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Projective modules with prime endomorphism rings

By

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Introduction. In [3] Chatters proves the following theorem: If R is a Noetherian PP-ring then R is a ring direct sum of prime rings and Artinian rings. This theorem is a generalization of his previous theorem concerning hereditary, Noetherian rings ([2]).

The notions of hereditary and PP extend naturally to projective modules. For example, a hereditary (PP) module is a projective module with every (cyclic) submodule projective. The main objective of this paper is to extend his theorem to endomorphism rings of finitely generated PP-modules over Noetherian rings. The theorem as it now stands is primarily an existence theorem. We give a method for constructing the prime and Artinian rings that occur in the decomposition of the endomorphism ring of a finitely generated PP-module over a Noetherian ring, by means of an equivalence relation defined on its indecomposable summands.

In Sect. 1, a basic criterion is given in order that the endomorphism ring of a projective module be prime. In Sect. 2, some results about PP-modules are proved. As we shall see, PP-modules share many of the same properties as hereditary modules. For example, direct sums of PP-modules are PP. However, in contrast to hereditary modules, the endomorphism ring of a finitely generated PP-module is seldom PP. For example, see ([4], Theorem 2.3). However, the main justification for PP-modules comes in Sect. 3: The endomorphism ring of a finitely generated PP-module over a Noetherian ring decomposes into prime rings and Artinian rings.

In this paper all rings considered are associative with identity, and all modules are unital. All conditions will be assumed to hold on both sides unless otherwise stated. For example, a Noetherian ring is both left and right Noetherian. The following notation will be used: The symbol J will always be used to denote the Jacobson radical of R. For an R-module M, $M^{(A)}$ will be used to denote the direct sum of A copies of M.

For basic properties of hereditary modules we refer to [6], and for PP-rings we refer to [9].

1. Endomorphism rings of projective modules. Some necessary and sufficient conditions on a projective module are determined which imply that its endomorphism ring is prime. But first the following lemma and corollary whose proofs are immediate and will be left to the reader.

1.1 Lemma. *Suppose R is a ring direct sum of rings R_i $(i = 1, \ldots, k)$ and P a module over R. Then $P = P_1 \oplus \ldots \oplus P_k$ (some of the P_i may be 0) with each P_i a module over R_i. Moreover,* $\operatorname{End}_R(P) = S_1 \oplus \ldots \oplus S_k$ $(S_i = \operatorname{End}_R(P_i))$.

1.2 Corollary. *Suppose R is a ring direct sum of rings R_1 and R_2 with R_1 Artinian and P a finitely generated projective module. Then $P = P_1 \oplus P_2$ (P_i a module over R_i $(i = 1, 2)$), and $\operatorname{End}_R(P)$ is a ring direct sum of S_1 and S_2, with $S_i = \operatorname{End}_R(P_i)$, $(i = 1, 2)$. Moreover, S_1 is Artinian.*

1.3 Proposition. *Suppose R is prime, and P a projective R-module. Then* $\operatorname{End}_R(P)$ *is prime.*

Proof. Since for any prime ring R, eRe is also prime whenever e is an idempotent, it suffices to show that the endomorphism ring S of any free R-module F is prime as $\operatorname{End}_R(P) \cong eSe$.

Suppose that $aSb = 0$, $a, b \in S$ and $b \neq 0$. Therefore, there exists a canonical projection $\pi_\gamma \colon F \to R$, and a canonical injection $i_\alpha \colon R \to F$ along the γ and α'th coordinates respectively such that $\pi_\gamma b i_\alpha \neq 0$. Thus for π_τ, i_ϱ with τ and ϱ arbitrary, we obtain

$$0 = \pi_\tau a S b i_\alpha = \pi_\tau a i_\varrho \pi_\varrho S i_\gamma \pi_\gamma b i_\alpha .$$

Since $\pi_\varrho S i_\gamma \cong \operatorname{End}_R(R) \cong R$, and $\pi_\gamma b i_\alpha \neq 0$, we obtain $\pi_\tau a i_\varrho = 0$. Since π_τ and i_ϱ are arbitrary, $a = 0$.

The following corollary is well known and first proved by Robson ([8], Lemma 4.1).

1.4 Corollary. *Let P be a finitely generated projective module and R a hereditary, Noetherian, prime ring. Then* $\operatorname{End}_R(P)$ *is a hereditary, Noetherian, prime ring.*

Proof. This is a consequence of 1.3 and ([6], Theorem 2.5).

