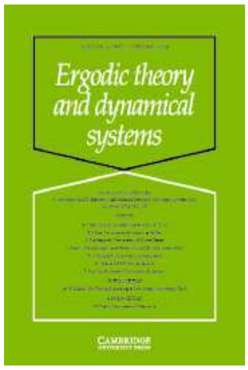

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Fast mixing for attractors with a mostly contracting central direction

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Abstract. Bonatti and Viana introduced a robust (non-empty interior) class of partially hyperbolic attractors of C^2-diffeomorphisms on a compact manifold, for which they construct Sinai–Ruelle–Bowen measures. For some such robust examples, we prove the exponential decay of correlations and the central limit theorem, in the space of Hölder continuous functions. For the proof, we adapt the techniques (backward inducing, redundancy elimination algorithm) we have previously developed.

1. *Introduction*

The purpose of this paper is to study the speed of mixing and related statistical properties of a certain robust (non-empty interior) class of diffeomorphisms on a compact manifold M. More specifically, each diffeomorphism f we consider presents some partially hyperbolic attractor Λ. This means that the tangent bundle $T_\Lambda M$ over the attractor has an invariant dominated splitting into two subbundles, one of which is uniformly hyperbolic—in our case, uniformly expanding. The precise definitions are given in the next section. In rough terms, we are interested in studying those partially hyperbolic attractors which can be partitioned into two regions as follows. The diffeomorphism restricted to one of such regions seems to be hyperbolic, at least for one iteration of f. In the other region, the hyperbolicity breaks down. A kind of weak Markov condition, as well as a control assumption on the derivative of f make it possible for the first region to counterbalance the non-hyperbolic effects of the time spent by the orbits in the second region.

We recall that an f-invariant probability measure μ_0 is *physical* or *Sinai–Ruelle–Bowen* (SRB) if the set $B(\mu_0)$ of points $z \in M$ which satisfy

$$\mu_0 = \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(z)}, \quad \delta_p = \text{Dirac measure at } p$$

has positive Lebesgue measure. This set $B(\mu_0)$ will be called the *basin* of μ_0.

The hypotheses we assume in the next section imply that each attractor Λ we study here admits a unique SRB measure μ_0, supported on Λ. For such a measure, we prove the exponential decay of correlations (that is, exponential mixing) and the central limit theorem on the space of Hölder continuous functions.

1.1. *Assumptions and definitions.* In this work, we study the statistical properties of a partially hyperbolic attractor of a map $f : M \to M$ belonging in a non-empty interior set of the space of C^2-diffeomorphisms on a compact manifold M.

By a partially hyperbolic attractor we mean an invariant compact subset $\Lambda \subset M$ with the following properties.

1. There exists an open neighborhood $\mathcal{N}$ of Λ such that closure $f(\mathcal{N}) \subset \mathcal{N}$ and $\Lambda = \bigcap_{n=0}^{\infty}(f^n(\mathcal{N}))$.
2. Λ is *partially hyperbolic*, meaning that there exists a continuous Df-invariant splitting
$$T_\Lambda M = E^{cs} \oplus E^{uu}, \quad \dim(E^{uu}) > 0$$
of the tangent bundle restricted to Λ with the following properties with respect to some adapted Riemannian metric:
   (a) E^{uu} is uniformly expanding
$$\|Df^{-1}|E^{uu}_{f(x)}\| \leq \lambda_u, \quad \text{for every } x \in \Lambda;$$
   (b) E^{cs} is *dominated* by E^{uu}
$$\|Df|E^{cs}_x\| \cdot \|Df^{-1}|E^{uu}_{f(x)}\| \leq \lambda_u, \quad \text{for } x \in \Lambda,$$
   $0 < \lambda_u < 1$ is independent of $x \in \Lambda$.
   These conditions imply (cf. [**5, 10**]) that there exists a unique foliation (or lamination) $\mathcal{F}^{uu}$ of Λ which is tangent to the strong-unstable bundle E^{uu}_x, at every $x \in \Lambda$. Its leaves are C^2 submanifolds immersed in M and the attractor Λ consists of entire leaves.

On the other hand, in our setting we also suppose the following.

3. The unstable dimension $\dim(E^{uu}) = 1$.

This last condition is necessary to state the (weak) Markov assumption in item 4(iii) below.

Since f is a C^2-diffeomorphism and M is compact, Df is globally Lipschitz; that is, there exists $c > 0$ such that
$$d(Df(x), Df(y)) < cd(x, y), \quad \text{for all } x, y \in M.$$

In the statement of the next condition, we consider Λ endowed with the induced topology. By a *region* of Λ we mean a non-empty (not necessarily connected) open subset of Λ for the induced topology. In some concrete contexts, the term *region* may also refer to the closure of such an open subset of Λ. We also need to define the *pseudo-product structure region*.

Definition 1.1. (Pseudo-product structure region) Let Λ be a partially hyperbolic attractor. We say that D is a *pseudo-central disk* if it is the exponentiated image of a disk centered in the origin of $E^{cs}(x)$, for some $x \in \Lambda$. We say that a region $R \subset \Lambda$ is a *pseudo-product structure region* if there exists a continuous family of pseudo-central disks $\mathcal{C} = \{D_c\}$ and a continuous family of unstable disks $\mathcal{F}^u = \{\Gamma\}$ such that:

- the Γ-disks are transversal to the D_c-disks with the angles between them bounded away from zero;
- each Γ-disk meets each D_c-disk in exactly one point; and
- $R = \bigcup \Gamma \cap \bigcup D_c$.

We define the *central diameter* of R as

$$\mathrm{diam}_c(R) := \sup\{\mathrm{diam}(R \cap D_c);\ D_c \in \mathcal{C}\}$$

Note that the central diameter may depend on the family of pseudo-central disks chosen.

