Free resolutions of certain codimension three perfect radical ideals

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Introduction. The original conception of this work included a complete study of certain codimension three perfect ideals that arise naturally in a "determinantal" set-up, with the purpose of enriching the present collection of such ideals. The ones we consider are always radical and their syzygy-theoretic properties are relatively easy to describe. From the viewpoint of the theory of graded resolutions, they share common numerical features with (the projecting cones of) certain codimension two and three varieties in projective space. Their resolutions being seldom pure, they seemed largely convenient for testing the sharpness of some calculations made in the pure case (cf. [19], [25], [26]). Moreover, they yield a large class suited for testing recently developed concepts, such as the notion of strong obstruction, that have its natural place in the theory of linkage ([loc. cit.]) or the more general theory of residual intersection developed in [16]. The authors did not pursue this line of thought, but it seemed worthwhile pointing it out.

A brief description of the present contents is as follows. In the first section one obtains the explicit free resolution of a large class of codimension three ideals that appear as presentation ideals of normal cones associated to determinantal ideals fixing a submatrix. As it turns out, the resolutions of these presentation ideals are given by a mapping cylinder of a suitable map to the resolution of the original determinantal ideals.

The second section is concerned with deriving some ideal-theoretic results from the knowledge of the presentation given in section one. This procedure, by and large, follows the techniques developed by various authors ([17], [14], [23] and [24]).

In the third section one gives the resolution of the determinantal ideal J_r associated to a map $R^m \to R^n (m \geq n)$, fixing $R^r \subset R^m$, in the case of the data $r = n - 2$ and $m = n + 2$. This case not only extends the previous known ones [1], [2], but mainly gives some genuine clue for the general situation. The construction follows closely the spirit of the "scandinavian" complex [9] in that one is led to tensor suitable complexes and then to cut down the resulting size by an use of "trace" maps. This method seems to generalize well in various contexts [21].

Last one establishes exact values for the codimension of the complete intersection locus of the ideals considered in the previous sections. The reason for doing so is that the known estimates to present [11], [27] would give no real grasp, in this case, of the locus.

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1. Free resolutions of normal cones associated to determinantal loci fixing a submatrix.
Let (X) be an $n \times m$ $(n \leq m)$ matrix of indeterminates over a field k. Set $R := k\,[X]$ and let $J_r = J_r(X)$ stand for the ideal of R generated by the minors of (X) that fix the first r columns. A thorough study of J_r was the theme of [2], while a detailed account of the properties of the Rees algebra $R\,(J_r)$ and of the associated graded ring $\mathrm{gr}_{J_r}(R)$, in the cases $r = n - 1$ and $m = n + 1$, was provided in [6]. Thus, for instance, if $r = n - 1$, $R\,(J_r)$ is given by the determinantal ring of maximal minors of the matrix

$$X/T := \begin{bmatrix} X \\ 0 \ldots 0 \; T_n \, T_{n+1} \ldots T_m \end{bmatrix}.$$

From this one sees that the Eagon-Northcott complex yields a free resolution of $R\,(J_r)$ over the polynomial ring $R\,[T]$. The resolution of $\mathrm{gr}_{J_r}(R)$ over $R\,[T]$ is a little harder.

In this section one sticks to the case where $r = n - 1$ and $m = n + 2$ both because of its simplicity and the fact that the presentation ideal over $R\,[T]$ has codimension three. On the other hand, the original set-up will be slightly extended. Namely, let R be a noetherian ring, let $f: R^{n+2} \to R^n$ be a map and let there be given a splitting $R^{n+2} = R^{n-1} \oplus R^3$. Let f' stand for the restriction of f to R^{n-1} and let $J_{n-1} = J_{n-1}(f) \subset \Lambda^n R^n = R$ denote the image of the map $\Lambda^n f$ restricted to the module $\Lambda^{n-1} R^{n-1} \otimes \Lambda^1 R^3$.

According to [1], if $\mathrm{grade}\,(I_{n-1}(f')) \geq 2$ and $\mathrm{grade}\,(I_n(f)) \geq 3$ then R/J_{n-1} admits a finite free resolution:

$$\mathscr{F}: \; 0 \longrightarrow \Lambda^{n+2} R^{n+2} \otimes S_1(R^n) \xrightarrow{\varphi_3} \Lambda^{n+1} R^{n+2} \xrightarrow{\varphi_2} R^3 \xrightarrow{\varphi_1} R,$$

with the identifications $\Lambda^{n-1} R^{n-1} \otimes \Lambda^1 R^3 = R^3$ and $\varphi_1 = \Lambda^n f$ restricted to R^3.

One also has the acyclic (provided $\mathrm{grade}\,(I_{n-1}(f')) \geq 2$) complex

$$\mathscr{G}: \; 0 \longrightarrow R \xrightarrow{\delta} R^{n*} \xrightarrow{f'^*} R^{n-1*}$$

where $\delta = (\Lambda^{n-1} f'^*) = \Lambda^{n-1} f'$, $R^{n*} = \Lambda^{n-1} R^n$, $R = \Lambda^{n-1} R^{n-1}$.

The maps φ_3 and φ_2 are easy to describe and one refers to [1] for the details. The exact knowledge of φ_3 and φ_2 are inessential for the construction we are about to consider – it suffices to realize that the restriction of φ_2 to $\Lambda^{n-1} R^{n-1} \otimes \Lambda^2 R^3 \subset \Lambda^{n+1} R^{n+2}$ is given by the Koszul map $\Lambda^2 R^3 \to R^3$ on the three generators of J_{n-1}.