1.5. Proposition. *Let $P_\alpha (\alpha \in A)$ be a set of modules, and $P = \oplus \Sigma P_\alpha (\alpha \in A)$, $S = \operatorname{End}_R(P)$. Then S is prime if and only if for all $i, j, k, l \in A$ (i, j, k, l not necessarily distinct) and non-zero maps $\varphi_{ij} \colon P_i \to P_j$, $\varphi_{kl} \colon P_k \to P_l$, there exists a map $\varphi_{jk} \colon P_j \to P_k$ such that $\varphi_{kl}\varphi_{jk}\varphi_{ij} \neq 0$.*

Proof. Suppose S is prime and let φ_{ij} and φ_{kl} be as given in the hypothesis. Then φ_{ij} and φ_{kl} have natural extensions φ'_{ij}, $\varphi'_{kl} \in S$ which are defined to be zero on the complement of P_i, P_k respectively. Since S is prime, $\varphi'_{kl} S \varphi'_{ij} \neq 0$. Thus there exists $\varphi \in S$ such that $\varphi'_{kl}\varphi\varphi'_{ij} \neq 0$. Define φ_{jk} to be the restriction of φ to P_j followed by the projection onto P_k. Since the image of φ'_{ij} is contained in P_j and φ'_{kl} is non-zero only on P_k, $\varphi_{jk} \neq 0$.

Now suppose the condition holds, and let a, b non-zero homomorphisms in S. Then for certain $i, j, k, l \in A$, the restriction of a to P_i and b to P_k followed by the projection of a to P_j and b to P_l yield non-zero homomorphisms $a_{ij} \colon P_i \to P_j$, $b_{kl} \colon P_k \to P_l$. By hypothesis, there exists a homomorphism $\varphi_{jk} \colon P_j \to P_k$ such that $b_{kl}\varphi_{jk} a_{ij} \neq 0$. Thus, it is easily seen that φ_{jk} can be extended to a map $\varphi \in S$ such that $b\varphi a \neq 0$. Thus, $bSa \neq 0$ and so S is prime.

2. *PP*-modules. A ring is called *left (right) PP* if and only if every cyclic left (right) ideal is projective. If a ring is both left and right *PP*, it is called *PP*. The concept of a *PP*-ring leads naturally to the module theoretic generalization. Thus, we have the following definition:

Definition. Let P be a projective module over a ring R. Then P is said to be *PP* in case every cyclic submodule of P is projective.

Remark. Hereditary modules are examples of *PP*-modules as are left and right *PP*-rings. A number of results are true with only this hypothesis. See for example ([7], Theorem 4.3).

It is clear that any direct summand of a *PP*-module is also *PP*. The same is true for direct sums of *PP*-modules:

2.1 Lemma. *Let R be a ring and $P_\alpha (\alpha \in A)$ a set of PP-modules over R. Then $P = \oplus \Sigma P_\alpha (\alpha \in A)$ is PP.*

Proof. We will first show that a finite direct sum of *PP*-modules is *PP* using a finite induction argument. For this it will suffice to show that for P_1 and P_2 *PP*-modules, $P_1 \oplus P_2$ is *PP*.

So let $x = (x_1, x_2) \in P_1 \oplus P_2$, and consider the canonical projections $\pi_i : R x \to R x_i$ $(i = 1, 2)$. Since $R x_1 \subseteqq P_1, \pi_1$ splits and $R x = K_1 \oplus K_2$ with $K_1 \cong R x_1$ and $K_2 = \ker(\pi_1) = \{r x \mid r x_1 = 0\}$. Now K_2 is cyclic and so the image of K_2 under π_2 is a cyclic submodule of $R x_2$, which is necessarily projective. But the kernal of π_2 restricted to K_2 is the set of $r x$ with $r x_1 = r x_2 = 0$, which is zero. So K_2 is projective, and this shows that $R x$ is also projective.

The case when A is arbitrary is now obvious since for $x \in \oplus \Sigma P_\alpha$, $R x$ is contained in a direct sum of a finite number of P_α which is *PP*.

2.2 Proposition. *Let P_α $(\alpha \in A)$ be a set of indecomposable projective modules and $P = \oplus \Sigma P_\alpha (\alpha \in A)$, $S = \mathrm{End}_R(P)$. If the P_α are hereditary (or cyclic and PP), then S is prime if and only if $\mathrm{Hom}_R(P_\alpha, P_\beta) \neq 0$ for all $\alpha, \beta \in A$.*

Proof. Suppose $\mathrm{Hom}_R(P_\alpha, P_\beta) = 0$ for some α and β. Let π_α and π_β be the natural projections from P to P_α and P_β respectively. Therefore,

$$S \pi_\beta S \pi_\alpha = S(\pi_\beta S \pi_\alpha) = S \cdot 0 = 0$$

with $S \pi_\alpha \neq 0$, $S \pi_\beta \neq 0$.

