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ON THE GRADE OF SOME IDEALS

J.F. Andrade(*), A. Simis and W. Vasconcelos(**)

An "additivity" formula is obtained for the grade of an ideal in a residue ring R/I, where I is a perfect ideal. This result is then applied to compute the grade of ideals of linear (inhomogeneous) polynomials. Further results on the homological rigidity of the conormal module I/I^2 are pointed out. Finally, an elementary proof is given of a result of Buchsbaum concerning the grade of ideals of minors of a matrix.

Introduction

Throughout R is a noetherian ring. We stick to the term grade(I) (or $\text{grade}_R(I)$) to denote the length of a maximal R-sequence contained in the ideal I. Thus, grade I = I-depth(R) in the sense of [16]. Now, given a second ideal $J \supseteq I$, one may ask whether grade(I) and J-depth(R/I) add up to anything meaningful in terms of the ring R. While the answer may be complicated in general, we get a simple one imposing restrictions on I. Thus, for example, if I is a perfect ideal - meaning that grade(I)= = $\text{pd}_R(R/I)$ - then grade(I) and J-depth(R/I) add up exactly to grade(J). This (cf. Cor. 1.3) and other technicalities are developed in section 1.

Section 2 contains our first application, namely, to computing the grade of ideals generated by (inhomogeneous) linear polynomials. This we get by means of a reduction to

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the homogeneous case, subsequently drawing upon the estimates obtained in [20, Th. 2.1]. To be precise, we only compute the grade of the (inhomogeneous) ideal when we are given sufficient grade bounds on the ideals of minors of the matrix of the associated linear system of equations. The exact value of the grade, in general, seems to depend on certain "gaps" that are hard to control.

In section 3 we display some consequences of (or rather, some variations on) the material of the first section. The first one is the fact that a certain Ext module is reflexive under suitable conditions (cf. Cor. 3.2), this having been observed earlier by Kunz in the context of canonical modules [12, 7.4]. This fact could be used to provide a minor simplification in a paper of Lipman-Sathaye [14, Fact($\underset{\sim}{A}$)]. Further, using a recent result of Platte [17], we add some more evidence to the homological rigidity of I/I^2.

The appendix is an elementary proof of Proposition 4.1 in [2]. Roughly, that result asserts that the homology of the so called generalized Koszul complexes is "rigid" in a local ring. After rephrasing that proposition in terms of ideals of minors of a matrix, we prove a more general result on the grade of such ideals which easily yields the proposition (cf. Proposition).

The non-explained terminology and basic results are to be found in [13] and/or [16].

1. Residual grade

Let $I \subset R$ be an ideal and let $a \in R$ be a non-zero-divisor modulo I. We want to compare the grades of I and (I,a). It is well known that in case R is a local ring and $(I,a) \neq R$, then one has $\text{grade}(I,a) \leq 1 + \text{grade}(I)$ [13, Theorem 127]. Apart from this, the grade can jump quite arbitrarily (e.g., let $R = k[X_1,\ldots,X_n,Y]$, $I = (1-Y)\cdot(X_1,\ldots,X_n)$ and $a = Y$) and it may not increase at all (e.g., let $\dim R = 2$, $\text{depth}\, R = 1$ and I a height 1 prime non-associated to (0)).

Our first result is to the effect that, under certain conditions, the grade in fact increases and, sometimes, by the correct amount.

PROPOSITION 1.1. *Let* $I \subset R$ *be an ideal and* $a \in R$ *a non-zero-divisor on* R/I.

(i) *If* R *is Cohen-Macaulay, then* $\text{grade}(I,a) \geq 1 + \text{grade}(I)$.

(ii) *If* $\text{p.d. } R/I < \infty$, *then* $\text{grade}(I,a) \geq 1 + \text{grade}(I)$.

(iii) *If* I *is perfect and* $(I,a) \neq R$, *then* $\text{grade}(I,a) = 1 + \text{grade}(I)$.

(iv) *If* I *is a homogeneous ideal generated by elements of positive degree* (R *being graded and* $a = 1+a_1+a_2+\ldots$, *where* a_i *has degree* i $(i \geq 1)$, *then* $\text{grade}(I,a) \geq 1 + \text{grade}(I)$.