Next, let $T := T_n, T_{n+1}, T_{n+2}$ be a set of three indeterminates over R, so numbered for convenience. Set $B := R\,[T]$. We will define a (degree preserving) map of complexes $\eta: \mathscr{G} \otimes_R B \to \mathscr{F} \otimes_R B$. For this, one uses the Koszul complex on T, whose differentials will be denoted χ_i $(i = 1, 2, 3)$, and the map

$$f/T := \begin{bmatrix} f \\ 0 \ldots 0 \; \chi_1 \\ \underbrace{}_{n-1} \end{bmatrix}: B^{n-1} \oplus B^3 \to B^n \oplus B.$$

We will also use the following identifications in addition to the above:

$$(B^{n-1})^* = \Lambda^{n-2} B^{n-1}$$

$$= (\Lambda^{n-2} B^{n-1} \otimes \Lambda^3 B^3) \otimes \Lambda^{n+1} B^{n+1} \subset \Lambda^{n+1} B^{n+2} \otimes \Lambda^{n+1} B^{n+1}$$

$$(B^3)^* = \Lambda^2 B^3; \quad B = \Lambda^{n-1} B^{n-1}; \quad \Lambda^2 B^3 = \Lambda^{n-1} B^{n-1} \otimes \Lambda^2 B^3.$$

One is now ready for the definition of the map η, which is very simple. Namely, set

$$\eta_0 := \text{restriction of } \Lambda^{n+1}(f/T) \quad \text{to} \quad (B^{n-1})^*$$

$$\eta_1 := \varkappa_2 \circ \iota_2^* \circ (f^* \otimes B), \quad \text{where} \quad \iota_2 : B^3 \to B^{n+2} \quad \text{is the inclusion}$$

$$\eta_2 := \varkappa_3$$

Here is the main result of this section:

Proposition 1.1 (i) $\eta : \mathcal{G} \otimes_R B \to \mathcal{F} \otimes_R B$ is a map of complexes.

(ii) If grade $(I_{n-1}(f')) \geqq 2$ and grade $(I_n(f)) \geqq 3$, then the mapping cylinder $C(\eta)$ is a free B-resolution of $S(J_{n-1}/J_{n-1}^2)$, the symmetric algebra of the conormal module of J_{n-1}.

P r o o f. First we show that η is indeed a map of complexes, i.e., that

$$(1) \qquad \Lambda^n f \otimes B|_{B^3} \circ \varkappa_2 \circ \iota_2^* (f^* \otimes B) = (\Lambda^{n+1} f/T)|_{(B^{n-1})^*} \circ (f \otimes B)|_{B^{n-1}}^*$$

and

$$(2) \qquad \varphi_2 \otimes B|_{(\Lambda^{n-1} B^{n-1} \otimes \Lambda^2 B^3)} \circ \varkappa_3 = \varkappa_2 \circ \iota_2^* \circ (f^* \otimes B) \circ (\delta \otimes B).$$

Now, (1) is just a fancy way of writing the Cramer-Laplace relations for the $(n + 1) \times (n + 1)$ minors of the map f/T, along its rows. As to (2), it expresses the elementary fact to the effect that, in order to write down the $(n + 1) \times (n + 1)$ minors of f/T fixing the first $n - 1$ columns, one can first expand by Laplace along the $(n + 1)$-th row and then expand the corresponding cofactors ($=$ generators of J_{n-1}) along the column complementary to the first $n - 1$ columns. Incidentally, of course, (1) and (2) are also expressions of higher order Laplace relations or expansions.

Next, since $\mathcal{G} \otimes B$ and $\mathcal{F} \otimes B$ are cyclic under the given hypotheses, it follows that the mapping cylinder $C(\eta)$ is also acyclic. Therefore, $C(\eta)$ is a free B-resolution of coker $(B^3 \oplus (B^{n-1})^* \xrightarrow{(\varphi_1 \otimes B, \eta_0)} B)$. An easy inspection shows that this cokernel is isomorphic to $S(J_{n-1}/J_{n-1}^2)$. $\quad\square$

Corollary 1.2. Let $f : R^{n+2} \to R^n$ be a generic (say, over a field) map. Then $C(\eta)$ is a free B-resolution of $\mathrm{gr}_{J_{n-1}}(R)$.

P r o o f. By [2] (cf. also [6]), J_{n-1} is of linear type, i.e., its symmetric and Rees algebras coincide. $\quad\square$

R e m a r k. As a graded B-module – where B is graded in the usual way, R being its piece of degree zero – $S(J_{n-1}/J_{n-1}^2)$ admits a graded resolution

$$0 \to B^n \oplus B(-3) \to B^{n+2} \oplus B(-2)^n \to B^3 \oplus B(-1)^{n-1} \to B.$$

Perhaps more interesting is the generic case: one regrades B so as to have the entries of the map f of degree one. Then $S(J/J^2) = \mathrm{gr}_J(R)$ admits the graded homogeneous resolution

$$0 \to B(-(2.n+1))^{n+1} \to B(-2n)^{n+2} \oplus B(-(n+2))^n$$

$$\to B(-n)^{3^*} \oplus B(-(n+1))^{n-1} \to B.$$

The case $n = 2$ gives a presentation ideal $I \subset B$ which is an almost complete intersection. Moreover, I admits a minimal B-sequence which is part of a minimal set of generators. Therefore, I is linked to a Gorenstein ideal, necessarily generated by the Pfaffians of a skew-symmetric matrix over B [4]. This ideal was explicitly given in [22] in order to test a formula for the multiplicity of B/I. For $n \geq 4$, I has deviation $n + 2 - 3 = n - 1 \geq 3$. It seems natural to ask whether I is strongly obstructed – i.e., whether the twisted conormal module $w_{B/I} \otimes I$ is Cohen-Macaulay – or, even more, whether I is in the linkage class of a complete intersection.

One even lacks the information as to whether the first Koszul homology module on the generators of I is Cohen-Macaulay – a negative answer to this would discard any hopes to extending Vasconcelos conjecture (A) [26] beyond the realm of pure resolutions.