Now suppose that $\mathrm{Hom}_R(P_\alpha, P_\beta) \neq 0$ for all $\alpha, \beta \in A$. We apply 1.5. Let $\varphi_{ij} : P_i \to P_j$ and $\varphi_{kl} : P_k \to P_l$ be non-zero homomorphisms with $i, j, k, l \in A$. Applying the hypothesis, there exists a non-zero map $\varphi_{jk} : P_j \to P_k$. Since the P_α are hereditary (or cyclic *PP*), φ_{ij}, φ_{jk}, φ_{kl} are all monic. Thus the composition $\varphi_{kl}\, \varphi_{jk}\, \varphi_{ij} \neq 0$.

Remark. The "only if" part of the proposition is true for any set of modules.

Since the composition of monomorphisms is always a monomorphism, the proof of 2.2 yields the following corollary:

2.3 Corollary. *Suppose that P is an indecomposable, hereditary (cyclic and PP) module. Then* $\mathrm{End}_R(P)$ *is an integral domain.*

We investigate under what conditions an indecomposable PP-module has a prime endomorphism ring.

Let M be a module and x_α $(\alpha \in A)$ a set of generators of M. We say that the set x_α $(\alpha \in A)$ is *minimal* in case no x_α is an R-linear combination of the others. Clearly every finitely generated module has a minimal set of generators.

For the subsequent results the following notation will be introduced: Let P be a PP-module with a minimal generating set x_α $(\alpha \in A)$, and $F = \oplus \Sigma R x_\alpha$ $(\alpha \in A)$. Then there exists a natural map $\varphi: F \to P$ defined as follows: Let $x \in F$ whose α'th coordinate is $r_\alpha x_\alpha$ $(r_\alpha \in R)$. Then let $\varphi(x) = \Sigma r_\alpha x_\alpha \in P$ where addition is addition of elements in P. Clearly φ is well defined and onto. So φ splits, and $F = \ker(\varphi) \oplus P'$, $P \cong P'$, $\varphi(P') = P$. Let $g: F \to P'$ be the natural projection and note that $\ker(g) = \ker(\varphi)$.

2.4 Lemma. *Let P be a PP-module and suppose that P has a minimal set of generators x_α $(\alpha \in A)$ with each $R x_\alpha$ indecomposable, F, P', φ, and g as defined previously. For each $\alpha \in A$ let π_α be the natural projection of F onto $R x_\alpha$. Then for each $\alpha \in A$, $g \pi_\alpha g \in \mathrm{End}_R(P')$ is non-zero.*

Proof. Suppose there exists $\beta \in A$ such that $g \pi_\beta g = 0$. Consider $x \in P'$ such that $\varphi(x) = x_\beta$, and let $c_\beta x_\beta = \pi_\beta(x)$. Then $x \in P'$ and g the identity on P' imply that $g \pi_\beta g(x) = g(c_\beta x_\beta) = 0$. So $c_\beta x_\beta \in \ker(g) = \ker(\varphi)$. Thus letting $\pi_\alpha(x) = c_\alpha x_\alpha$ $(\alpha \in A)$, we obtain

$$x_\beta = \varphi(x) = \sum_{\alpha \neq \beta} c_\alpha x_\alpha \in P,$$

a contradiction to the minimality of the x_α.

2.5 Theorem. *Let P be a PP-module and suppose P has a minimal set of generators x_α $(\alpha \in A)$ with each $R x_\alpha$ indecomposable. Set $F = \oplus \Sigma R x_\alpha$ $(\alpha \in A)$. Then the following statements are equivalent:*

(a) $\mathrm{End}_R(P)$ *is prime.*
(b) $S = \mathrm{End}_R(F)$ *is prime.*
(c) $\mathrm{Hom}_R(R x_\alpha, R x_\beta) \neq 0$ *for all* $\alpha, \beta \in A$.

Proof. That (b) is equivalent to (c) is just Proposition 2.2.

(b) implies (a): Suppose S is prime. Since P is projective, $\mathrm{End}_R(P) \cong gSg$ $g \in S$, g idempotent. By a well known argument $\mathrm{End}_R(P)$ is prime.

(a) implies (b): Suppose S is not prime. Then by 2.2, there exists $i, j \in A$ such that $\mathrm{Hom}_R(R x_i, R x_j) = 0$. Let π_i and π_j be the natural projections of F onto $R x_i$ and $R x_j$ respectively. Applying 2.4 $g \pi_i g$ and $g \pi_j g$ are non-zero. Thus, $\pi_j S \pi_i = 0$ implies that $g \pi_j g \cdot gSg \cdot g \pi_i g = 0$. So $gSg \cong \mathrm{End}_R(P)$ is not prime.

Since any indecomposable, hereditary module has a prime endomorphism ring by 2.3, we have the following corollary:

2.6 Corollary. *Let P be a finitely generated indecomposable hereditary module with x_i $(i = 1, \ldots, n)$ a minimal set of generators, and each $R x_i$ indecomposable. Then* $\mathrm{Hom}_R(R x_i, R x_j) \neq 0$ *for all* $1 \leqq i, j \leqq n$.

