k[T_1, T_2]$. But the contraction of (I, J) to $k[y, T_1, T_2]$ is the ideal (y^5T_2, y^6), so that G is not unmixed and therefore I is not complete.

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