2. Analytic behaviour of the normal cone. The following generality will serve our purpose in this section: B is a normal quasi-unmixed domain and $I \subset B$ is an ideal of finite homological dimension and codimension at least two.

We recall that, quite generally, for a normal domain B and an ideal $I \subset B$ of codimension at least two, such that the Rees algebra $R(I)$ is normal, there is the so called exact sequence of divisor class groups [23], [18]:

$$0 \to \mathbb{Z}^r \to \mathrm{Cl}(R(I)) \to \mathrm{Cl}(B) \to 0,$$

where r is the number of height one primes of the exceptional divisor $IR(I)$.

As a notation, $v(E)$ will stand for the minimal number of generators of a finitely generated module over a local ring.

For the reader's convenience, we collect out of various sources the pertinent results.

Theorem 2.1. *Let B be a normal quasi-unmixed domain and let $I \subset B$ be a radical ideal of codimension at least two and finite homological dimension. If I is of linear type then the following conditions are equivalent:*

(i)   *The symmetric algebra $S(I)$ is a normal domain and $\ker(\mathrm{Cl}(S(I)) \to \mathrm{Cl}(B))$ is (a free group) of rank equal to the number of associated primes of B/I.*
(ii) $\mathrm{gr}_I(R)$ *is R/I-torsion free.*
(iii) $\mathrm{gr}_I(R)$ *is reduced.*
(iv) $v(I_p) \leqq \max\{\mathrm{ht}(I_p), \mathrm{ht}(P) - 1\}$, *for every prime $P \supset I$.*
(v)   *For a presentation $F_2 \xrightarrow{\psi} F_1 \longrightarrow I \longrightarrow 0$, one has $\mathrm{ht}(I_t(\psi)) \geq \mathrm{rk}(\psi) - t + 3$, for $1 \leqq t \leqq \mathrm{dev}(I) := \mathrm{rk}(F_1) - \mathrm{ht}(I)$.*

P r o o f. The equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are in [17] (cf. also [23] for further generality in (i) $\Leftrightarrow$ (ii)). We refer to [14] for the equivalences (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v). $\square$

Lemma 2.2. *Let $I \subset B := R[T]$, $R = k[X]$, stand for the presentation ideal of the ring $\mathrm{gr}_{J_{n-1}}(R)$ as in Section 1 (generic case). Then $v(I_p) \leqq \max\{\mathrm{ht}(I_p), \mathrm{ht}(P) - 1\}$ for every $P \supset I$.*

P r o o f. By the results of Section 1 as applied in this case, one has a minimal presentation

$$B^{2n+2} = B^{n+2} \oplus B^n \xrightarrow{\psi} B^{n+2} = B^3 \oplus B^{n-1} \longrightarrow I \longrightarrow 0,$$

$$\psi = \left[\begin{array}{c|c} \varphi_2 \otimes B & \eta_1 \\ \hline 0 & (f' \otimes B)^* \end{array} \right], \quad f' := f|_{R^{n-1}}.$$

First observe that $v(I) = n + 2$, so one may assume that $\mathrm{ht}(P) \le n + 2$. Induct on n. Let $n \ge 3$. Then $\mathrm{ht}(I_1(f' \otimes B)^*) = n(n-1) > n + 2$, hence, say, $X_{11} \notin P$. By inverting this element and performing elementary transformations on the matrix X/T, one obtains a matrix of the same form on in new indeterminates $Y_{ij} = X_{ij} - X_{1i} X_{i1} X_{11}^{-1}$ in a new ring $\tilde{B} \cong B[X_{11}^{-1}]$ and, moreover, $I[X_{11}^{-1}] \cong \tilde{I} \subset \tilde{B}$, where $\tilde{I}$ is the ideal of maximal minors of the new matrix. Also, the ring $\mathrm{gr}_{\tilde{J}_{n-1}}(\tilde{R})$ is presented over $\tilde{B}$ by $\tilde{I}$, where $\tilde{J}_{n-1} \subset \tilde{R} \cong R[X_{11}^{-1}]$ is the corresponding ideal of maximal minors of (Y) fixing the initial $(n-1) \times (n-2)$ submatrix. Letting $\tilde{P} := P\tilde{B}$, we have $I_P \cong \tilde{I}_{\tilde{P}}$ and $\mathrm{ht}(P) = \mathrm{ht}(\tilde{P})$.

In this way, we have reduced the entire argument to the case $n = 2$. Here, we are assuming that $\mathrm{ht}(P) = 4$. Looking closer at the presentation of I, one sees that $I_1(\varphi_2 \otimes B | \eta_1)$ contains the cofactors of the 3×3 matrix

$$\begin{bmatrix} X_{12} & X_{13} & X_{14} \\ X_{22} & X_{23} & X_{24} \\ T_2 & T_3 & T_4 \end{bmatrix}.$$

Further, $I_1(f' \otimes B)^* = (X_{11}, X_{21}) B$, a regular sequence modulo the previous cofactors. Therefore, one can invert a coefficient of some relation among the generators of I. This shows our contention.   □

Corollary 2.3. *Keeping the hypotheses of Lemma 2.2, if $n = 2, 3$ then $S(I)$ is normal and $\mathrm{Cl}(S(I)) \cong \mathbb{Z}^2$.*

P r o o f. For $n = 2, 3, I$ has deviation at most 2, hence is strongly Cohen-Macaulay [26]. Applying the above lemma, it follows that I is of linear type [14]. The result now follows from Proposition 2.1.   □

R e m a r k. Even if I happens not to be of linear type, one can still derive similar results for the Rees algebra. However, the authors expect Corollary 2.3 to be valid for any value of n. A question remains as to whether one can prescribe, in the non-generic case, sufficient estimates for the grades of the Fitting ideals in order to obtain results along the same line.