2.7 Remark. It is easy to see that for any Noetherian module a minimal generating set can always be found whose elements generate indecomposable modules.

2.8 Proposition. *Suppose R is a Noetherian ring and P an indecomposable PP-module over R. Then* $\mathrm{End}_R(P)$ *is prime.*

Proof. By factoring out the annihilator of P we may assume that P is faithful. Let g be a primitive idempotent of R. Then $g x \neq 0$ for some $x \in P$. Thus $R g \cong R g x$ so $R g$ is PP. Applying 2.1 R is a Noetherian PP-ring. By 1.1 R is indecomposable. Thus by Chatter's theorem ([3]) R is either prime or Artinian. If R is prime apply 1.3. If R is Artinian then P is isomorphic to $R f$ for some primitive idempotent f. Thus P is cyclic so 2.3 applies. This completes the proof.

2.9 Corollary. *Let P be a finitely generated indecomposable PP-module over a Noetherian ring R, with a minimal set of generators x_i $(i = 1, \ldots, n)$ and each $R x_i$ indecomposable. Then* $\mathrm{Hom}_R(R x_i, R x_j) \neq 0$ *for all* $1 \leqq i, j \leqq n$.

2.10 Lemma. *Suppose P_1 and P_2 are PP-modules with minimal generating sets x_α $(\alpha \in A)$ and y_β $(\beta \in B)$ respectively, and such that the $R x_\alpha$ and $R y_\beta$ are indecomposable. Then* $\mathrm{Hom}_R(P_1, P_2) \neq 0$ *if and only if there exist an $\alpha \in A$ and $\beta \in B$ such that* $\mathrm{Hom}_R(R x_\alpha, R y_\beta) \neq 0$.

Proof. Suppose that $\mathrm{Hom}_R(P_1, P_2) \neq 0$. Then there exists an $\alpha \in A$ and a non-zero map $\varphi_\alpha : R x_\alpha \to P_2$. Since there always exists an embedding of P_2 in $\oplus \Sigma R y_\beta$ $(\beta \in B)$, we can obtain a $\beta \in B$ and a non-zero map $\varphi_\beta : \mathrm{Im}(\varphi_\alpha) \to R y_\beta$. The composition of these two maps yields a non-zero homomorphism $\varphi_{\alpha\beta} : R x_\alpha \to R y_\beta$.

Now suppose there exists an $\alpha \in A$ and $\beta \in B$ and a non-zero map $\varphi_{\alpha\beta} : R x_\alpha \to R y_\beta$. Identifying P_1 with its image in $\oplus \Sigma R x_\alpha$ $(\alpha \in A)$ and letting g_1 be the natural projection of $\oplus \Sigma R x_\alpha$ onto P_1, and π_α the projection of $\oplus \Sigma R x_\alpha$ onto $R x_\alpha$, Lemma 2.4 implies that $\pi_\alpha g_1 \neq 0$. Since $\varphi_{\alpha\beta}$ is monic, $\varphi_{\alpha\beta} \pi_\alpha g_1 : P_1 \to R y_\beta \subseteqq P_2$ is non-zero.

2.11 Theorem. *Let R be Noetherian and P a direct sum of finitely generated indecomposable PP-modules P_α $(\alpha \in A)$. Then* $\mathrm{End}_R(P)$ *is prime if and only if* $\mathrm{Hom}_R(P_\alpha, P_\beta) \neq 0$ *for all* $\alpha, \beta \in A$.

Proof. Since R is Noetherian, we may apply 2.7 to obtain a minimal generating set for each P_α whose elements generate indecomposable submodules. The union of these sets is a minimal generating set for P whose elements generate indecomposable submodules.

The 'if' part of the theorem is an easy consequence of 2.5, 2.9, and 2.10, and we leave the details to the reader. For the 'only if' part apply the observation following 2.2.

3. The theorem of Chatters. We consider projective modules which are direct sums of indecomposable projectives. Projective modules having such a decomposition occur naturally as for example, all finitely generated projectives over Noetherian rings.

Suppose $P = \oplus \Sigma P_\alpha (\alpha \in A)$, P projective, and each P_α indecomposable. One can define a relation $\sim$ on the P_α as follows: $P_\alpha \sim P_\beta$ if and only if $\mathrm{Hom}_R(P_\alpha, P_\beta) \neq 0$. Given $\alpha \in A$, let

$$A_\alpha = \{\beta \in A \mid \mathrm{Hom}_R(P_\alpha, P_\beta) \neq 0\}$$

and set $T_\alpha = \mathrm{End}_R(\oplus \sum P_\beta, \beta \in A_\alpha)$. (It is clear that $P_\alpha \neq 0$ always implies that $\alpha \in A_\alpha$). However the relation is seldom symmetric or transitive. When P is hereditary or a direct sum of indecomposable cyclic PP-modules, the relation is transitive, since the composition of non-zero monomorphisms is non-zero.