Proof. (i) One may clearly assume $(I,a) \neq R$. Thus, let Q be a prime ideal containing (I,a) and such that $\text{ht}(Q) = \text{ht}(I,a)$. Then, there exists a prime ideal P minimal over I such that $P \subseteq Q$. As $a \notin P$, $\text{ht}(Q) \geq 1 + \text{ht}(P) \geq 1 + \text{ht}(I)$. Since R is Cohen-Macaulay, $\text{grade}(I,a) \geq 1 + \text{grade}(I)$ follows as well.

(ii) (We assume (iii) which will be proved independently of (ii)). We set $J = (I,a)$. Say, $\text{grade}(I) = \ell$. Then $\text{grade}(J) \geq \ell$ and we want to show that $\text{Ext}_R^{\ell}(R/J,R) = (0)$. If we assume otherwise, there is a prime $P \subset R$ associated to $\text{Ext}_R^{\ell}(R/J,R)$. But $\text{grade}(J) = \ell$ says that $\text{Ext}_R^{\ell}(R/J,R) \simeq \text{Hom}_R(R/J, R/(\underline{x}))$, where $(\underline{x})$ is a maximal R-sequence in J. Therefore, $J \subseteq P$ and P is associated to the ideal $(\underline{x})$. This implies that $\text{grade}(P_P) = \text{grade}(P) = \ell$, hence $\text{grade}(I_P) = \text{grade}(I)$. But, since $\text{pd}_{R_P} R_P/I_P < \infty$, one gets $\text{pd}_{R_P} R_P/I_P \leq \text{grade}(P_P)$. It follows that I_P is perfect. Therefore, by (iii) one obtains

$$\text{grade}(J) = \text{grade}(J_P) \geq \text{grade}(I_P) + 1 = \ell + 1.$$

Contradiction.

(iii) The argument in this case rests on the fact that, for I a perfect ideal of grade ℓ, every associated prime of $\text{Ext}^{\ell}(R/I,R)$ is an associated prime of R/I.

Indeed, assume this fact. The exact sequence

$$0 \to R/I \xrightarrow{a} R/I \to R/(I,a) \to 0$$

induces the exact sequence

$$0 \to \mathrm{Ext}^{\ell}(R/(I,a),R) \to \mathrm{Ext}^{\ell}(R/I,R) \xrightarrow{a} \mathrm{Ext}^{\ell}(R/I,R).$$

If $\mathrm{Ext}^{\ell}(R/(I,a),R) \neq (0)$, let P be any of the associated primes of this module. Then P is associated to $\mathrm{Ext}^{\ell}(R/I,R)$, hence to R/I, by the assumption made. Localizing the above Ext sequence at P, we get that the map

$$\mathrm{Ext}^{\ell}(R/I,R)_P \xrightarrow{a\otimes R_P} \mathrm{Ext}^{\ell}(R/I,R)_P$$

is an isomorphism, hence $\mathrm{Ext}^{\ell}(R/(I,a),R)_P = (0)$; contradiction. Thus, we must have $\mathrm{Ext}^{\ell}(R/(I,a),R) = (0)$ i.e., $\mathrm{grade}(I,a) \geq \mathrm{grade}(I) + 1$. The reverse inequality follows from [18, Th.1.4].

Thus, we are reduced to showing the inclusion $\mathrm{Ass}(\mathrm{Ext}^{\ell}(R/I,R)) \subseteq \mathrm{Ass}(R/I)$ whenever I is perfect of grade ℓ. Since $\mathrm{Ext}^{\ell}(R/I,R)$ it too is then a perfect module of grade ℓ - a consequence of dualizing a projective resolution of length ℓ of R/I - and since, similarly, $\mathrm{Ext}^{\ell}(\mathrm{Ext}^{\ell}(R/I,R),R) \simeq R/I$, it suffices to prove the following

LEMMA 1.2. *Let R be a noetherian ring and M a perfect module of grade ℓ. Then* $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(\mathrm{Ext}^{\ell}(M,R))$.