3. The free resolution of the maximal minors fixing a submatrix: the unmixed case.
Referring to the notation of Section 1, recall that $J_r = J_r(f)$ stands for the ideal $(\Lambda^n f)(\Lambda^r R^r \otimes \Lambda^{n-r} R^{m-r}) \subset R$, where $f : R^m \to R^n$ is a map and there is given a splitting $R^m = R^r \oplus R^{m-r}$ $(m \ge n + 1, n \ge r + 1)$. If f is "generic" then one of the results of [2] is to the effect that the codimension of J_r equals the minimum value between $m - n + 1$ and $n - r + 1$, while the homological dimension of R/J_r is $m - r$.

In general, one would like to prove:

Conjecture. *(R noetherian) If* grade $I_n(f) \ge m - n + 1$, grade $I_r(f | R^r) \ge n - r + 1$ *and if $I_n(f) + I_{n-2}(f |_{R^r})$ is a proper ideal of grade at least $m - r$, then R/J_r has homological dimension $m - r$.*

We observe that in earlier works [1], [2] the third grade requirement was automatically subsumed in the first two. In general, however, this additional bound is essential, as the following example shows: let $R = k[X, Y, Z]$ and let f be represented by the 3×5 matrix, where we fix the first column:

$$M := \begin{bmatrix} X & Y & Z & 0 & 0 \\ Y & X & Y & Z & 0 \\ Z & 0 & X & Y & Z \end{bmatrix}.$$

Here, $I_1(M') = (X, Y, Z)$ and $I_3(M) = (X, Y, Z)^3$ have grade three, but of course, the homological dimension of any R-module is at most three, whereas $5 - 1 = 4$.

Clearly, one would like to conjecture about the actual structure of a finite free resolution of J_r under the above conditions. Once an explicit presentation of J_r is at hand, finding the codimension of J_r is, in principle, an easy task. A good candidate is given by a variation of the Cramer map (cf. the proof of Theorem 3.1). It is much harder to guess the remaining maps of a (potential) free resolution. The authors believe that there is such a resolution obtained by tensoring well-known free complexes and by cutting the faulty sizes by suitable trace maps.

In this section, we illustrate this expectation in a particular case which, nevertheless, is believed to keep the main features of the general unmixed case, that is to say, the case where $m - n + 1 = n - r + 1$. The reason we focus on unmixed ideals, at this stage, is that, among other nice properties, they enjoy the (rare?) phenomenon that their associated graded ring is Gorenstein while not being a domain (a general account of such phenomena is to be found in [15]).

First, one has the well-known Buchsbaum-Rim complexes that, under the present hypotheses, resolve the cokernel of the maps $R^{n+2} \xrightarrow{f} R^n$ and $(R^n)^* \xrightarrow{f^*} (R^{n-2})^*$, respectively (cf. [5]). We are particularly interested in the tail maps of these complexes, namely, $0 \to R^n \to \Lambda^{n+1} R^{n+2}$ and $0 \to (R^{n-2})^* \to \Lambda^{n-1} (R^n)^*$. The tensor product of these maps yields a complex

$$0 \to R^n \otimes (R^{n-2})^* \xrightarrow{\varphi_4} (R^n \otimes \Lambda^{n-1} (R^n)^*)$$

$$\oplus ((R^{n-2})^* \otimes \Lambda^{n+1} R^{n+2}) \xrightarrow{\varphi_3} \Lambda^{n+1} R^{n+2} \otimes \Lambda^{n-1} (R^n)^*$$

which we modify as follows: project the module $\Lambda^{n+1} R^{n+2}$ onto its direct summand $\Lambda^{n-3} R^{n-2} \otimes \Lambda^4 R^4$ and compose with the trace map $(V, W) \to \operatorname{tr} V + \operatorname{tr} W$, where the module $R^n \otimes \Lambda^{n-1} (R^n)^*$ (respectively, $(R^{n-2})^* \otimes \Lambda^{n-3} R^{n-2}$) is identified with the free R-module whose elements are the $n \times n$ (respectively, $(n - 2) \times (n - 2)$) matrices over R. The result of this composition clearly maps onto R, hence its kernel K is a free R-module of rank $2n^2 - 5$. It is easy to see that $\operatorname{im}(\varphi_4) \subset K$.

This takes care of the "trace" modification of the tensor product of the maps. We further introduce the following variation of the "Cramer map" associated to the map $f: R^{n+2} \to R^n$:

$$\Lambda^{n+1} R^{n+2} \otimes (\Lambda^{n-1} R^n)^* \to \Lambda^2 R^{n+2}$$

$$w_1 \Lambda \dots \Lambda w_{n+1} \otimes \Phi \mapsto \sum_{1 \leq j_1 \leq j_2 \leq n+1} \sigma(j_1, j_2) \, \Phi(f(w_{i_1}) \Lambda \dots \Lambda f(w_{i_{n-1}})) \, w_{j_1} \Lambda w_{j_2},$$

where $\sigma(j_1, j_2)$ is the sign of the permutation $1 \to i_1, \ldots, n \to j_1, n + 1 \to j_2$, with $\{1, \ldots, n + 1\} = \{i_1, \ldots, i_{n-1}, j_1, j_2\}$.

Composing this map with the canonical projection of $\Lambda^2 R^{n+2}$ onto its summand $\Lambda^2 R^4 \cong \Lambda^{n-2} R^{n-2} \otimes \Lambda^2 R^4$, we obtain a map φ_2. Finally, set $\varphi_1 := $ restriction of $\Lambda^n f$ to $\Lambda^{n-2} R^{n-2} \otimes \Lambda^2 R^4$.