When $\sim$ is an equivalence relation, there is a partition of A in equivalence classes. If the cardinality of A is also finite, an easy argument shows that $S = \mathrm{End}_R(P)$ is a ring direct sum of the T_α, α varying over each equivalence class.

The main theorem of this section shows that for a finitely generated PP-module over a Noetherian ring, the relation $\sim$ is an equivalence relation on the non-Artinian indecomposable direct summands.

We start with some lemmas. The following lemma was first proved by Colby and Rutter ([4], Lemma 2.5), under the hypothesis that R is a PP-ring. Their proof works under the slightly weaker hypothesis that Re is a PP-module.

3.1 Lemma. *Suppose e is an idempotent and Re a PP-module. Then eRe is left PP-ring.*

3.2 Lemma. *Suppose R is Noetherian and P a direct sum of finitely generated indecomposable PP-modules P_α $(\alpha \in A)$. Then the relation $\sim$ is transitive.*

Proof. The result follows easily from 2.10 and 2.9.

3.3 Lemma. *Let R be Noetherian and $Re = Re_1 \oplus Re_2$ a PP-module, e_1 and e_2 primitive orthogonal idempotents. Suppose that $e_2 Re_2$ is not Artinian. Then $e_1 Re_2 \neq 0$ if and only if $e_2 Re_1 \neq 0$.*

Proof. Suppose $e_1 Re_2 \neq 0$, and $e_2 Re_1 = 0$. Then eRe is isomorphic to the ring of matrices of the form,

$$\begin{bmatrix} e_1 Re_1 & e_1 Re_2 \\ 0 & e_2 Re_2 \end{bmatrix}$$

Applying 3.1, eRe is an indecomposable, Noetherian PP-ring which is not Artinian. So by Chatters theorem ([3]), eRe is prime, clearly a contradiction. Thus $e_2 Re_1 \neq 0$. The 'if' part of the lemma uses the same argument.

We will need the following concept: A projective module P is said to be *local* in case it possesses a unique maximal submodule. This definition is equivalent to $P \cong Re$, e a primitive idempotent with Je the unique maximal submodule and eRe/eJe a skew-field ([10]). For a discussion of local modules, see [10].

3.4 Lemma. *Suppose R is Noetherian and P a finitely generated indecomposable PP-module. Then $S = \mathrm{End}_R(P)$ Artinian implies that $P \cong Re$, e a primitive idempotent and Re a local module.*

Proof. Applying 2.8, S is prime. Thus S is a simple Artinian ring. Since P is indecomposable, S is a skew-field. Applying ([10], Theorem 4.2) of Ware, P is a local module. Thus, $P \cong Re$ with Je the unique maximal submodule.

3.5 Lemma. *Let R be Noetherian. Then R has only a finite number of isomorphism classes of local modules. Furthermore, if P is local and $P \not\cong Re$, e a primitive idempotent, then there exists a primitive idempotent f such that $P \cong Rf$ with e and f orthogonal.*

Proof. We may assume there exists a set of primitive orthogonal idempotents $e_i (i = 1, \ldots, n)$ such that $e_1 + \ldots + e_n = 1$. Suppose P is a local module. Then $P \cong Rg$, g a primitive idempotent with unique maximal submodule Jg. Therefore, since $g = e_1 g + \ldots + e_n g$, there exists some $i \leqq n$ such that $e_i \notin Jg$. So there exists a homomorphism $\varrho_g : Re_i \to Rg$ given by right multiplication by g where the image is not contained in Jg. As Jg is the unique maximal proper submodule of Rg, ϱ_g is epic. This implies that $Rg \cong Re_i$. Since at most only n of the Re_i are local, the first assertion is proved. The second assertion follows from the above proof observing that there always exists a complete set of primitive orthogonal idempotents containing one that is given.

