EQUIVALENCE OF SELF-CONSISTENCY AND CONSERVATION OF ENERGY IN THE SCATTERING OF AN ELECTROMAGNETIC WAVE BY A PLANE OF DIPOLES

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The scattering of a harmonic electromagnetic plane wave by a plane of induced dipoles has been rigorously treated by means of the Hertz vector. We have shown that only a self-consistent solution, which takes into account cooperative effects among scatterers, is able to conserve energy.

1. The scattering problem of an electromagnetic plane wave by a plane of induced dipoles has previously been treated in the scalar approximation by Darwin [1], using the Fresnel construction, and in the vector approximation by Pearson [2], for right angle incidency. In both works the scattering was regarded as due to radiation from classical dipoles, activated by an external field exclusively.

In section 2 of this paper we discuss the more general situation in which the unique source of the field forcing the dipole oscillation is an electromagnetic plane wave incident at an arbitrary glancing angle θ. Thus, we are able to reconsider the earlier calculations, without making any approximations. In section 3 of the paper we extend these considerations to the self-consistent solution.

2. In an orthonormal coordinate system, where the wave is incident at a general angle θ and the scattering plane is the plane defined by $x=0$, we calculate the outgoing dipole fields for a point $\boldsymbol{P}=(x_P, y_P, z_P)$ outside that plane. We assume that the dipole plane is uniformly filled up by scattering electrons which behave as classical harmonic oscillators, vibrating without energy loss. The surface density of electron distribution is described by σ_0 and the electric part of the incident electromagnetic wave A^{I} is given by

$$\boldsymbol{E}_{\mathrm{A}}^{\mathrm{I}}(\boldsymbol{r},t)=\exp\left[\mathrm{i}\left(\phi_0+\omega t-\boldsymbol{k}_{\mathrm{A}}^{\mathrm{I}}\cdot\boldsymbol{r}\right)\right]\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}, \tag{1}$$

where $\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}=A_{\mathrm{E}}^{\mathrm{I}}(0,0,1)$ is the polarisation vector of the incident wave and $\boldsymbol{k}_{\mathrm{A}}^{\mathrm{I}}=k(\sin\theta, \cos\theta, 0)$ is the wave vector with magnitude $k=2\pi/\lambda$.

The combined field due to dipoles oscillating in the same direction and with the same frequency can be most conveniently handled by means of the Hertz vector. The infinitesimal dipole moment at the point $\boldsymbol{Q}=(0, y, z)$ is given by

$$\mathrm{d}\boldsymbol{M}(\boldsymbol{Q})=-\left(e^2\sigma_0/mc^2k^2\right)\times\exp\left[\mathrm{i}\left(\phi_0+\omega t-ky\cos\theta\right)\right]\,\mathrm{d}y\,\mathrm{d}z\,\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}, \tag{2}$$

and the resultant Hertz vector at the observation point $\boldsymbol{P}$, due to the whole plane of dipole oscillation, is defined by

$$\boldsymbol{Z}(\boldsymbol{P})=-r_{\mathrm{e}}k^{-2}\sigma_0\exp\left[i\left(\phi_0+\omega t\right)\right]\times\iint_{-\infty}^{+\infty}R^{-1}\exp\left[\mathrm{i}\left(R+\cos\theta\right)\right]\,\mathrm{d}y\,\mathrm{d}z\,\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}} \tag{3}$$

where r_{e} is the classical electron radius and $\boldsymbol{R}=\boldsymbol{P}-\boldsymbol{Q}$. Through two consecutive substitutions

$y-y_P=v$,

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$$z - z_P = w, \quad w = \left(x_P^2 + v^2\right)^{1/2} \sinh t,$$

the relevant double integral is rewritten as

$$2 \exp(-\mathrm{i}ky_P \cos \theta) \times \iint_{-\infty}^{+\infty} \exp\left\{\mathrm{i}k\left[v \cos \theta + \left(x_P^2 + v^2\right)^{1/2} \times \cosh t\right]\right\} \mathrm{d}t \, \mathrm{d}v. \quad (4)$$

Subsequently, using the solution for the integral in parentheses given by Watson [3], the above double integral is transformed into a single integral

$$-\mathrm{i}\pi \exp(-\mathrm{i}ky_P \cos \theta) \int_{-\infty}^{+\infty} H_0^{(2)} k\left(x_P^2 + v^2\right)^{1/2} \times \exp(-\mathrm{i}kv \cos \theta) \, \mathrm{d}v. \quad (5)$$

Finally, under the conditions $\pi/2 > \theta > 0$ and $x_P \neq 0$, according to McLachlan [4], the resultant Hertz vector is given by

$$\boldsymbol{Z}(\boldsymbol{P}) = \mathrm{i}k^{-2} r_e (\lambda/\sin \theta) \sigma_0 \times \exp\left\{\mathrm{i}\left[\phi_0 + \omega t - k\left(\sin \theta \, |x_P| + \cos \theta \, y_P\right)\right]\right\} \boldsymbol{A}_\mathrm{E}^\mathrm{I}. \quad (6)$$