Proof. Let $(\underline{x})$ be a maximal R-sequence inside $I = \mathrm{ann}\, M$. Then $\mathrm{Ext}^{\ell}(M,R) \simeq \mathrm{Hom}(M,R/(\underline{x}))$, so that $\mathrm{Ass}(\mathrm{Ext}^{\ell}(M,R)) = \mathrm{Spec}(R/I) \cap \mathrm{Ass}(R/(\underline{x}))$. Let $P \in \mathrm{Ass}(M)$. Then, as M is perfect, $\mathrm{grade}(P) = \mathrm{grade}(I)$. Therefore, $(\underline{x})$ is a maximal R-sequence inside P, so that $P \subseteq Q$ for some $Q \in \mathrm{Ass}(R/(\underline{x}))$. But, for such Q we have

$$\ell = \mathrm{grade}(Q) = \mathrm{grade}(Q_Q) \geq \mathrm{p.d.}_{R_Q} M_Q \geq \mathrm{p.d.}_{R_P} M_P = \mathrm{grade}(P_P).$$

But, since at any rate $\mathrm{p.d.}_{R_P}(R/(\underline{x}))_P = \ell$, we get $P_P\text{-depth}(R/(\underline{x}))_P = 0$. Therefore, $P_P \in \mathrm{Ass}(R/(\underline{x}))_P$, hence $P \in \mathrm{Ass}(R/(\underline{x}))$, as required.

(iv) We want to show $\mathrm{Ext}^{\ell}(R/(I,a),R) = (0)$. But since $\mathrm{Ext}^{\ell}(R/I,R)$ has a structure of graded R/I-module and

$a = 1+a_1+\dots$, with a_i of positive degree, it follows that no non-zero element of $\mathrm{Ext}^{\ell}(R/I,R)$ can be annihilated by a.

REMARK. (iv) is also a consequence of the so-called "avoiding lemma" in the graded case; cf. e.g., [1, Ch. III, §1, Prop. 8].

As a consequence, we obtain the following additivity result.

COROLLARY 1.3. *Let* I *be a perfect ideal of a noetherian ring* R. *For any ideal* $J \supseteq I$,

$$\mathrm{grade}(J) = \mathrm{grade}(I) + J\text{-depth}(R/I).$$

Proof. Let $\underline{a} = \{a_1,\dots,a_r\}$ be a maximal R/I-sequence in J. From Prop. 1.1 (iii), by induction, we get

$$\mathrm{grade}(I,\underline{a}) = \mathrm{grade}(I)+r.$$

Therefore, $\mathrm{grade}(J) \geq \mathrm{grade}(I)+r$.

On the other hand, since every element of J is a zero-divisor on $R/(I,\underline{a})$, J is contained in some associated prime of $(I,\underline{a})$. Since (I,a) is a perfect ideal, we get $\mathrm{grade}(J) \leq \mathrm{grade}(I,a)$. Thence, the equality.

2. The grade of an ideal generated by linear polynomials

Let $\ell_j = a_{1j} + \sum_{i=2}^{n} a_{ij} X_i$, $j = 1,\dots,m$, be (inhomogeneous) linear polynomials in the polynomial ring $R[X_2,\dots,X_n]$. Let $J = (\ell_j)$, the ideal generated by these polynomials. The $n\times m$ matrix of the coefficients of the ℓ's gives rise to a map $R^m \xrightarrow{\varphi} R^n$. It is very reasonable to expect to describe all the invariants of J by means of those of $\mathrm{coker}(\varphi)$. Thus, if one considers the associated homogeneous linear system, i.e., ${}^h\ell_j = a_{1j}X_1 + \sum_{i=2}^{n} a_{ij}X_i$ $(j=1,\dots,m)$ and sets $J' = ({}^h\ell_j) \subset R[X_1,X_2,\dots,X_n]$ (attention: J' is not the homogenization of J!), then it has been shown in [20] that the grade of J' is at most the rank of φ. Moreover, suitable conditions on the grades of the determinantal ideals of φ imply that $\mathrm{grade}(J') = \mathrm{rank}(\varphi)$ [ibid; Th. 2.1].

We will now see that suitable conditions on the determinantal ideals of φ imply that $\mathrm{grade}(J)$ is maximal possible (i.e., J is a complete intersection in $R[X_2,\ldots,X_n]$).

As a matter of notation, $I_t(\varphi)$ (or simply, I_t) will stand for the ideal of $t\times t$ minors of (a matrix representing) φ.

THEOREM 2.1. *Let* $J = (\ell_j)_{1\le j\le m} \subset R[X_2,\ldots,X_n]$ *be as above and assume* $m \ge n$. *If* $\mathrm{grade}(I_t) \ge m-t+1$ *for* $1 \le t \le n$, *then* $\mathrm{grade}(J) \ge m$.