3.6 Lemma. *Suppose R is Noetherian and Re an indecomposable PP-module, e a primitive idempotent. Then Re is non-Artinian if and only if eRe is non-Artinian.*

Proof. In one direction the result is obvious. So suppose that eRe is Artinian. We first show that Re has descending chain condition on indecomposable cyclic submodules. Suppose Re has a proper descending chain of indecomposable cyclic submodules:

$$Re = Rx_o \supset Rx_1 \supset \ldots \supset Rx_n \supset \ldots.$$

We claim that for some $i > 0$, $\mathrm{End}_R(Rx_i)$ is not Artinian. Suppose not. With each Rx_i indecomposable and $\mathrm{End}_R(Rx_i)$ Artinian it is immediate from 2.3 that $\mathrm{End}_R(Rx_i)$ is a skew-field and from 3.4 that Rx_i is local. Using 3.5 there are only a finite number of isomorphism classes of Rx_i. So there exists an i and j such that $j > i$ and $Rx_i \cong Rx_j$. Now let e_i be a primitive idempotent such that $Rx_i \cong Re_i$. Since $Rx_j \subseteqq Jx_i$, $e_i J e_i \neq 0$. This contradicts $e_i Re_i \cong \mathrm{End}_R(Rx_i)$ being a skew-field. Therefore we conclude that for some $i > 0$, $\mathrm{End}_R(Rx_i)$ is not Artinian. Now $Rx_i \cong Re_i$ implies that $Re_i \not\cong Re$ as eRe is assumed to be Artinian. So applying 3.5 we assume that e and e_i are orthogonal. Thus, $f = e + e_i$ is an idempotent and so fRf is a Noetherian PP-ring by 3.1. Observing that $Rx_i \subseteqq Re$, yields that $e_i Re \neq 0$. Applying 3.3, we obtain $eRe_i \neq 0$. Since $Rx_i \subseteqq Je$, this means that,

$$0 \neq eRx_i \subseteqq eJe$$

a contradiction to eRe a skew-field. Thus the above chain must terminate after a finite number of steps. Now Re must be Artinian. Suppose not. Since Re has descending chain condition on indecomposable cyclic submodules, Re has a submodule Rx minimal with respect to being cyclic indecomposable and non-Artinian. Therefore every proper cyclic submodule of Rx is Artinian which implies that Rx is Artinian a contradiction.

3.7 Lemma. *Let R be Noetherian and P a finitely generated indecomposable PP-module, with a minimal generating set* $Rx_i (i = 1, \ldots, n)$ *and each* Rx_i *indecomposable. Then* $\mathrm{End}_R(P)$ *non-Artinian implies that all the* $\mathrm{End}_R(Rx_i)$ *are non-Artinian* $(i = 1, \ldots, n)$.

Proof. Applying 3.6, $\mathrm{End}_R(Rx_i)$ is non-Artinian if and only if Rx_i is non-Artinian. So it suffices to show that all the Rx_i are non-Artinian. It is clear from the hypothesis that at least one of the Rx_i is non-Artinian say Rx_1. By 2.9, Rx_1 embeds in each Rx_i so that all the Rx_i must be non-Artinian.

3.8 Theorem. *Let R be a Noetherian ring and* $P = P_1 \oplus \ldots \oplus P_n$ *a finitely generated PP-module with each* P_i *indecomposable. Then* $S = \mathrm{End}_R(P)$ *is a ring direct sum of Artinian rings and Noetherian, prime rings whose decomposition is determined as follows:*

$$S = S_1 \oplus S_2,\ S_i = \mathrm{End}_R(Q_i), \quad (i = 1, 2)$$

where Q_1 *is the direct sum of all the Artinian indecomposable modules that occur in the decomposition of P, and* Q_2 *is the direct sum of the non-Artinian indecomposable summands. The ring* $S_2 = T_1 \oplus \ldots \oplus T_k$ $(k \leqq n)$ *where each* T_i *is the endomorphism ring of the direct sum of all the non-Artinian indecomposable projectives which belong to the i'th equivalence class determined by the relation:* $P_\alpha \sim P_\beta$ *if and only if* $\mathrm{Hom}_R(P_\alpha, P_\beta) \neq 0$. *Furthermore,* S_1 *is Artinian and the* T_i *are all non-Artinian, Noetherian, prime rings.*

Proof. Applying Corollary 1.2, it is clear that S_1 is Artinian. We first observe that if P_i is not Artinian then $\mathrm{End}_R(P_i)$ is not Artinian. Suppose $\mathrm{End}_R(P_i)$ is Artinian. Then by 3.4 $P_i \cong \mathrm{Re}$, Re a local module. An application of 3.6 yields a contradiction. This proves our claim.