4. There exists a pseudo-product structure region $R_0 \subset \Lambda$ such that we have the following.
   (i) There exists $\lambda_s < 1$ satisfying
   $$\|Df|E^{cs}_x\| < \lambda_s, \quad \text{for all } x \in R_0.$$
   (ii) Now fix a constant ς, $\lambda_s < \varsigma < 1$. We also assume that the *central diameter* (with respect to some specific continuous family of pseudo-central disks) of R_0 is less than some constant
   $$r_0 < \lambda_s \cdot \varsigma^{-1} \cdot (1 - \varsigma)/(10c),$$
   where $c = \mathrm{Lip}(Df)$. We also take r_0 sufficiently small so that for each point $x \in R_0$, the ball $B(x, r_0)$ lies in the image of a single exponential chart of M. The other (technical) conditions on the value of r_0 are established in equations (1) and (2). Although somewhat technical, we stress that such conditions are on Df and so they can be determined *a priori*.
   (iii) There exist $E > 0$ and $c_0 \in (0, 1)$ such that, given any segment Γ in the unstable foliation with $2E \geq \mathrm{length}(\Gamma) \geq E$, we may partition $f(\Gamma)$ into segments $\Gamma_1, \ldots, \Gamma_l$ such that $E \leq \mathrm{length}(\Gamma_i) \leq 2E$, for every $i = 1, \ldots, l$, and the total length of those Γ_i that intersect $\Lambda \setminus R_0$ is less than $c_0\, \mathrm{length}(f(\Gamma))$.
5. For some sufficiently small $\epsilon_0 > 0$, $\epsilon_0 < (1 - \varsigma)/2$, we have
   $$\|Df|E^{cs}_x\| < (1 + \epsilon_0), \quad \text{for } x \in \Lambda \setminus R_0.$$

In Proposition 2.25, we prove that if these conditions 1–5 hold, then each global unstable leaf is dense in Λ. Therefore, it follows from [**4**] (see also other references below) that if the conditions 1–5 hold for f, then f admits a unique ergodic SRB measure μ_0 supported in Λ. μ_0 is also the unique SRB measure for any iterate f^j, $j > 0$.

We say that (f, μ_0) has *exponential decay of correlations* in $\mathcal{H}$ if there exists $\tau < 1$ and for each $\varphi, \psi \in \mathcal{H}$ there exists $K = K(\varphi, \psi) > 0$ so that

$$|C_n(\varphi, \psi)| \leq K\tau^n, \quad \text{for all } n \geq 1.$$

Our main result, Theorem A below, states that the SRB measure existing in a system satisfying conditions 1–5 presents the exponential decay of correlations in the space of Hölder functions. We explain this in precise terms in the next section.

Let us mention that we have obtained similar results in [**7**] and [**8**]. There, even though we do not have any restriction on the unstable subbundle (here we suppose the unstable dimension equals one), we needed to assume the existence of invariant central manifolds and the existence of a Markov partition for the attractor. In this paper, we do not suppose the existence *a priori* of invariant central manifolds, but we construct them by adapting Pesin theory. Moreover, we prove here that over a positive (SRB) measure subset of the attractor, such manifolds are long and they behave as classical stable manifolds.

Similar results were also obtained by Dolgopyat [**9**], independently and through a very different approach, for another class of partially hyperbolic attractors with mostly contracting central direction. More recently, Alves *et al.* [**3**] proved the subexponential decay of correlations for what they called *non-uniformly expanding maps*, also through different techniques.

1.2. *Statement of main results.* We consider a C^2-diffeomorphism $f : M \to M$ on a compact manifold M, admitting a compact invariant subset $\Lambda \subset M$ satisfying conditions 1–5.

THEOREM A. *Under assumptions 1–5 of §1.1, the SRB measure associated with the diffeomorphism f and supported on Λ exhibits the* exponential decay of correlations *in the space of Hölder continuous functions.*

We say that (f, μ_0) satisfies the *central limit theorem* in $\mathcal{H}$ if for every $\varphi \in \mathcal{H}$

$$\Phi_n = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} \left(\varphi \circ f^j - \int \varphi \, d\mu_0 \right)$$

converges in distribution to a Gaussian law $N(0, \sigma)$:

$$\mu_0(\{x \in \mathcal{N} : \Phi_n(x) \leq a\}) \to \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right) dt, \quad \text{for each } a \in \mathbb{R}.$$

As a consequence of the proof of Theorem A we also obtain the following

THEOREM B. *The system (f, μ_0) satisfies the central limit theorem in the space of Hölder continuous functions.*

Remark 1.2. Since the exponential decay of correlations (and also the central limit theorem) for f is equivalent to the same theorem for some positive iterate f^j, $j > 0$, we obtain the same results as above if we replace f by some positive iterate f^j in hypotheses 1–5 of §1.1.

Remark 1.3. The results here also hold if f is $C^{1+\beta}$, with $0 < \beta \leq 1$, instead of C^2. The proofs are almost the same and we choose to state our theorems in the C^2 case for the sake of simpler calculations and constants.

Remark 1.4. As a by-product of our construction (see §2.3), we also obtain that there exists a set $\mathcal{A} \subset \Lambda$ such that:

- $\bigcup_{j=0}^{\infty} f^j(\mathcal{A})$ contains Λ except for a zero set (with respect to the SRB measure or any uu-Gibbs measure—see [**12**]);
- any point $x \in \mathcal{A}$ presents a central stable disk $\gamma(x) \ni x$ such that the radius of $\gamma(x)$ is greater than r_0 and

$$\|Df^n_{E^{cs}}(y)\| \leq \prod_{j=0}^{n-1} \|Df_{E^{cs}}(f^j(y))\| < (\sqrt{\varsigma})^n, \quad \text{for all } y \in \gamma(x) \text{ and all } n > 0.$$

1.3. *Examples.* Carvalho [**6**] proved the existence and uniqueness of SRB measures for certain partially hyperbolic attractors (in dimensions greater than or equal to 3) obtained by modifying an Anosov diffeomorphism by isotopy (along the stable direction) in a neighborhood of some periodic saddle point p, in such a way that this periodic point goes through a Hopf (respectively, saddle-node or period-doubling) bifurcation. The diffeomorphisms one gets in this way have a non-hyperbolic attractor satisfying conditions 1–3 of §1.1.

This procedure generalizes to general nontrivial hyperbolic attractors (in the place of Anosov diffeomorphisms), e.g. Smale's solenoid (see [**14**, ch. 4]). The systems derived from a solenoid have quite simple combinatorics, which makes it easier to verify the hypotheses of our theorems. The approach we present here provides open classes of systems to which our theorems apply, as we state as a corollary (Corollary 1.5) below.

Let us begin with a solenoid Λ of a C^2-diffeomorphism $f_0 : Q \mapsto Q$ (in fact, f_0 is not surjective), $Q = S_m \times B_d$, where S_m is the circle of radius m and B_d is a ball in $\mathbb{R}^2$ with radius d, with $d \ll m$. Such a diffeomorphism has one expanding and two contracting directions. We suppose that the norm of Df_0 along the stable subbundle is bounded by λ_s and the norm of Df_0^{-1} along the unstable bundle is bounded by a constant $\frac{1}{2} > \lambda_u > \lambda_s^2$.

Let $p \in \Lambda$ be a fixed point of f_0, with complex contractive eigenvalues Just for simplify the arguments, let us suppose that Df_0 has small Lipschitz constant.