Proof. The case where $I_n = R$ is obvious since $I_n \subset J$ (Cramer's rule). Thus, assume $I_n \neq R$.

Then, as $\mathrm{grade}(I_n) \ge m-n+1$ by assumption, I_n is a perfect ideal of grade $m-n+1$ by the Macaulay-Eagon-Northcott result. Thus, if we let $\bar\varphi = \varphi \bmod I_n$, where $\varphi\colon R^m \to R^n$ is given by the coefficients of the ℓ's, then by Prop. 1.1 (iii)

$$\begin{aligned}\mathrm{grade}\ I_t(\bar\varphi) &= \mathrm{grade}\ I_t - \mathrm{grade}\ I_n\\ &\ge m-t+1 - (m-n+1) = (n-1) - t+1.\end{aligned}$$

Note that, in particular, $\mathrm{rank}(\bar\varphi) = n-1$.

Now, let $\bar J = J/I_n[X_2,\ldots,X_n] = (\bar\ell_j)$. For each j, let ${}^h\bar\ell_j$ denote the homogenized of $\bar\ell_j$ in $R/I_n[X_1,\ldots,X_n]$ and set ${}^h\bar J = ({}^h\bar\ell_j)$. Thus, $R/I_n[X_1,\ldots,X_n]/{}^h\bar J \simeq \mathrm{Sym}(\mathrm{coker}(\bar\varphi))$, the symmetric algebra of the R/I_n-module $\mathrm{coker}(\bar\varphi)$. We can therefore apply [20, Th. 2.1] to get $\mathrm{grade}({}^h\bar J) = n-1$. On the other hand, as $\bar J = ({}^h\bar J, X_1-1)/(X_1-1)$, we get $\mathrm{grade}(\bar J) \ge n-1$ by Proposition 1.1 (iv). At the outset, again by perfectness of I_n, we get

$$\begin{aligned}\mathrm{grade}(J) &= \mathrm{grade}(\bar J) + \mathrm{grade}(I_n)\\ &\ge n-1 + m-n+1 = m.\end{aligned}$$

QUESTION. What is the exact value of $\mathrm{grade}(J)$ in general?

If one just requires $\mathrm{grade}(I_n) \ge m-n+1$, then $\mathrm{grade}(J)$ can go down (this is seen by means of easy examples). Note, however, that this condition alone implies that all the symmetric powers $S_r(\mathrm{coker}\ \varphi)$ are perfect modules [4, Theorem 2.3].

3. On a result of Platte and the homological rigidity of I/I^2

Let M be a finitely generated module over the noetherian ring R. The following conditions are easily seen to be equivalent, for each integer $q \geq 0$:

$(i)_q$ $\operatorname{depth}(M_P) \geq \inf\{q, \operatorname{grade}(P)\}$ for every prime $P \subset R$,

$(ii)_q$ every R-sequence of length q is an M-sequence.

If, for a given $q \geq 1$, M satisfies any of the above conditions we say that M is $(\tilde{S}_q)$ (the traditional notion has $\operatorname{ht}(P)$ instead of $\operatorname{grade}(P)$; cf. [19, Prop. 6] for a proof of the equivalence of $(i)_q$ and $(ii)_q$ for R Macaulay. Also note that the ring itself is $(\tilde{S}_q)$ for every $q \geq 0$, while in the (S_q) terminology it must be necessarily Macaulay).

From Prop. 1.1 (or rather from its proof) emerges

PROPOSITION 3.1. *Let* $I \subset R$ *be an ideal of grade* ℓ. *Suppose any one of the following conditions is satisfied:*

(i) R *is a Cohen-Macaulay local ring;*

(ii) R *is a local ring and* $\text{p.d. } R/I < \infty$;

(iii) I *is a perfect ideal.*

Then, the R/I-*module* $\operatorname{Ext}^{\ell}_R(R/I,R)$ *is* $(\tilde{S}_2)$.

COROLLARY 3.2. *Assumptions as in Prop. 3.1. If, moreover,* I *is a complete intersection at the primes* P *for which* $\operatorname{depth}(R/I)_P \leq 1$, *then* $\operatorname{Ext}^{\ell}(R/I,R)$ *is a reflexive* R/I-*module.*

REMARK. The above corollary can be used to provide a simplification in the proof of Theorem 2 of [15]. Namely, it gives a pretty much elementary proof (without using preduality) of Fact A [ibid].