We now show that $\sim$ is an equivalence relation. By 3.2 we need only show reflexivity. Let P_i be non-Artinian, and suppose there exists $P_j (j \neq i)$ such that $\mathrm{Hom}_R(P_i, P_j) \neq 0$. By 2.10, there exists an $Rx_i \subseteqq P_i$ and an $Rx_j \subseteqq P_j$, Rx_i and Rx_j indecomposable and such that $\mathrm{Hom}_R(Rx_i, Rx_j) \neq 0$. Applying 3.7, $\mathrm{End}_R(Rx_i)$ is non-Artinian. Since $Rx_i \cong Re_i$ and $Rx_j \cong Re_j$, e_i and e_j primitive idempotents, we have $e_i Re_j \neq 0$. Suppose $e_j e_i \neq 0$, then $\mathrm{Hom}_R(Rx_j, Rx_i) \neq 0$. So we may assume that $e_j e_i = 0$. Then applying elementary properties of idempotents, we may assume that e_j and e_i are orthogonal. So by 3.3, $0 \neq \mathrm{Hom}_R(Rx_j, Rx_i) \cong e_j Re_i$. Applying 2.10 again $\mathrm{Hom}_R(P_j, P_i) \neq 0$. This means that the relation $\sim$ is an equivalence relation, and so $S_2 = \oplus \Sigma T_i (i = 1, \ldots, k)$ as a ring. The T_i are non-Artinian Noetherian, prime rings is a consequence of Theorem 2.11 and $\mathrm{End}_R(P_i)$ non-Artinian.

Since the endomorphism ring of a finitely generated hereditary module is always hereditary ([6], Theorem 2.5), we have the following corollary:

3.9 Corollary. *We assume the hypothesis of Theorem* 3.8. *Also suppose the* $P_i (i = 1, \ldots, n)$ *are hereditary. Then* $\mathrm{End}_R(P)$ *is a direct sum of hereditary, Noetherian, prime rings and Artinian hereditary rings with the decomposition as determined in* 3.8.

3.10 Corollary. *Suppose R is Noetherian and $P \cong Re$, e an idempotent with Re a PP-module. Then eRe is a direct sum of Artinian PP-rings and Noetherian prime PP-rings with the decomposition as determined in* 3.8.

3.11 Corollary. *Let R be a Noetherian PP-ring and P a finitely generated projective module over R. Then $\mathrm{End}_R(P)$ is a direct sum of Artinian rings and Noetherian prime rings with the decomposition as determined in* 3.8.

Remarks. (1) Let R be a Noetherian PP-ring and $e_i (i = 1, \ldots, n)$ a set of primitive orthogonal idempotents with $e_1 + \ldots + e_n = 1$. Let Re be the sum of all the Re_i with Re_i Artinian. Then since $\mathrm{End}_R(P) \cong R$ canonically, Theorem 3.8 says that $R = eRe \oplus (1-e) R (1-e)$ as a ring with $eRe = Re R$, $(1-e) = R(1-e) R$, and $(1-e) R (1-e)$ non-Artinian. The Noetherian prime rings can becomputed as follows: For each $e_i \in (1-e) R (1-e)$ apply 3.8 to obtain the Noetherian prime ring T_i which is just the sum of all Re_j such that $e_i Re_j \neq 0$.

(2) Observe that the proof of 2.8 and 3.3 requires Chatters Theorem for PP-rings. We suspect that a module theoretic proof of Theorem 3.8 can be given without requiring the use of Chatters Theorem.

4. Examples and applications. We now give some examples and applications which will serve to place the major results of sections 2 and 3 into perspective. The first example shows that an Artinian PP-ring need not be hereditary.

1. Let F be a field and R the ring of matrices of the form

$$R = \begin{bmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{bmatrix}.$$

Then R is a hereditary left serial algebra (That is every indecomposable projective left ideal has a unique composition series). Using the well known duality $\mathrm{Hom}_F(\ , F)$, R has an indecomposable injective left module E with composition series of length 3 and $E/JE \cong F \oplus F \oplus F$, $JE \cong F$. In fact if we let

$$e_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

and write $Je_2 = R v$, $Je_3 = Rw$ for some $v, w \in R$, and let

$$D = \{(\alpha v, \alpha w) \mid \alpha \in R\}$$

then ${}_RE$ may be written as ${}_RE = (Re_2 \oplus Re_3)/D$.

Since E is in a natural way a right F-vector space, we may define S to be the ring of matrices of the form

$$S = \begin{bmatrix} R & {}_RE_F \\ 0 & F \end{bmatrix}$$

with addition and multiplication of elements being the obvious ones.

Then S is a PP-ring but is not hereditary. The left ideal

$$\begin{bmatrix} 0 & {}_RE_F \\ 0 & 0 \end{bmatrix}$$

is not projective over S since ${}_RE_F$ is an indecomposable proper quotient of $Re_2 \oplus Re_3$.

2. The next construction gives examples of PP-modules whose endomorphism rings are not PP.

Using the proof of the 'if' part of ([4], Theorem 2.3), their result can be sharpened as follows:

4.1 Proposition. *Let R be a left PP-ring and I a left ideal with n generators such that I is not projective. Then* $\mathrm{End}_R(R^{(n)})$ *is not left PP.*

Consider the non-projective left ideal on two generators of example 1 namely

$$\begin{bmatrix} 0 & {}_RE_F \\ 0 & 0 \end{bmatrix}.$$

By the above proposition $\mathrm{End}_S(S^{(2)})$ is not PP, although S is a Noetherian PP-ring.