It will be convenient to define a scattering factor f_p characterising the scattering power of the plane of dipoles:

$$f_\mathrm{p} = r_e (\lambda/\sin \theta) \sigma_0. \quad (7)$$

The respective resultant electric and magnetic vectors are readily derived from the known relations:

$$\boldsymbol{E}(\boldsymbol{P},t) = \nabla(\nabla \cdot \boldsymbol{Z}) - (1/c^2)(\partial^2 \boldsymbol{Z}/\partial t^2) = \mathrm{i}f_\mathrm{p} \exp\left\{\mathrm{i}\left[\phi_0 + \omega t - k\left(\sin \theta \, |x_P| + \cos \theta \, y_P\right)\right]\right\} \boldsymbol{A}_\mathrm{E}^\mathrm{I}, \quad (8)$$

$$\boldsymbol{B}(\boldsymbol{P},t) = (1/c)(\partial/\partial t) \nabla \times \boldsymbol{Z} = \mathrm{i}f_\mathrm{p} \exp\left\{\mathrm{i}\left[\phi_0 + \omega t - k\left(\sin \theta \, |x_P| + \cos \theta \, y_P\right)\right]\right\} \boldsymbol{A}_\mathrm{B}^\mathrm{I}, \quad (9)$$

where $\boldsymbol{A}_\mathrm{B}^\mathrm{I} = A_\mathrm{E}^\mathrm{I}(\cos \theta, \mathrm{sign}(x) \sin \theta, 0)$.

The solutions obtained above represent two electromagnetic plane waves due to the oscillating dipoles. The first dipole wave A^0, for $x > 0$, is travelling in the direction of the incident wave A^I since we have that $\boldsymbol{k}_\mathrm{A}^0 = \boldsymbol{k}_\mathrm{A}^\mathrm{I}$. When A^0 is added to A^I, a refracted or forward scattered wave A^S is generated according to

$$\boldsymbol{E}_\mathrm{A}^\mathrm{S}(\boldsymbol{r},t) = (1 + \mathrm{i}f_\mathrm{p}) \boldsymbol{E}_\mathrm{A}^\mathrm{I}(\boldsymbol{r},t), \quad \boldsymbol{B}_\mathrm{A}^\mathrm{S}(\boldsymbol{r},t) = (1 + \mathrm{i}f_\mathrm{p}) \boldsymbol{B}_\mathrm{A}^\mathrm{S}(\boldsymbol{r},t). \quad (10)$$

The second resultant dipole wave B^0, for $x < 0$, is leaving symmetrically the scattering plane at the same angle as follows from $\boldsymbol{k}_\mathrm{B}^0 = k(-\sin \theta, \cos \theta, 0)$, and is equivalent to the reflected or scattered wave B^S:

$$\boldsymbol{E}_\mathrm{B}^0(\boldsymbol{r},t) = \mathrm{i}f_\mathrm{p} \boldsymbol{E}_\mathrm{A}^\mathrm{I}(\boldsymbol{r},t), \quad \boldsymbol{B}_\mathrm{B}^\mathrm{S}(\boldsymbol{r},t) = \mathrm{i}f_\mathrm{p} \boldsymbol{B}_\mathrm{A}^\mathrm{I}(\boldsymbol{r},t). \quad (11)$$

Both dipole waves A^0 and B^0 are formed immediately and suffer a -phase shift $\delta = \pi/2$ in relation to the incident wave A^I. It is readily noted from fig. 1 that in this solution the energy is not conserved, since

$$|E_\mathrm{A}^\mathrm{I}|^2 < |E_\mathrm{A}^\mathrm{S}|^2 + |E_\mathrm{B}^\mathrm{S}|^2. \quad (12)$$

3. Due to this defect in the theory we look for the self-consistent solution, in which fields of neighbouring radiating dipoles are also included. It can be most easily done if we use the conclusion from the second section of the paper that the combined dipole fields take a plane waveform immediately. Extending this result over the scattering plane itself we assume now that the total forcing field includes a mean value of self-consistent dipole fields $\boldsymbol{E}_{\mathrm{A,self}}^0$ and $\boldsymbol{E}_{\mathrm{B,self}}^0$ which are observed in two points separately below and above the scattering plane, respectively, as travelling dipole waves. At the plane itself, however they are counted as a part of the total forcing, self-consistent field $\boldsymbol{F}_\mathrm{self}$. It means that in the same calculation as made in section 2, we should substitute $\boldsymbol{A}_\mathrm{E}^\mathrm{I}$ in eq. (2) by the following sum:

$$\boldsymbol{A}_\mathrm{E}^\mathrm{I} + \tfrac{1}{2}\left(\boldsymbol{A}_{\mathrm{E,self}}^0 + \boldsymbol{B}_{\mathrm{E,self}}^0\right), \quad (13)$$

where $\boldsymbol{A}_{\mathrm{E,self}}^0 = A_{\mathrm{E,self}}^0(0, 0, 1)$ and $\boldsymbol{B}_{\mathrm{E,self}}^0 = B_{\mathrm{E,self}}^0(0, 0, 1)$ are the looked for dipole polarisation vectors. On the other hand the Hertz-vector