Proof of Prop. 3.1. Let $\{a,b\}$ be an R/I-sequence. The exact sequence $0 \to R/I \xrightarrow{a} R/I \to R/(I,a) \to 0$ induces the exact sequence

$$0 \to \operatorname{Ext}^{\ell}(R/(I,a),R) \to \operatorname{Ext}^{\ell}(R/I,R) \xrightarrow{a} \operatorname{Ext}^{\ell}(R/I,R) \to$$
$$\to \operatorname{Ext}^{\ell+1}(R/I,a),R).$$

By Proposition 1.1, $\mathrm{Ext}^{\ell}(R/(I,a),R) = (0)$, hence a is a non-zero divisor on $\mathrm{Ext}^{\ell}(R/I,R)$. On the other hand,

$$\mathrm{Ext}^{\ell}(R/I,R)/a\cdot\mathrm{Ext}^{\ell}(R/I,R) \hookrightarrow \mathrm{Ext}^{\ell+1}(R/(I,a),R).$$

Thus, we will be done if we show that b is a non-zero divisor on $\mathrm{Ext}^{\ell+1}(R/(I,a),R)$. But this follows again by Proposition 1.1 as applied to the ideal (I,a) and the element b.

Our main application in this section is to the question of the so-called homological rigidity of the conormal module. This question was raised in [22] and [23] as a

CONJECTURE. Let R be a local ring and $I \subset R$ an ideal of finite projective dimension. If $\mathrm{p.d.}_{R/I}(I/I^2) < \infty$ then I is generated by an R-sequence.

The known general cases where this conjecture is a fact are: $\mathrm{p.d.}I/I^2 = 0$ ([21] or [7]) and $\mathrm{p.d.}I/I^2 = 1$ ([22]; essentially subsumed in [9] by a close reading of the proof of Proposition 1.4.9 there). It follows from this last case that the assumption $\mathrm{p.d.}_{R/I}(I/I^2) < \infty$ at least implies that I is a complete intersection at the primes P for which $\mathrm{depth}(R/I)_P \le 1$.

We now add further evidence to the above conjecture in the following result. This result has also been obtained by Herzog [11, Theorem 1.3 and Theorem 3.1].

THEOREM 3.3. *Let* R *be a regular local ring (assume* $1/2 \in R$*). Let* $I \subset R$ *be such that* $\mathrm{p.d.}_R(I) \le 2$ *and* $\mathrm{p.d.}_{R/I}(I/I^2) < \infty$. *Then* I *is generated by an R-sequence.*

Proof. From [22], R/I is Cohen-Macaulay. If we can prove that R/I is Gorenstein, the result will follow from [22, Corollary 1] or [10, Satz 2.8].

For this, we consider the canonical homomorphisms

$$\mathrm{Ext}^{\ell}(R/I,R) \to \mathrm{Ext}^{\ell}(R/I,R)^{**} \to (\Lambda^{\ell}(I/I^2))^{*}.$$

(The second one comes from duality Theory, cf. [8, supplement]). By the remark on the lower case of $\mathrm{p.d.}I/I^2 < \infty$ and by Corollary 3.2, we have that $\mathrm{Ext}^{\ell}(R/I,R)$ is a reflexive R/I-module, hence the first map above is an isomorphism. But $(\Lambda^{\ell}(I/I^2))^*$ is also reflexive (a dual!), hence the same remark applies to show that the second map is an isomorphism, being so at primes P where R/I has depth ≤ 1.

Thus, we have $\mathrm{Ext}^{\ell}(R/I,R) \simeq (\Lambda^{\ell}(I/I^2))^*$.

To conclude, we use the following charming result:

LEMMA 3.4 ([17, Hilfssatz]). *Let $0 \to N \to F \to M \to 0$ be an exact sequence of finitely generated R-modules (R noetherian), where F is free and M has a rank and is free in depth ≤ 1. Then $(\Lambda^r M)^{**} \simeq (\Lambda^s N)^*$, where r (resp. s) is the rank of M (resp. N).*

Applying this result successively to the syzygies in a finite free resolution of I/I^2, we arrive at an isomorphism $(\Lambda^{\ell}(I/I^2))^{**} \simeq R/I$, hence also $(\Lambda^{\ell}(I/I^2))^* \simeq R/I$, as required.