Let $\delta > 0$ be such that the $\mathbb{R}^3$-ball $B(p, \delta) \subset Q$. For the sake of simpler calculations, rescaling Q if necessary, suppose that $d = 1 \geq \delta$. Denote $V_0 = B(p, \delta)$. We deform f_0^{-1} inside V_0 by a isotopy obtaining a continuous family of maps f_ν, $0 < \nu < 2$ in such a way that we have the following.

(i) The continuation p_{f_ν} of the fixed point p goes through a Hopf bifurcation and becomes a repeller for values of ν between one and two (all the time staying inside V_0). At $\nu = 1$ we have the first Hopf bifurcation, with f_1 topologically conjugate to f_0. We suppose that the derivative $Df_1|_{E^{cs}}$ does not expand vectors. Finally, we suppose that $Df_1|_{E^{cs}}(p_{f_\nu})$ exhibits complex eigenvalues with norm 1.

(ii) In the process, there always exist (continuous extensions of) a strong-unstable cone field C^{uu} (cf. [**15**] for definitions) and a center-stable cone field C^{cs}, defined everywhere in Q.

(iii) Moreover, the width of the cone fields C^{uu} and C^{cs} are bounded by a small constant $\alpha > 0$.

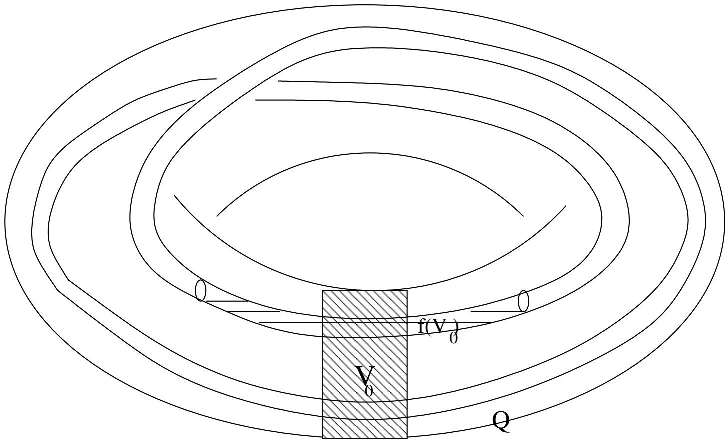

FIGURE 1. System derived from a solenoid.

(iv) The maps f_ν are $\delta - C^1$ close to f_0 outside V_0 so that $\|(Df_1|_{E^{cs}})\| < \lambda_s < \frac{1}{3}$ outside V_0. Furthermore, we suppose that the norm of Df_ν^{-1} remains bounded by λ_u and that $f_\nu(\overline{Q}) \subset Q$, for all $0 \leq \nu \leq 2$.

(v) The constants of Lipschitz of Df_ν are less then $c = \tilde{c} \cdot (1 - \lambda_s)$, with $\tilde{c} = \tilde{c}(\delta)$ not depending on λ_s.

Note that the properties stated in conditions (i)–(v), which are valid for $f_\nu, 0 \leq \nu \leq 2$, are also valid for a whole C^2-neighborhood $\mathcal{U}$ of the set of diffeomorphisms $\{f_\nu, 0 \leq \nu \leq 2\}$. In particular, these conditions imply that all $f \in \mathcal{U}$ also exhibit an unstable foliation varying continuously with the diffeomorphism.

Let us take a C^2-neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of $\{f_\nu, 1 < \nu < 2\}$ such that each $f \in \mathcal{U}_1$ exhibits a fixed point $p_f \in V_0$ which is a repeller. For each $f \in \mathcal{U}_1$, we define

$$\Lambda_f := \bigcap_{n=0}^{\infty} (f^n(Q \setminus \{p_f\})),$$

which clearly satisfies conditions 1–3 of §1.1.

Fixing $\varsigma = \sqrt{\lambda_s}$, we obtain

$$r_0 = \frac{\sqrt{(}\lambda_s) \cdot (1 - \sqrt{(}\lambda_s))}{10\tilde{c} \cdot (1 - \lambda_s)},$$

where r_0 is the constant in condition 4. Since the derivative of f remains contractive by a rate of (at least) λ_s on the central subbundle restricted to $Q \setminus V_0$, $f(Q \setminus V_0) \cap \Lambda_f$ is (fundamentally) a pseudo-product structure region with central diameter less than $\lambda_s \cdot d = \lambda_s$. As long as we take λ_s sufficiently small, we obtain that $r_0 > \lambda_s$. (For instance, the last inequality can be easily verified for the numerically reasonable values $c = 2$, $\lambda_s = 1/1000, \delta = 1$.)

By taking a slightly larger region $V_f \supset V_0$, we define $\tilde{R}_0 := f(Q \setminus V_0) \setminus V_f$ and $R_0 := \tilde{R}_0 \cap \Lambda_f$. Note that $\tilde{R}_0$ is not necessarily connected. At this point, we may take

$E > 0$ in condition 4 to be some large fraction of the infimum of the length of the unstable segments contained in R_0 (see Figure 1).

This implies condition 4 for any diffeomorphism f in the open set $\mathcal{U}_1$ in the space of C^2-diffeomorphisms. By restricting $\mathcal{U}_1$ to a subset of it, if necessary, we can assume that condition 5 also holds for any $f \in \mathcal{U}_1$.

COROLLARY 1.5. *There exists an open set of non-hyperbolic diffeomorphisms $f : Q \to Q$ satisfying conditions 1–5 of §1.1.*

Proof. Proposition 2.25 implies that any global strong unstable manifold in Λ_f is dense in Λ_f and so $\Lambda_f = \overline{W^u(p_f)} \setminus W^u(p_f)$ is a transitive partially hyperbolic attractor for $f \in \mathcal{U}_1$. Just as in [**4**], there is a C^2-neighborhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of the set $\{f_\nu, 1 < \nu \leq 2\}$ such that for all $f \in \mathcal{U}_2$, Λ_f is a partially hyperbolic attractor which is not hyperbolic, because it is transitive and contains a (normally hyperbolic) invariant circle. A transitive set containing an invariant circle cannot be hyperbolic, since it has points with different indexes. Just take the open set in the statement of the corollary to be $\mathcal{U}_2$. □

The same arguments above may be used to provide robust examples in other classes of systems close to Axiom-A attractors (e.g. systems derived from Anosov as in [**6**]). Recently, Bonatti and Viana [**4**] extended the results of [**6**] to general partially hyperbolic attractors with a mostly contracting central direction: there always exist SRB measures supported in the attractor and they are finitely many. The last section of their paper also contains several robust examples, not necessarily close to hyperbolic systems, to which our Theorems A and B apply.

1.4. *Structure of the proof.* Let us present the ideas used in the proof of our results.