Lemma 3.1. *The above construction yields a complex*

$$\mathscr{C}: 0 \longrightarrow R^n \otimes (R^{n-2})^* \xrightarrow{\varphi_4} K \xrightarrow{\varphi_3|_K} \Lambda^{n+1} R^{n+2}$$

$$\otimes (\Lambda^{n-1} R^n)^* \xrightarrow{\varphi_2} \Lambda^{n-2} R^{n-2} \otimes \Lambda^2 R^4 \xrightarrow{\varphi_1} R.$$

Proof. We are to verify that $\varphi_2 \circ \varphi_3|_K$ and $\varphi_1 \circ \varphi_2$ are the zero maps. The fact that $\varphi_1 \circ \varphi_2 = 0$ is a base-free restatement of the straightening relations of the $n \times n$ minors of f fixing the submodule $R^{n-2} \subset R^{n+2}$. As for the other composition, the argument goes as follows. Choose bases $f_1, \ldots, f_n$ and $e_1, \ldots, e_{n+2}$ of R^n and R^{n+2}, respectively. Then, a basis of K is given by the following collection of elements:

(1) $f_i \otimes f_1^* \Lambda \ldots \Lambda \hat{f}_l^* \Lambda \ldots \Lambda f_n^*$, $i \neq l$, $1 \leq i, l \leq n$

(2) $e_j^* \otimes e_1 \Lambda \ldots \Lambda \hat{e}_k \Lambda \ldots \Lambda e_{n+2}$, $j \neq k$, $1 \leq j \leq n - 2$, $1 \leq k \leq n + 2$

(3) $f_i \otimes f_1^* \Lambda \ldots \Lambda \hat{f}_i^* \Lambda \ldots \Lambda f_n^* - f_1 \otimes \hat{f}_1^* \Lambda f_2^* \Lambda \ldots \Lambda f_n^*$, $i \neq 1$, $1 \leq i \leq n$

(4) $e_j^* \otimes e_1 \Lambda \ldots \Lambda \hat{e}_j \Lambda \ldots \Lambda e_{n+2} - f_1 \otimes \hat{f}_1^* \Lambda f_2^* \Lambda \ldots \Lambda f_n^*$, $1 \leq j \leq n - 2$.

To prove our contention, we compute the value of $\varphi_2 \circ \varphi_3$ on each class as above of basis elements. Leaving out the details of the computation for space reasons, we obtain that the vanishing of $\varphi_2 \circ \varphi_3$ on class (1) expresses the Cramer-Laplace relations of the $(m-1) \times (m-1)$ minors of an $(m-1) \times m$ submatrix of f with rows $1, \ldots, \hat{l}, \ldots, n$, along the i-th row $(i \neq l)$; on elements of class (2), it expresses the same provided "row" is changed to "column". On class (3), the vanishing of $\varphi_2 \circ \varphi_3$ translates into the fact that an $n \times n$ minor of f can be developed along the i-th row $(i \neq 1)$ or along the first row. A similar translation applies in the case of class (4). $\square$

Lemma 3.2. *If* grade $I_n(f) \geq 3$, grade $I_{n-2}(f_{|R^{n-2}}) \geq 3$ *and* grade $(I_n(f) + I_{n-2}(f_{|R^{n-2}}))$ ≥ 4, *then the complex* $\mathscr{C}$ *is acyclic.*

Proof. By the "lemme d'acyclicité" [20], it suffices to establish the acyclicity of $\mathscr{C}_P$ for a prime ideal $P \subset R$ of height at most three. For such a prime, by hypothesis, we must have either $I_n(f) \not\subset P$ or else $I_{n-2}(f') \not\subset P$ (with $f' = f_{|R^{n-2}}$).

Let $I_n(f) \not\subset P$ and let Δ denote an $n \times n$ minor not lying in P. If such a minor involves the first $n - 2$ columns of f, then $(\varphi_1)_P$ splits and, therefore, $\mathscr{C}_P$ is split exact all the way through. Thus, we assume Δ does not involve all of the first $n - 2$ columns of f'. In this case, by applying suitable elementary column transformations on f that do not affect the ideals im (φ_1) and $I_{n-2}(f')$, one can assume that Δ is the $n \times n$ minor with columns $2, \ldots, n + 1$ or the one with columns $3, \ldots, n + 2$. Then, by inverting Δ (in R_P) and applying further elementary transformations, one can represent f_P, respectively, by the matrices

$$\begin{bmatrix} a_{11} & & & & & 1 & \\ a_{21} & & & & 1 & & \\ a_{31} & & & 1 & & & \\ \hline & & 1 & & & & \\ & & & \ddots & & & \\ & 1 & & & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_{11} & a_{12} & & & & & 1 \\ a_{21} & a_{22} & & & & 1 & \\ a_{31} & a_{32} & & & 1 & & \\ a_{41} & a_{42} & & 1 & & & \\ \hline & & & & \ddots & & \\ & & 1 & & & & \end{bmatrix}$$

where all unspecified entries are zero. Next, a convenient choice of bases sets the complex $\mathscr{C}_P$ in a form suitable for applying the criterion of Buchsbaum-Eisenbud [3]. Thus, if $I(\psi)$ denotes the ideal of $(\text{rk}\,\psi) \times (\text{rk}\,\psi)$ minors of a matrix ψ, one obtains in both cases that, up to radical, $I(\varphi_i)_P$ and $I_{n-2}(f'_P)$ are one and the same ideal for $1 \leqq i \leqq 3$ and $I(\varphi_4)_P = R_P$. Therefore, $\mathscr{C}_P$ is acyclic.

Let now $I_{n-2}(f') \not\subset P$. Assume, as we may, that the $(n-2) \times (n-2)$ minor of f with rows $1, \ldots, n-2$ and columns $1, \ldots, n-2$ does not belong to P. By a similar token, the matrix of f_P can be set in the form

$$
\left[
\begin{array}{c|cccc}
I_{n-2} & & & & \\
\hline
 & a_{n-1,n-1} & a_{n-1,n} & a_{n-1,n+1} & a_{n-1,n+2} \\
 & a_{n,n-1} & a_{n,n} & a_{n,n+1} & a_{n,n+2}
\end{array}
\right].
$$

Again, by computing the depths of $I(\varphi_i)_P$, one sees that $\mathscr{C}_P$ is acyclic. □

The main theorem of this section can be stated as follows.