In the same vein of thought, we can settle the above conjecture for ideals of height two. This will come out as a consequence of the above techniques along with the affirmative solution to the so called "syzygy problem" [6] - hence, the need for the extra hypothesis that R contain a field.

THEOREM 3.5. *Let R be a regular local ring containing a field. If $I \subset R$ is an ideal of height at most two such that $\mathrm{p.d.}_{R/I}(I/I^2) < \infty$, then I is generated by an R-sequence.*

Proof. We may assume I has height 2, as otherwise the result is easy. As above, we see that $\mathrm{Ext}^2(R/I,R) \simeq R/I$ (no need for R/I to be Macaulay at this stage). On the other hand, applying $\mathrm{Hom}(\ ,R)$ to a minimal resolution

$$R^q \xrightarrow{\gamma} R^m \xrightarrow{\beta} R^n \xrightarrow{\alpha} I \to 0$$

of I, yields the exact sequences

$$0 \to R \simeq \mathrm{Hom}(I,R) \to (R^n)^* \to B \to 0$$

$$0 \to Z \to (R^m)^* \to (R^q)^*$$

$$0 \to B \to Z \to \mathrm{Ext}^1(I,R) \to 0,$$

with $B = \mathrm{coker}(\alpha^*)$, $Z = \ker(\gamma^*)$.

But, $\mathrm{Ext}^1(I,R) \simeq \mathrm{Ext}^2(R/I,R) \simeq R/I$ and, by the first sequence, B is a module of rank n-1, generated by n elements. Therefore, by the third sequence, Z has rank n-1 and is generated by n+1 elements. If $0 \to M \to R^{n+1} \to Z \to 0$ is a presentation of Z, we then have that M has rank 2 and (by the second exact sequence) is a third syzygy.

By [6], M must be free. Therefore, p.d. Z = 1 and, since p.d. B = 1, it follows that p.d. $R/I \le 2$. Thus, I is perfect. Together with $\mathrm{Ext}^2(R/I,R) \simeq R/I$, this says that I is Gorenstein, as required.

APPENDIX

The result we wish to give another proof of is Proposition 4.1 in [2]. Translated into the language of matrices it simply reads [3, Theorem 8.4]: "let R be a local ring and let A be a $n \times m$ $(m \ge n)$ matrix with entries in the maximal ideal of R. If grade $I_n(A) = m-n+1$ then, for every $n \times k$ $(n \le k \le m)$ submatrix A' of A, grade $I_n(A') = k-n+1$."

Here, as before, $I_n(-)$ denotes the ideal of the $n \times n$ minors of the matrix in question. It is well known that grade $I_n(-) \le m-n+1$ as soon as $I_n(-) \ne R$. Thus, one sees that the above result is an immediate consequence of the following more general.

PROPOSITION. *Let R be a noetherian ring and let A be a $n \times m$ $(m \ge n)$ matrix with entries in the Jacobson radical of R. If A' is a submatrix of A obtained by deleting a single column then* $\mathrm{grade}(I_n(A)) \le \mathrm{grade}(I_n(A')) + 1$.

Before embarking in the proof of the Proposition proper, we list some simple facts that will be used.

(1) For any ideal I in a noetherian ring R,

$$\mathrm{grade}(I) = \inf\{\mathrm{grade}(I_{\mathfrak{m}}) \mid \mathfrak{m} \supset I,\ \mathfrak{m} \text{ maximal}\}.$$

A proof consists in collecting Theorems 134 and 135 of [13].

(2) If $R \to S$ is a faithfully flat extension of local

rings, then $\operatorname{grade}(I) = \operatorname{grade}(IS)$ for any ideal $I \subset R$.

This is a simple exercise in Ext's.

(3) Let $R, \mathfrak{m}$ be a local ring and let $(a_1,\ldots,a_q) \subset \mathfrak{m}$ be an R-sequence. If X is an indeterminate over R then, for any $b_1,\ldots,b_q \in \mathfrak{m}$, $a_1+b_1X,\ldots,a_q+b_qX$ is an $R[X]_{\mathfrak{m}[X]}$-sequence.