To deal with the non-hyperbolic behavior of our maps (the fact that the bundle E^{cs} may fail to be contracting), we begin by constructing a new dynamical system F *induced* from the original f. That is, F is given locally by an (variable) iterate of f. This is now a standard tool in ergodic theory. However, our method is novel in that inducing (in fact, a tower construction) is carried out *backwards*. This is related to the fact that it is along the center-stable direction that hyperbolicity breaks down. More precisely, given a point $x \in \Lambda$, we analyse the negative orbit $f^{-n}(x)$ until finding some $n(x) \geq 1$ so that f^j is (uniformly) hyperbolic at $f^{-n(x)}(x)$, for every $1 \leq j \leq n(x)$. Then we construct a *tower space*

$$\mathcal{T} = \{(f^{-i}(x), -i) : 0 \leq i < n(x)\}$$

and we lift f^{-1} to a map G on $\mathcal{T}$: $\mathrm{proj} \circ G = f^{-1} \circ \mathrm{proj}$, where $\mathrm{proj} : \mathcal{T} \to \Lambda$ is the canonical projection $\mathrm{proj}(x, -i) = x$. For $x \in \Lambda$, each element $(x, j) \in \mathcal{T}$ is called a *copy* of x. A key point is that G is uniformly hyperbolic with respect to some metric in the tower $\mathcal{T}$. The strategy is to deduce properties of f from properties of G by means of the projection proj.

However, proceeding from this, we are faced with a serious problem: since the map G is not injective, in general, f cannot be lifted to a map on the tower (although it lifts to a multivalued relation). In order to completely bypass this difficulty, we introduced a *redundancy elimination algorithm.* In brief terms, the algorithm picks exactly one (convenient) copy of each point in the tower and removes all the others. The *reduced* tower obtained after the algorithm is applied is isomorphic to a full conditional Lebesgue measure (or any m_u measure) subset of Λ and f lifts to a map G^{-1} in it, which is an embedding.

The negative iterates of the remaining points in $\mathcal{T}$ which are contained in the 0th floor of the reduced tower exhausts (using f^{-1} iterates) Λ, up to a zero measure (for the conditional Lebesgue measure of any unstable leaf) subset of Λ. The advantage of this algorithm is that we decompose almost every orbit into the minimal possible segments so that the left endpoint is a hyperbolic time (cf. Definition 2.2) for the right endpoint. Both endpoints are in the 0th floor (that is, the subset of points $(y, 0) \in \mathcal{T}$) of the tower.

Sometimes we identify sets in the tower with their isomorphic images (under proj) in Λ. The 0th floor of the reduced tower and its isomorphic image are both denoted by $\mathcal{A}$.

Due to the way we construct our tower, this set $\mathcal{A}$ is hyperbolic in a strong sense: there is $\varsigma < 1$ such that, given any positive iterate i and $x \in \mathcal{A}$, we have $\|Df^i|_{E^{cs}}(x)\| \leq \prod_{j=0}^{i-1} \|Df|_{E^{cs}}(f^j(x))\| \leq \varsigma^i$. We call such points whose derivative contracts in the center-stable direction for all positive iterates *infinitely hyperbolic points* (see Definition 2.15).

However, the set $\mathcal{A}$ does not necessarily have a local product structure. At this point, the fact that the points in $\mathcal{A}$ are infinitely hyperbolic points enables us to adapt Pesin theory and transform graph techniques, as exposed in [**13**], in order to obtain a kind of Pesin stable manifold passing in each point $x \in \mathcal{A}$. In contrast to the classical Pesin stable manifold, the stable manifolds we build are uniformly contractive for points in $\mathcal{A}$. Such manifolds have uniform lower bounds for their sizes and they are long enough to cross the r_0-neighborhood of R_0 in M (that is, the set $R_1 := \{x \in M, d(x, R_0) < r_0\}$). The choice of r_0 is important at this point as it bounds the diameter of R_0. Our choice of r_0 also guarantees the existence of a local strong unstable manifold crossing R_1, obtained by means of graph transform techniques. We are then able to use these local manifolds to define a bracket map $[\cdot, \cdot] : \mathcal{A} \times \mathcal{A} \to \Lambda$. We use such a bracket map to 'complete' $\mathcal{A}$ up to a set $\mathcal{A}'$ with local product structure.

From this point on, the main results of our work can be derived along fairly well-known lines, from the properties of the set $\mathcal{A}'$.

In §3, we deduce that (f, μ_0) has an exponential decay of correlations in the space of Hölder continuous functions and satisfies the central limit theorem.

Besides the results we prove here, we expect these methods of backward inducing and the redundancy elimination algorithm to be useful in much more generality, in particular in studying systems whose prospective stable direction fails to be contracting.

2. *Construction of the set $\mathcal{A}$*

2.1. *Backward inducing.* As we have seen in the introduction, by [**4**], conditions 1–5 imply the existence of finitely many SRB measures supported on (subsets of) Λ. As a consequence of our construction we obtain (indeed, as a consequence of Proposition 2.25 and [**4**]) that there exists a unique SRB measure μ_0 supported on Λ. Since *a priori* we do not know the uniqueness of the SRB measure, up to Proposition 2.25 we use μ_0 to denote any SRB measure supported on Λ. By a μ_0-zero set we mean a zero set with respect to any SRB measure with support contained in Λ.

We recall that $\epsilon_0 > 0$, $\varsigma + 2\epsilon_0 < 1$ is some small constant such that

$$\sup_{x \in R_0^c} (\|Df|E^{cs}(x)\|) < 1 + \epsilon_0,$$

where R_0 is the region from assumptions 4 and 5 of §1.1 and R_0^c is $\Lambda \setminus R_0$. The exact conditions on the diameter r_0 of R_0 will be determined below. We note that all the conditions (including those we establish in this section) on the value of r_0 can be verified *a priori*, since they are conditions on Df. The value of ϵ_0 will be determined later in this section (see Proposition 2.7 and equations (5) and (6)).