Theorem 3.3. *The following conditions are equivalent:*

(i)   grade $I_n(f) \geqq 3$, grade $I_{n-2}(f \mid_{R^{n-2}}) \geqq 3$ *and* grade $(I_n(f) + I_{n-2}(f \mid_{R^{n-2}})) \geqq 4$.
(ii) *The complex $\mathscr{C}$ is acyclic.*

Moreover, in that case, the homological dimension of $R/J_{n-2}(f)$ is exactly four if and only if the ideal $I_n(f) + I_{n-2}(f \mid_{R^{n-2}})$ is proper.

P r o o f. Lemma 3.2 above takes care of the implication (i) $\Rightarrow$ (ii). To prove the converse implication we argue as follows. Let $\tilde{f}$ be the corresponding generic map whose entries are indeterminates over R; let $\tilde{f}'$, $\tilde{J}$, etc. stand for the corresponding data. Since the conditions in (i) are well-known to be satisfied in this generic situation, the corresponding complex $\tilde{\mathscr{C}}$ is acyclic by the first implication and, moreover, grade $\tilde{J} = 3$ as $\tilde{J} = I_n(\tilde{f}) \cap I_{n-2}(\tilde{f}')$. This shows easily that we have equality, up to radicals, of the ideals $I(\tilde{\varphi}_3)$ and $I(\tilde{\varphi}_2)$ respectively taken "at the rank" of the matrices). By specialization, we have at least $I(\varphi_3) \subset \sqrt{I(\varphi_2)}$, hence grade $I(\varphi_2) \geqq$ grade $I(\varphi_3) \geqq 3$ (the latter by exactness of $\mathscr{C}$). Since, in any case, grade $J_{n-2} =$ grade $I(\varphi_2)$, the upshot is that grade $J_{n-2} \geqq 3$. Therefore, also grade $I_n(f) \geqq 3$ and grade $I_{n-2}(f') \geqq 3$.

In order to finish the proof of the implication (ii) $\Rightarrow$ (i), we claim that $\sqrt{I(\varphi_4)} = \sqrt{I_n(f) + I_{n-2}(f')}$. Again, by a specialization argument, it suffices to look at the generic situation. Let P be a minimal prime of $I_n(f) + I_{n-2}(f')$ and assume $n \geqq 4$. Clearly, $I_{n-3}(f') \not\subset P$ (recall we are now dealing with the generic case). By the usual procedure of inverting a minor in $I_{n-3}(f')$ and applying elementary transformations to f over $R[X]_P$, we reduce the question to verifying that $I(\varphi_4)$ is P-primary when $n = 3$. But in this case, a straightforward computation yields directly that $\sqrt{I(\varphi_4)} = I_3(f) + I_1(f')$.

The claim about the homological dimension is standard and we leave to the reader the verification that it follows simply from the preceding arguments. □

We close this section with some results about the analytic behaviour of the ideal J. We let $R := k[X]$, where k is a field and X an $n \times (n+2)$ generic matrix. Set $J := J_{n-2}(X)$.

As before, $\nu(\ )$ will denote minimal number of generators. The analytic spread of an ideal I in a local ring will be denoted by $l(I)$.

Proposition 3.4. (i) $\nu(J_P) \leq \max \{\mathrm{ht}(P) - 1, \mathrm{ht}(J_P)\}$ *for every prime ideal P distinct from* $I_n(X) + I_{n-2}(X')$, *while* $\nu(J_P) = 5$ *for* $P = I_n(X) + I_{n-2}(X')$.

(ii) $l(J_P) \leq \max \{\mathrm{ht}(P) - 1, \mathrm{ht}(J_P)\}$.

P r o o f. Since J itself is generated by six elements, we may assume that $\mathrm{ht}(P) \leq 6$. We first prove (i): if $n \geq 4$ then $\mathrm{ht}(I_1(X')) = n(n-2) \geq 8 > \mathrm{ht}(P)$, so by inverting an entry of X' and applying suitable elementary transformations to the resulting matrix, one reduces the question to the case $n = 3$ – we leave out the case $n = 2$ which is well-known to be a complete intersection in the punctured spectrum.

Now, for $n = 3$ we may assume that $P \supset I_1(X') + I_3(X)$ as otherwise $J_P = I_1(X')_P$ or $J_P = I_3(X)_P$, which are well-known complete intersections (the latter because $P \not\supset I_2(X)$ as $\mathrm{ht}(I_2(X)) > 6$). Since even $P \not\supset I_2(X'')$, where X'' denotes the submatrix complementary to X' in X, we may further assume that the 2×2 minor δ corresponding to rows $1, 2$ and columns $2, 3$ does not belong to P. Inverting δ and applying suitable elementary transformations to X over $R[\delta^{-1}]$, we obtain a matrix

$$
Y := \begin{bmatrix} Y_1 & 1 & 0 & 0 & 0 \\ Y_2 & 0 & 1 & 0 & 0 \\ Y_3 & 0 & 0 & Y_4 & Y_5 \end{bmatrix},
$$

where $Y_i, (i = 1, \ldots, 5)$ are indeterminates in a new ring $\tilde{R}$. But $J(Y) = J(X)_\delta$ as Y_1, Y_2, Y_3 are the result of elementary transformations applied solely to X_{11}, X_{21}, X_{31}. Therefore, $J(X)_P = J(Y)_{\tilde{P}}$, where $\tilde{P} := P\tilde{R}$. A direct calculation shows that $J(Y) = (Y_1 Y_4, Y_1 Y_5, Y_2 Y_4, Y_2 Y_5, Y_3)$, which proves our contention.