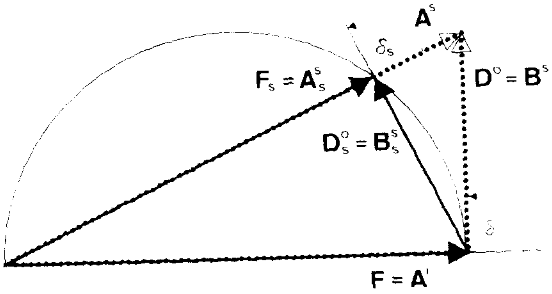

Fig. 1. Polarisation vectors of the relevant waves are represented in the phase space by the dotted and full lines, respectively.

approach provides a set of two equations derived for the mentioned observation points as

$$\mathrm{i}f_{\mathrm{p}}\left[\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}+\tfrac{1}{2}\left(\boldsymbol{A}_{\mathrm{E,self}}^{0}+\boldsymbol{B}_{\mathrm{E,self}}^{0}\right)\right]=\boldsymbol{A}_{\mathrm{E,self}}^{0},$$
$$\mathrm{i}f_{\mathrm{p}}\left[\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}+\tfrac{1}{2}\left(\boldsymbol{A}_{\mathrm{E,self}}^{0}+\boldsymbol{B}_{\mathrm{E,self}}^{0}\right)\right]=\boldsymbol{B}_{\mathrm{E,self}}^{0}. \quad (14)$$

The solution is readily obtained in the form

$$\boldsymbol{A}_{\mathrm{E,self}}^{0}=\boldsymbol{B}_{\mathrm{E,self}}^{0}=\left[\mathrm{i}f_{\mathrm{p}}/(1-\mathrm{i}f_{\mathrm{p}})\right]\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}(0,0,1)=\boldsymbol{D}_{\mathrm{self}}^{0}. \quad (15)$$

It may noted that the self-consistent dipole field $\boldsymbol{D}^0_{\mathrm{self}}$ is related to the self-consistent forcing field $\boldsymbol{F}_{\mathrm{self}}=\boldsymbol{A}^{\mathrm{I}}_{\mathrm{E}}+\boldsymbol{D}^0_{\mathrm{self}}$ in the same way as $\boldsymbol{D}^0$ is related to $\boldsymbol{F}=\boldsymbol{A}^{\mathrm{I}}_{\mathrm{E}}$, that is, with the same ratio of amplitude and the same phase shift, as follows from

$$\frac{\boldsymbol{D}_{\mathrm{self}}^{0}}{\boldsymbol{F}_{\mathrm{self}}}=\frac{\boldsymbol{D}_{\mathrm{self}}^{0}}{\boldsymbol{A}^{\mathrm{I}}+\boldsymbol{D}_{\mathrm{self}}^{0}}=\frac{\left[\mathrm{i}f_{\mathrm{p}}/(1-\mathrm{i}f_{\mathrm{p}})\right]\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}}{\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}+\left[\mathrm{i}f_{\mathrm{p}}/(1-\mathrm{i}f_{\mathrm{p}})\right]\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}}$$
$$=\frac{\mathrm{i}f_{\mathrm{p}}\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}}{\boldsymbol{A}_{\mathrm{E}}^{\mathrm{I}}}=\frac{\boldsymbol{D}}{\boldsymbol{F}}. \quad (16)$$

The above situation is illustrated in fig. 1. The right angle is now in the new position represented as δ_{S} in the figure, and refers to the self-consistent forcing field $\boldsymbol{F}_{\mathrm{self}}$. This assures that energy is conserved according to the formula

$$|E_{\mathrm{A}}^{I}|^2=|E_{\mathrm{A}}^{\mathrm{S}}|^2+|E_{\mathrm{B}}^{\mathrm{S}}|^2, \quad (17)$$

where $A^{\mathrm{S}}_{\mathrm{self}}=A^{I}+A^0_{\mathrm{self}}$ and $B^{\mathrm{S}}_{\mathrm{self}}=B^0_{\mathrm{self}}$.

Thus for the scattering problem discussed here, the self-consistency of the fields is equivalent to the conservation of energy.

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