We start by fixing continuous extensions of the subbundles E^{cs}, E^{uu} to some neighborhood V of Λ. We also denote these extensions by E^{cs}, E^{uu}. We do not require such extensions to be invariant under Df. Then, given $0 < a < 1$, we define the *center-stable cone field $C_a^{cs} = (C^{cs}(x))_{x\in V}$ of width a* by

$$C_a^{cs}(x) := \{v_1 + v_2 \in E^{cs}(x) \oplus E^{uu}(x) \text{ such that } |v_2| \geq a \cdot |v_1|\}.$$

The *(strong) unstable cone field $C_a^{uu} = (C^{uu}(x))_{x\in V}$ of width a* is defined in a similar way, by just exchanging the roles of the subbundles in the expression above. We fix $a > 0$ and V sufficiently small such that, up to slightly increasing $\lambda_u < 1$, the domination condition 2 of §1.1 holds for any pair of vectors in the two cone fields:

$$|Df(x)v^{cs}| \cdot |Df^{-1}(f(x))v^u| \leq \lambda_u |v^{cs}| \cdot |v^u|,$$

for every $v^{cs} \in C_a^{cs}(x)$, $v^u \in C^{uu}(x)$ and any point $x \in V \cap f^{-1}(V)$. Note that the center-stable cone field is negatively invariant. Indeed, the domination property, together with the invariance of the subbundle E^{cs} (restricted to Λ), imply

$$Df^{-1}(x)C_a^{cs}(x) \subset C_{\lambda_u a}^{cs}(f^{-1}(x)) \subset C_a^{cs}(f^{-1}(x)),$$

for every $x \in \Lambda$, and this extends to any $x \in V \cap f(V)$. Just as in [**2**], we take V sufficiently small such that the last expression holds for every $x \in V$. We also take $\rho_0 > 0$ such that

$$|Df(y)v| \leq \frac{1}{\sqrt{\varsigma}} |Df|_{E^{cs}(x)}| \cdot |v|, \tag{1}$$

whenever $x \in \Lambda$, $d(x, y) < \rho_0$ and $v \in C_a^{cs}(y)$. Such ρ_0 is taken small enough that the ρ_0-neighborhood of Λ is contained in V. We then take $r_0 > 0$ such that

$$r_0 < \min\{\rho_0/2, \lambda_s \cdot \varsigma^{-1} \cdot (1 - \varsigma)/(10c)\}. \tag{2}$$

Paraphrasing [**2**], we say that an embedded submanifold $N \subset V$ is *tangent to the center-stable cone field C_a^{cs}* if the tangent subspace to N at each point $x \in N$ is contained in V.

As we have seen in the introduction, we are going to construct a kind of tower structure from the original map f. Let us say what we mean by *tower* in this work.

Definition 2.1. (Tower space) A *tower space $\mathcal{T}$* or simply a *tower $\mathcal{T}$* is a countable disjoint union $\bigcup_i (T_i, i)$ of copies of Borelian sets T_i in Λ, where each i is a non-positive integer. Fix a non-positive integer i' and the set $(T_{i'}, i')$ is called the i'th *floor* of $\mathcal{T}$. The 0th *floor* is also called the *ground floor*.

First of all let us fix a constant ς' such that $0 < \lambda_s < \varsigma' < \varsigma < 1$.

Now we can begin the construction of our tower. This will be done by means of *backward inducing*, as explained in the introduction.

In the 0th floor of the tower, we include a copy $(x, 0)$ of each point x in Λ. So, the 0th floor is, in fact, a copy of Λ. In the following, we often identify x and $(x, 0)$. We associate with each $(x, 0)$ the function $\omega_0((x, 0)) = 1$. Now, suppose the ith-floor is constructed and that we have associated with it a function $\omega_i((x, i))$. Then, for each $z = f^{-1}(x)$, where (x, i) is contained in the ith floor of our tower, we calculate

$$E(z) = \omega_i((x, i)) \cdot \kappa(z) \cdot \varsigma^{-1},$$

where $\kappa(z) = \lambda_s$ if z belongs in R_0, and $\kappa(z) = 1 + \epsilon_0$ otherwise.

If $E(z)$ is less than or equal to one, (this means that we returned to the 0-th floor of the tower) we do nothing, otherwise we include a copy $(z, i-1)$ of z in the $(i-1)$st floor. In other words, the set T_{i-1} (see Definition 2.1) is precisely the union of pre-images $z = f^{-1}(x)$ such that $(x, i) \in T_i$ and for which $E(z) > 1$. Then, for those z, we set

$$\omega_{i-1}(z, i-1) = E(z).$$

This procedure defines our tower space.

In a natural manner, we lift f^{-1} to a tower map G on the tower, that is, G is such that

$$\operatorname{proj}(G(x, i)) = f^{-1}(\operatorname{proj}(x, i)),$$

where $\operatorname{proj}(x, i) = x$.

In fact, G is defined as

$$G(x, i) = \begin{cases} (f^{-1}(x), i-1), & \text{if } \omega_i(x, i) \cdot \kappa(f^{-1}(x)) \cdot \varsigma^{-1} > 1, \\ (f^{-1}(x), 0), & \text{otherwise.} \end{cases}$$

However, observe that the inverse G^{-1} is not well defined because G is not injective.

In the definition below, we adapt to our context a notion introduced in [**1**].

Definition 2.2. (Hyperbolic time) Given $z \in \Lambda$, $-p$ is a *hyperbolic time* for z if there exists $p \in \mathbb{N}$ such that $f^{-p}(z) = y$ and for $i = 1 \dots p$,

$$\varsigma^{-i} \cdot \prod_{j=0}^{i-1} \kappa(f^j(y)) \leq 1 \tag{3}$$

holds. This means, in particular, that $Df^i|E^{cs}(y)$ is contracting for $i = 1 \dots p$ by a factor of contraction of at least ς^i.

We say $-p$ is the *first hyperbolic time* for z, if none of $-p+1 \cdots -1$ is hyperbolic time for z.

Sometimes, by a slight abuse of language, we say that $y = f^{-p}(z)$ is the first hyperbolic time for z to denote that $-p$ is the *first hyperbolic time* for z.

Our next step is proving that given any local unstable manifold Γ supplied with its respective Lebesgue measure m_u, the measure m_u of (the projection of) the nth floor of the tower intersected with Γ decreases exponentially fast with n.

For this purpose, we need the following result of Pliss, which assures that the hyperbolic times are quite common (see, for example, [**11**] for a proof).

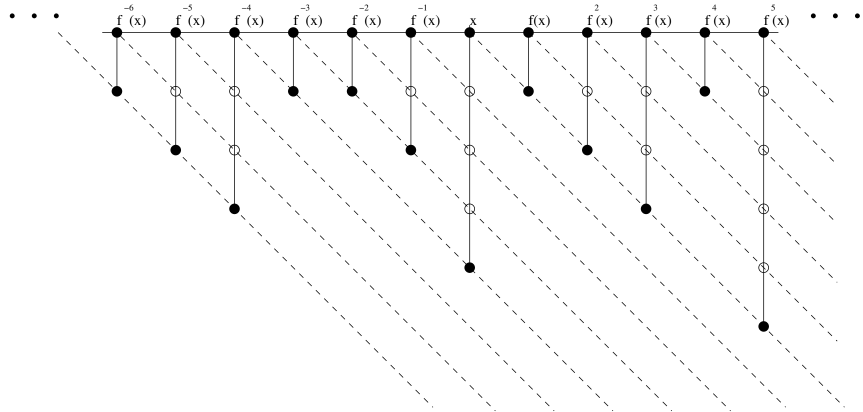

FIGURE 2. Backward inducing. In the horizontal line, we draw the orbit of a point x in Λ and in each vertical segment below a point $y = f^k(x)$ of such orbit, we draw the chain of y. Each filled circle represents a point in the 0th floor of the tower. All copies of a point y stand on the dashed diagonal line beginning in y. If a point x is the top of a maximal chain, then its corresponding diagonal line does not cross any vertical segment. If its diagonal line intersects a vertical line, such a point of intersection will necessarily be a bottom.