In order to prove (ii) it suffices, by (i), to show that $l(J_P) \leq 4$ for the prime $P = I_n(X) + I_{n-2}(X')$. As above, after the standard identifications, we have $J(X)_P = ((Y_1, Y_2) \cap Y_4, Y_5) + (Y_3))_{(Y)}$. The relation of analytic dependence given by $Y_1 Y_4 \cdot Y_2 Y_5 = Y_1 Y_5 \cdot Y_2 Y_4$ shows that $l(J_P) \leq 4$. $\square$

Corollary 3.5. *With same notation as above, one has:*

(i) $\dim S(J) = \dim R + 1$ *and $S(J)$ is n o t Cohen-Macaulay.*
(ii) $\dim S(J/J^2) = \dim R$.
(iii) $\mathrm{gr}_J(R)$ *is R/J-torsion free.*

P r o o f. (i) According to the formula of [13], the inequalities $\nu(J_P) \leq \mathrm{ht}(P) + 1$, for every prime P, already imply the required value for $\dim S(J)$. By [14], in this case, $S(J)$ cannot be Cohen-Macaulay unless it is a domain. However, the generators of J are analytically dependent, a relation of dependence being given by straightening the product of the minors corresponding, respectively, to columns $1, \ldots, n-2, n-1, n+2$ and $1, \ldots, n-2, n, n+1$.

(ii) This follows again from [13] and the earlier proposition.

(iii) From the estimates for the local analytic spreads given in the proposition and from [10], one has the required torsion-freeness. For a different proof, in greater generality, see [7]. □

The following points seemed to us worthwhile addressing:

1. Is there an "expected" free resolution of $S(J/J^2)$ over $R[T]$ that comes from natural subcomplexes of (graded) Koszul complexes?
2. Is $S(J/J^2)$ Cohen-Macaulay?

The authors do not have the answer to the questions.

4. The complete intersection locus. For a noetherian ring R and an ideal $I \subset R$, we consider the complete intersection locus of I. This is given by an open set $D(L) \subset \operatorname{Spec} R$ such that $P \in D(L)$ if and only if I_P is generated by an R_P-sequence. Assuming some "regularity" condition, such ideal is easy to describe.

Lemma 4.1. *Let R be a Cohen-Macaulay noetherian ring and let $I \subset R$ be an ideal of finite homological dimension. If $F_1 \xrightarrow{\varphi} F_0 \longrightarrow I \longrightarrow 0$ is a free presentation of I, then the open set $D(I_\delta(\varphi)) \subset \operatorname{Spec} R$ is the complete intersection locus of I, where δ stands for the deviation of $I := \operatorname{rank}(F_0) - \operatorname{ht}(I)$.*

P r o o f. The argument can be transcribed from the proof of Corollary (1.2) in [27], by recalling that, in the presence of finite homological dimension, I_P is a complete intersection if and only if I_P/I_P^2 is R/P-free. □

The estimates for $\operatorname{ht}(L)$ given in the literature ([8], [11], [27]), however sharp for some classes of ideals, are nearly inocuous in our present context. We shall derive the exact value of the codimension of the complete intersection locus, as well as a complete description of the generic components of $\operatorname{Spec} R \backslash D(L)$, for the ideals studied in the earlier sections. Following the notation of [27], we set $c(I) := \operatorname{ht}(L) = \operatorname{ht}(I_\delta(\varphi))$, where $\delta = $ deviation of I.

For completeness, we treat other cases of ideals of maximal minors fixing a set of columns. Only the generic case is considered, but presumably one could also deal with the general case as long as enough grade bounds are given.

Proposition 4.2. *Let $R := k[X]$ and let $J := J_{n-1}(X) \subset R$ be the ideal generated by the maximal minors of the $n \times m$ matrix X fixing the $n \times (n-1)$ initial submatrix X'. Then:*

(i) $\sqrt{I_\delta(\varphi)} = \ldots = \sqrt{I_1(\varphi)} = I_{n-2}(X') \cap I_n(X)$
(ii) $c(J) = \min\{6, m - n + 1\}$ *if* $n \geq 3$; $c(J) = m - 1$ *if* $n = 2$.

P r o o f. (i) We refer to the presentation given in [1], to wit,

$$\Lambda^{n+1} R^m \xrightarrow{\varphi} \Lambda^{n-1} R^{n-1} \otimes \Lambda^1 R^{m-n+1} = R^{m-n+1} \longrightarrow J \longrightarrow 0,$$

where φ is the composition of the Cramer map $\varLambda^{n+1} R^m \to R^m$, given by

$$w_1 \varLambda \dots \varLambda w_{n+1} \to \sum_t (-1)^t (X(w_1) \varLambda \dots \varLambda \widehat{X(w_i)} \varLambda \dots \varLambda X(w_{n+1})) w_t,$$

with the canonical projection $R^m = R^{n-1} \oplus R^{m-n+1} \to R^{m-n+1}$.

Now, obviously, $I_1(\varphi) \subset I_n(X)$. A closer inspection in the map φ shows that also $I_1(\varphi) \subset I_{n-2}(X')$. Clearly, then $I_\delta(\varphi) \subset I_{n-2}(X') \cap I_n(X)$. Conversely, let $P \subset R$ be a prime such that $I_\delta(\varphi) \subset P$. Suppose $I_{n-2}(X') \not\subset P$. Then, by inverting a minor in $I_{n-2}(X')$ and performing suitable elementary transformations, one reduces the problem to the case where $n = 2$. But, here it is easy to see that $I_2(X) = I_1(\varphi) \subset P$. This means that the original prime had to contain $I_n(X)$ if it did not contain $I_{n-2}(X')$. We must conclude that $\sqrt{I_\delta(\varphi)} = I_{n-2}(X') \cap I_n(X)$. Of course this implies $\sqrt{I_1(\varphi)} = \dots = \sqrt{I_\delta(\varphi)} = I_{n-2}(X') \cap I_n(X)$ all the way through.