Proof. Localize $R[X]$ at $(\mathfrak{m},X)$. Since $(a_1+b_1X,\ldots,a_q+b_qX,X) = (a_1,\ldots,a_q,X)$ as ideals, the grade is $q+1$. But, then $\{a_1+b_1X,\ldots,a_q+b_qX,X\}$ is necessarily an $R[X]_{(\mathfrak{m},X)}$-sequence, hence so is $a_1+b_1X,\ldots,a_q+b_qX$. Localize further at $\mathfrak{m}[X]$.

We can now prove the proposition.

By (1), we may assume R is local and $A = (a_{ij})$ has entries in the maximal ideal $\mathfrak{m}$.

We induct on n. For $n=1$, the proposition is just Theorem 127 of [13]. Assume $n \geq 2$ and the proposition true for $n-1$. There is no loss of generality in assuming that it is the last column that has been deleted.

We follow an idea in [5]. Namely, consider the matrix

$$\tilde{A} = \begin{bmatrix} a_{11}+X & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

with coefficients in $R[X]$ (X an indeterminate over R). Now, viewed as a matrix in $R[X]_{\mathfrak{m}[X]}$, $\tilde{A}$ can be brought by the so-called elementary transformations to the form

$$A' = \begin{bmatrix} 1 & 0 \; \cdots \; 0 \\ c_{21} & \\ \vdots & B \\ c_{n1} & \end{bmatrix}$$

Clearly, $I_{n-1}(B) = I_n(A') = I_n(\tilde{A})_{\mathfrak{m}[X]}$ (the latter viewed as ideal in $R[X]_{\mathfrak{m}[X]}$). Denoting $A_{(m)}$, $B_{(m)}$, etc. the corresponding submatrices with last column deleted, we have similarly $I_{n-1}(B_{(m)}) = I_n(A'_{(m)}) = I_n(\tilde{A}_{(m)})_{\mathfrak{m}[X]}$.

By induction, $\operatorname{grade} I_{n-1}(B) \leq 1+\operatorname{grade} I_{n-1}(B_{(m)})$, i.e., $\text{grade } I_n(\tilde{A})_{\mathfrak{m}[X]} \leq 1 + \text{grade } I_n(\tilde{A}_{(m)})_{\mathfrak{m}[X]}$.

On the other hand, since $R \to R[X]_{\mathfrak{m}[X]}$ is faithfully flat, we have by (2)

$$\text{grade}(I_n(A)) = \text{grade}(I_n(A)[X]_{\mathfrak{m}[X]})\,(\text{resp. grade}(I_n(A_{(m)})) = \\ = \text{grade}(I_n(A_{(m)})[X]_{\mathfrak{m}[X]})).$$

Therefore, it suffices to show that

$$\text{grade}(I_n(A)[X]_{\mathfrak{m}[X]}) = \text{grade}(I_n(\tilde{A})_{\mathfrak{m}[X]}),$$

and similarly when we delete last columns.

This last equality can be seen as follows. First, note that $I_n(A)[X] + (X) = I_n(\tilde{A}) + (X)$. Therefore, $P[X]$ is a minimal prime of $I_n(\tilde{A})$ for every minimal prime P of $I_n(A)$ [5, Lemma 4]. It follows that

$$\text{grade}(I_n(\tilde{A})_{\mathfrak{m}[X]}) \leq \\ \leq \inf\{\text{grade}(P[X]_{\mathfrak{m}[X]}) \mid P \text{ minimal over } I_n(A)\} = \\ = \text{grade } I_n(A)[X]_{\mathfrak{m}[X]}.$$

This gives inequality in one direction. The reverse inequality follows from (3) in a straightforward way.

REMARK. A proper formulation of the Proposition holds true for the ideal I_t of $t \times t$ minors, $1 \leq t \leq n$, the same proof going through almost verbatim. The interest in this case lies in that there is not yet a simple generalized Koszul complex for I_t (the one in [14] being rather intricate to follow).

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J.F. Andrade

Instituto de Matemática
Universidade Federal da Bahia
Rua Caetano Moura 99, Federação
40.000 Salvador, BA, Brasil

A. Simis

Instituto de Matemática Pura e Aplicada (IMPA)
Rua Luis de Camões 68, Centro
20.060 Rio de Janeiro, RJ, Brasil

W.V. Vasconcelos

Department of Mathematics
Rutgers University
New Brunswick, New Jersey 08903, USA

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