3. Let R be a Noetherian, semi-perfect ring and P a finitely generated PP-module over R. A large class of rings whose finitely generated projective modules satisfy these conditions are the Semi-perfect hereditary Noetherian rings.

The T_i can be computed as follows: Since R is semi-perfect, $P = \oplus \Sigma Re_j^{(n_j)}$ ($j = 1, \ldots, k$) with each Re_j a local PP-module, e_j a primitive idempotent, and satisfying $Re_i \ncong Re_j$ for $i \neq j$. Using 3.8 $\mathrm{End}_R(P)$ has a decomposition into Artinian and Noetherian prime rings.

If we let $f_1, \ldots f_t$ be those e_j that belong to the i'th equivalence class determined by the relation $\sim$ defined in 3.8 and s_j be the multiplicity of f_j in the decomposition of P, then each T_i can be expressed as a matrix in block form:

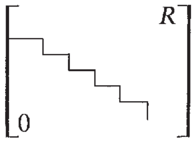
$$T_i = \begin{bmatrix} U_{11} \ldots U_{1t} \\ \vdots \quad\quad \vdots \\ U_{t1} \ldots U_{tt} \end{bmatrix}.$$

The qr'th block U_{qr} has entries taken from $f_q R f_r$ and dimensions $s_q \times s_r$ as shown below:

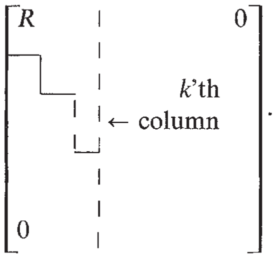
$$U_{qr} = \begin{bmatrix} f_q R f_r \ldots f_q R f_r \\ \vdots \quad\quad \vdots \\ f_q R f_r \ldots f_q R f_r \end{bmatrix}.$$

4. The following construction yields a class of Noetherian rings not PP but which have PP-modules. Let R be a Noetherian prime PP-ring, and S be the ring of $n \times n$ block

upper triangular matrices with entries from R above the blocks and zeros below as indicated in the figure.

$S =$ 

Then S has a PP-module which is the left ideal consisting of the upper left hand block and zeros elsewhere. This block is the maximal PP direct summand of S as the following argument shows: Denote by C_i the i'th column of S. Suppose for some $k \leqq n$, C_k is PP and C_k does not belong to the block in the upper left hand corner of S. Then for all $i \leqq k$, C_i is also PP since C_k contains an isomorphic copy of C_i. Thus S has a PP direct summand of the form

$T =$ 

.

Observe that $T = \operatorname{End}_S(T)$ (as rings). Now apply 3.8 to conclude that T must be prime. But this contradicts the existence of non-zero nilpotent ideals in T. Thus S is never PP unless it is the full matrix ring over R.

5. The following example yields indecomposable PP-modules which have minimal generating sets of more than one element.

Let R be a Dedekind domain that is not a principal ideal domain, and I a non-principal ideal of R. Since R is herditary and prime, I is an indecomposable hereditary module on two generators.

For example, in the above construction one could take $R = Z[\sqrt{-5}]$ and $I = 2R + (1 + \sqrt{-5})R$. Then I is an indecomposable hereditary module and has minimal generating set $\{2, 1 + \sqrt{-5}\}$.

References

[1] J. E. Bjork, Rings satisfying minimum condition on principal ideals. J. Reine Angew. Math. **236**, 466–488 (1969).

[2] A. W. Chatters, A decomposition theorem for Noetherian hereditary rings. Bull. London Math. Soc. **4**, 125–126 (1972).

[3] A. W. Chatters, Two results on P. P. rings. Comm. Algebra **4** (9), 881–891 (1976).

[4] R. R. Colby and E. A. Rutter Jr., Generalizations of *QF*-3 algebras. Trans. Amer. Math. Soc. **153**, 371–385 (1971).

[5] C. Faith, Algebra 11: Ring theory. Berlin-Heidelberg-New York 1976.

[6] D. A. Hill, Endomorphism rings of hereditary modules. Arch. Math. **28**, 45–50 (1977).

[7] L. S. Levy, Torsion-free and divisible modules over non-integral domains. Canad. J. Math. **15**, 132–151 (1963).

[8] J. C. ROBSON, Idealizers and hereditary Noetherian prime rings. J. Algebra **22**, 45–81 (1972).
[9] L. W. SMALL, Semi-hereditary rings. Bull. Amer. Math. Soc. **73**, 656–658 (1967).
[10] R. WARE, Endomorphism rings of projective modules. Trans. Amer. Math. Soc. **155**, 233–256 (1971).

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