LEMMA 2.3. (Pliss' lemma) *Given $\lambda > 0$, $\epsilon > 0$, $H > 0$, there exists $N_0 = N_0(\lambda, \epsilon, H)$, $\delta = \delta(\lambda, \epsilon, H) > 0$ such that, if $a_1, \dots, a_{N_1}$ are real numbers, $N_1 \geq N_0$ with*

$$\sum_{n=1}^{N_1} a_n \leq N_1 \cdot \lambda \quad \textit{and} \quad |a_n| \leq H, \quad \textit{for } n = 1 \dots, N_1,$$

then there are $1 \leq n_1 \leq \cdots \leq n_t \leq N_1$ such that

$$\sum_{i=n_j+1}^{n} a_i \leq (n - n_j)(\lambda + \epsilon), \quad \textit{for all } j = 1, \dots, t_1 \quad \textit{and} \quad n_j < n \leq N_1.$$

Furthermore, t_1 satisfies $t_1/N_1 \geq \delta$.

Let us explain the way we will use Pliss' lemma. If we fix any ς' such that $\lambda_s < \varsigma' < \varsigma$, then there exists N_0 (depending on ς, ς' and on bounds for $\|Df|E^{cs}\|$) such that for any point x satisfying

$$\|Df^{N_1}|E^{cs}(x)\| < (\varsigma')^{N_1}, \quad N_1 > N_0$$

we will have at least t_1 hyperbolic times between x and $f^{N_1}(x)$, t_1 being a fixed fraction of N_1.

We also need the following lemma.

LEMMA 2.4. (Bounded distortion in unstable directions) *Given $L_1 > 0$, there exists $\tilde{K} > 0$ such that given any C^2 disk Γ with tangent bundle inside an unstable cone field and with small curvature, and given any $n \geq 1$ such that* $\operatorname{diam}(f^n(\Gamma)) < L_1$, *we have*

$$\frac{1}{\tilde{K}} \leq \frac{|\det Df^n|_{T_x\Gamma}(x)|}{|\det Df^n|_{T_y\Gamma}(y)|} \leq \tilde{K},$$

for every $x, y \in \Gamma$.

Proof. Let us write $J_\Gamma^n(z) = |\det Df^n|_{T_z\Gamma}|$. Due to [**4**, Lemma 3.1] , the positive iterates $f^i(\Gamma)$ have bounded curvature, which implies that $J_{f^i(\Gamma)}$ is c'-Lipschitz continuous for some uniform constant $c' > 0$. On the other hand, f is uniformly expanding along any direction contained in the unstable cone field, so we have

$$d(f^i(x), f^i(y)) \le \lambda_u^{n-i} d(f^n(x), f^n(y)) \le \lambda_u^{n-i} L_1, \quad i = 0 \dots n.$$

Therefore, we obtain

$$\begin{aligned}\left|\log \frac{J_\Gamma^n(x)}{J_\Gamma^n(y)}\right| &\le \sum_{i=0}^{n-1} |\log J_{f^i(\Gamma)}(f^i(x)) - \log J_{f^i(\Gamma)}(f^i(y))| \\ &\le \sum_{i=0}^{n-1} c' d(f^i(x), f^i(y)) \le \sum_{i=0}^{n-1} c'(\lambda_u^{n-i} L_1).\end{aligned}$$

Hence, just take $\tilde{K} = \exp(c' \cdot L_1 \cdot \sum_{i=0}^{\infty} \lambda_u^i)$. □

Definition 2.5. (The chain of a point x in Λ) Given a point $(x, 0)$ in the tower, let $0 > l = l(x)$ be the first (greatest) negative integer such that $G^{-l}(x, 0) = (f^l(x), 0)$, if such an integer exists. Otherwise we put $l = -\infty$.

When $l > -\infty$, the *chain* of x, denoted by $T(x)$, is the set of points contained in the tower given by

$$T(x) := \bigcup_{0 \ge i > l} \{(f^i(x), i)\} \cup \{(f^l(x), 0)\}$$

The point $(x, 0)$ is called the *top* of the chain $T(x)$. On the other hand, the point $(f^l(x), 0)$ is called the *bottom* of the chain $T(x)$.

In the following proposition, we prove that each bottom corresponds to the first hyperbolic time for its respective top.

PROPOSITION 2.6. *Given $x \in \Lambda$, the function $l(x)$ defined above is the first hyperbolic time for x.*

Proof. Let us write $x_i = f^i(x)$, $l < i < 0$. Consider the chain of x expressed above. From the construction of our tower, we know that

$$\varsigma^i \cdot \prod_{j=0}^{-i-1} \kappa(f^j(x_i)) = \omega_i(f^i(x), i) > 1.$$

This means that we do not reach a hyperbolic time for $0 > i > l$.

Therefore, the only thing left to prove is that we have a hyperbolic time (which then will be the first) for l.

We know that if $l < i \le -1$, then

$$1 \ge \varsigma^l \cdot \prod_{j=0}^{-l-1} \kappa(f^j(x_l)) = \varsigma^i \cdot \prod_{j=0}^{-i-1} \kappa(f^j(x_l)) \cdot \omega_i((x_i, i)).$$

Since $\omega_i((x_i, i)) > 1$ and the product above is less than or equal to one, the factor $\varsigma^i \cdot \prod_{j=0}^{-i-1} \kappa(f^j(x_l))$ has to be less than or equal to one. So, we have

$$\varsigma^{-i} \cdot \prod_{j=0}^{i-1} \kappa(f^j(x_l)) \leq 1, \quad -1 \geq i \geq l. \qquad \square$$

PROPOSITION 2.7. *Given any local unstable manifold* Γ *and any* $n \in \mathbb{N}$*, the intersection of the (projection of the) $(-n)$th floor S_{-n} of the tower with Γ has exponentially small m_u measure, as long as we fix ϵ_0 in condition 5 sufficiently close to zero. That is, as long as we fix constant ϵ_0 sufficiently close to zero, there are constants $q > 0$ and $0 < u < 1$ which do not depend on Γ (except for its length), such that*

$$m_u(S_{-n} \cap \Gamma) < q \cdot u^n, \quad \textit{for all } n \in \mathbb{N}.$$

In particular, we have $\sum_{n=0}^{\infty} m_u(\Gamma \cap S_{-n}) < +\infty$.