(ii) This is just reading heights in (i).   □

Proposition 4.3. *Let* $J := J_{n-2}(X) \subset R := k[X]$ *be the ideal generated by the maximal minors of the* $n \times (n+2)$ *matrix* X, *fixing the initial* $n \times (n-2)$ *submatrix* X'. *If* $n \geq 3$, *one has*:

(i) $\sqrt{I_3(\varphi)} = I_{n-3}(X') \cap I_{n-1}(X) \cap (I_{n-2}(X') + I_n(X))$.
(ii) $c(J) = 5$.

P r o o f. We refer to the presentation of J given in Section 3. A close reading of the map φ in that presentation reveals that $I_1(\varphi) \subset I_{n-3}(X') \cap I_{n-1}(X)$. It is much subtler to see that $I_3(\varphi) \subset I_{n-2}(X') + I_n(X)$. Actually, we will prove a stronger statement, namely, that $I_2(\varphi) \subset I_{n-2}(X') + I_n(X)$. To see this, one writes the matrix of φ in convenient bases, thus detecting two kinds of 2×2 minors: the ones whose terms contain a factor which is an $(n-1) \times (n-1)$ minor of X fixing X' – these belong then to $I_{n-2}(X')$ – and the ones containing terms whose factors do not all fix X' – these can be expressed as sums of products of $(n-2) \times (n-2)$ minors of X by $n \times n$ minors of X (in the way of classical identities), hence belong to $I_n(X)$.

Summing up, we have $\sqrt{I_3(\varphi)} \subset I_{n-3}(X') \cap I_{n-1}(X) \cap (I_{n-2}(X') + I_n(X))$. In order to prove equality, since $I_3(\varphi)$ defines the complete intersection locus of J, it is sufficient to show that J is a complete intersection along those primes not containing any of the three above primes. Let P be such a prime. If $n \geq 4$, we may invert a minor belonging to $I_{n-3}(X')$ and perform elementary transformations on X, thus reducing the question to the case where $n = 3$. Here we are assuming that either $I_1(X') \not\subset P$ or $I_3(X) \not\subset P$. Note, however, that $J = I_1(X') \cap I_3(X)$. On the other hand, $I_1(X')$ is even globally a complete intersection, while the complete intersection locus of $I_3(X)$ is given by $I_2(X)$. Since $P \not\subset I_2(X)$ as well, one is through.

(ii) This is reading heights above: $\text{ht}(I_{n-3}(X')) = 8 = \text{ht}(I_{n-1}(X))$ $(n \geq 4)$ and $\text{ht}(I_{n-2}(X') + I_n(X)) = 5$ (cf., e.g., [12]).

We close the section with the computation of the complete intersection locus of the presentation ideal of the associated graded ring studied in sections one and two.

Proposition 4.4. *Let X be an $n \times (n + 2)$ matrix $(n \geq 2)$ and let $I \subset R[T]$ be the presentation ideal of the associated graded ring of the ideal $J_{n-1}(X) \subset R := k[X]$. Then:*

(i) $\sqrt{I_{n-1}(\varphi)} = (I_{n-1}(X') + I_n(X \mid T)) \cap (I_{n-2}(X') + (T))$,

(ii) $c(I) = 6$.

P r o o f. We refer to the presentation of I given in the earlier sections. As we saw there, the deviation of I is $n - 1$. From the presentation, one easily checks that $I_{n-1}(\varphi) \subset (I_{n-1}(X') + I_n(X \mid T)) \cap (I_{n-2}(X') + (T))$. Conversely, let $P \supset I$ be a prime ideal containing neither $I_{n-1}(X') + I_n(X \mid T)$ nor $I_{n-2}(X') + (T)$. Note that $I = I_{n+1}(X \mid T) + J_{n-1}(X)$. Therefore, we must have $I_{n-3}(X') \not\subset P$. If $n \geq 4$, invert a minor in $I_{n-3}(X')$, etc., so as to reduce the question to the case $n = 3$. We are now given a prime $P \supset I$ such that P does not contain either $I_2(X') + I_3(X \mid T)$ or $I_1(X') + (T)$. The case where $I_1(X') \not\subset P$ can be further reduced to $n = 2$; but, here an inspection shows that $I_1(\varphi) = I_1(X') + I_2(X \mid T)$ and we are through. Otherwise, let $I_1(X') \subset P$ (back to $n = 3$). In this case, we must have $(T) \not\subset P$ and $I_3(X'' \mid T) \not\subset P$, where X'' is the complementary matrix of X relative to X'. One can check that, by inverting suitable elements and performing elementary transformations, the matrix $X \mid T$ can be brought to one of the following forms

$$\begin{bmatrix} X' & \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \\ \hline 0 \quad 0 & \begin{matrix} 1 & U_4 & U_5 \end{matrix} \end{bmatrix}, \qquad \begin{bmatrix} X' & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{matrix} \\ \hline 0 \quad 0 & \begin{matrix} 1 & 0 & 0 \end{matrix} \end{bmatrix}$$

and, therefore, locally at P, I is generated by the minors $[23 \mid 12]$, $[1345]$ and $[2345]$.

(ii)   Reading heights once more, one has:

$\mathrm{ht}\,(I_{n-2}(X') + (T)) = 6 + 3 = 9 \quad (n \geq 3)$

$\mathrm{ht}\,(I_{n-1}(X') + I_n(X \mid T)) = 6$ (for this, note that $I_{n-1}(X') + I_n(X \mid T)$ is an associated prime of $I_n(X \mid T)$).

A side curiosity about the ideals considered in this work is that the codimension of their complete intersection locus is very nearly the minimum between their analytic spread and twice the codimension of their component of largest codimension – this is far better than the additive bound of [11].

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