Proof. There is no loss of generality in supposing that Γ has length between E and $2E$. Let us call ν_u the Borelian measure on Λ given by $\nu_u(B) = m_u(\Gamma \cap B)$, where B is any Borelian set.

Let us consider the partition $\mathcal{M}$ of Λ consisting of $\mathcal{M} = \{R_0, \Lambda \setminus R_0\}$. We call an *n-cylinder* a set $C_j \subset \Gamma$ whose points visit the same elements of the partition at the same times, for n iterates $n \in \mathbb{N}$.

Let $C_1, \ldots, C_v$ be all the *n-cylinders*. Given ς' such that $\lambda_s < \varsigma' < \varsigma$, we define a *good* cylinder as one which satisfies

$$(\varsigma')^{-n} \cdot \prod_{j=0}^{n-1} \kappa(f^j(C_q)) \leq 1, \quad q \in \{1 \ldots v\}. \tag{4}$$

If a cylinder is not good, we say it is *bad*.

Note that there exists $N_0 \in \mathbb{N}$ such that the $(-n)$th floor of the tower is contained in the union of the bad cylinders, if $n > N_0$. This follows from Pliss' lemma. If x is in a good cylinder, we know by Pliss' lemma that some positive iterate $f^j(x)$, $0 < j < n$, of x has already become a hyperbolic time (with ς instead of ς') for $y = f^n(x)$. Therefore, such an x cannot be in the $(-n)$th floor of the tower.

Of course the sum $\sum_{i=0}^{N_0} \nu_u(\{x : (x, i) \text{ is in the tower}\})$ of the ν_u-measure of the floors from the ground floor to $-N_0$th floor ($N_0 \in \mathbb{N}$ fixed as in the paragraph above) is finite. So, the only property to prove is that the ν_u-measure of the union of bad n-cylinders is exponentially small with n, $n > N_0$.

Now the proof is very close to that of Proposition 6.5, Lemma 6.6 and Corollary 6.7 in [**4**]. Let us recall the main steps.

As in [**4**], we decompose the successive iterates

$$f^n(\Gamma) = \bigcup_{i_1, \ldots, i_n} \Gamma(i_1, \ldots, i_n),$$

where each $\Gamma(i_1, \ldots, i_n)$ is a uu-segment as in condition 5(iii) of §1.1.

Let us fix $n \geq k \geq 1$ and $1 \leq t_1, \ldots, t_r < n$. We consider $M(t_1, \ldots, t_k)$, $0 \leq t_1 < \cdots < t_k \leq n$, the subset of points $x \in \Gamma$ such that $f^t(x)$ belongs in some segment

$\Gamma(i_1,\dots,i_t)$ that intersects $M\setminus R_0$, for all $t\in\{t_1,\dots,t_k\}$. Note that the union of the bad n-cylinders is contained in the union of such sets $M(t_1,\dots,t_k)$, with k/n near one. So, our strategy here is to bound the measure of the union of such sets $M(t_1,\dots,t_k)$.

Claim. The ν_u-measure of $M(t_1,\dots,t_k)$ is bounded by constant $\times\, u_2^k$, for some $0<u_2<1$. The inductive proof of this claim can be found in [**4**, Lemma 6.6].

Now, we are able to prove that the ν_u-measure of the union of the bad n-cylinders decays exponentially fast to zero as n goes to infinity.

For this, we group the n-cylinders into sets $M(t_1,\dots,t_k)$, $t_k\le n$, as above. The bad sets, which group the bad cylinders, consist of points that visit $M\setminus R_0$ for a number of times greater than or equal to $k>\alpha_0\cdot n$, where α_0 is a fixed fraction of n such that

$$\varsigma'\le(1+\epsilon_0)^{\alpha_0}\cdot\lambda_s^{1-\alpha_0}<1. \tag{5}$$

Each of them have exponentially small Lebesgue measure (bounded by constant $\times\, u_2^k$), and the number of them is bounded by:

$$\sum_{k>\alpha_0\cdot n}^{n}\binom{n}{k}.$$

By Stirling's formula, we know that

$$\binom{n}{k}=\frac{n!}{(n-k)!k!}\le\widehat{A}\cdot\frac{n^n}{(n-k)^{(n-k)}\cdot k^k},$$

for some universal constant $\widehat{A}$. Then, we have

$$\frac{n^n}{(n-k)^{n-k}\cdot k^k}=\left(\frac{n}{k}\right)^k\left(\frac{n}{n-k}\right)^{n-k}=\left[\left(1+\frac{n-k}{k}\right)\left(1+\frac{k}{n-k}\right)^{(n-k)/k}\right]^k.$$

Since $k>\alpha_0\cdot n$, by choosing α_0 close to one, the term in brackets above can be made (uniformly) as close to one as we want.

This implies that the ν_u-measure of the bad cylinders is bounded by

$$\text{constant}\times\sum_{k>\alpha_0\cdot n}\binom{n}{k}u_2^k<\text{constant}\times u_3^{\alpha_0\cdot n}, \tag{6}$$

for some $u_2<u_3<1$ as long as we take α_0 near one. (The term 'constant' does not necessarily represent the same positive constant in both sides above.) □

COROLLARY 2.8. *Let A_{-n}, $n\in\mathbb{N}$ be the set defined by*

$$A_{-n}:=\{x\in\Lambda\colon(x,0)\in\mathcal{T}\text{ and }f(x)\in S_{-n},(S_{-n},-n)\text{ is the }-n\text{th floor of }\mathcal{T}\}.$$

Then, given any local unstable manifold Γ, the intersection $A_{-n}\cap\Gamma$ also has exponentially small m_u measure, as long as we fix ϵ_0 as in condition 5 (see the introduction) sufficiently close to zero. That is, for ϵ_0 sufficiently small, there are constants $\tilde q>0$ and $0<\tilde u<1$ which do not depend on Γ (except for its length), such that

$$m_u(A_{-n}\cap\Gamma)<\tilde q\cdot\tilde u^n,\quad\text{for all } n\in\mathbb{N}.$$

Proof. Indeed, this is an immediate consequence of the definition of A_{-n}, the invariance of the unstable bundle and the fact that $Df|_{E^{uu}}$ is bounded. □

COROLLARY 2.9. *Let $B := \{x$, there is an infinite number of subscripts i such that (x, i) is in the tower$\}$. Then for any (global) unstable leaf Γ, $m_u(B \cap \Gamma) = 0$.*

Proof. We apply the Borel–Cantelli Lemma. Given any negative integer n, $B \subset \bigcup_{j=n}^{-\infty} S_j$, where (S_j, j) is the jth floor of our tower. Given any local unstable leaf $\Gamma_1 \subset \Gamma$, from the proof above we have that such union has exponentially small ν_u-measure (here, ν_u is the restriction of m_u to Γ_1). When $n \to -\infty$, we obtain $\nu_u(B) = 0$. Since this is valid for any $\Gamma_1 \subset \Gamma$, this implies that $m_u(B) = 0$. □

Remark 2.10. We note that if x belongs in B then $f^j(x)$ also belongs in B, for $j \in \mathbb{N}$. So, we have

$$\tilde{B} := \bigcup_{j \in -\mathbb{N}} f^j(B) = \bigcup_{j \in \mathbb{Z}} f^j(B).$$

Such a set $\tilde{B}$ is an f and f^{-1}-invariant set and still has zero μ_0 measure.

As a consequence of Proposition 2.7, we also obtain that the set

$$I = \{x \in \Lambda, x \text{ does not have any hyperbolic time}\}$$

is a zero set, as well as

$$\tilde{I} := \bigcup_{j \in \mathbb{Z}} f^j(I).$$

This means that almost every $x \in \Lambda$ has a (first) hyperbolic time and then the tower can be seen as the collection of all chains.

For the following arguments, we must discard the set $\mathcal{D} := \tilde{B} \cup \tilde{I}$. Then, *we will consider the tower built for $\Lambda \setminus \mathcal{D}$.*

2.2. *Redundance elimination algorithm.* As we have seen, f does not lift to a map on the tower. In this section we describe an algorithm that permits us to discard all the copies but one of each point (x, i) in the tower. In fact, the union of the remaining copies will be isomorphic to $\Lambda \setminus \mathcal{D}$, a full SRB subset of Λ. We close this section by analysing the action of f on the reduced tower.

First, we need some definitions.

Definition 2.11. (Chain inclusion) We say the chain of x is *contained* in the chain of x' if there is a copy (x, i), $i \neq 0$, of x in the chain of x'. In this case, we write $T(x) \prec T(x')$.

This terminology is justified by the following proposition.

PROPOSITION 2.12. *If $T(x) \prec T(x')$, then for every point $(\tilde{x}, i)$ in $T(x)$ there will be a copy $(\tilde{x}, j)$ in $T(x')$, with $i \geq j$ (recall that i, j are non-positive). If $i = j$, then both are zero.*

Proof. Suppose that $i \neq 0$. Let $(\tilde{x}, i) \in T(x)$ be as in the statement of the proposition. If $T(x) \prec T(x')$, there exists $k \neq 0$ such that $(x, k) \in T(x')$. Then, $f^{-i}(\tilde{x}) = x$ implies that $f^{-(i+k)}(\tilde{x}) = x'$. As $f^{-(i+k)}(\tilde{x}) = x'$ does not have any hyperbolic time until k, it

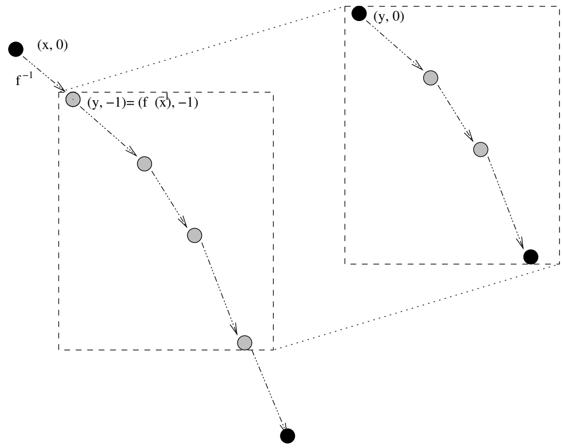

FIGURE 3. Chain inclusion and redundance elimination algorithm. Each small circle represents a point in the tower, and we draw two chains, such that the chain of y is contained in the chain of x. Tops and bottoms appear as black full circles.

cannot have hyperbolic time until $k + i$, because, if it had, such a hyperbolic time would correspond to a hyperbolic time for $f^{-i}(\tilde{x}) = x$ before i, and $f^{-i}(x)$ does not reach its first hyperbolic time before i.

Therefore, if we put $j = k + i$, we have $(\tilde{x}, j) \in T(x')$ by the rules of the chain construction. Then, we see that in this case $i \neq j$.

If $i = 0$, we can apply our last analysis for $(x^l, l) \supset f(\tilde{x}, 0)$, $(x^l, l) \in T(x)$, $l < 0$. (We can suppose $l < 0$, because, if $l = 0$, $x^l = x$ and the result is trivial).

Then, either $(\tilde{x}, 0)$ belongs in $T(x')$ (so $j = i = 0$) or $(\tilde{x}, l-1)$ belongs in $T(x')$ (in this case, $j = l - 1 \neq 0$). $\square$

Now, we can explain our method of redundance elimination. Given a chain $T(x)$ it can be contained in, at most, a finite number of other chains. This is because we discard the invariant set $\mathcal{D}$, which contains points with an infinite number of copies in the initial tower. So, every point in the tower has just a finite number of copies. If $T(x)$ was contained in an infinite number of other chains, x would have an infinite number of copies. In this case, x would belong in $\mathcal{D}$ and so would the points whose copies lie in its chain.

Therefore, given x, x' such that $T(x) \prec T(x')$, we just erase $T(x)$ from the tower, except possibly its bottom in the case when $T(x)$ and $T(x')$ have the same bottom. In other words, we only keep $T(x')$. We will not lose any information by proceeding, because, as we have seen in the last proposition, there is a copy of $T(x)$ in $T(x')$ with decreased subscripts (recall that the subscripts are negative) in relation to the original $T(x)$. Since $T(x)$ is contained in a finite number of chains, this process finishes in some steps. That is, given a chain $T(x)$, we have two possibilities: either $T(x)$ is not contained in any other chain, and then we keep $T(x)$ in the tower; or we take another (different) chain $T(x')$ that contains $T(x)$. In the last case, we ask the same question for $T(x')$: if there is some chain that contains $T(x')$ properly. If there is not such a chain, we stop; otherwise, we continue constructing a nested sequence of chains. Since chain inclusion is transitive, and

$T(x)$ is contained in just a finite number of other chains, this sequence must be finite, as its construction necessarily stops in a finite number of steps.

Of course, by our algorithm, we always keep at least one copy of each point y in the tower, as we only eliminate a copy if we keep at least one other as back-up.

In fact, there will be exactly one copy of each $y \in \Lambda \setminus \mathcal{D}$, at the end of the process, as we show in the next proposition.

Let us call *maximal* the chains that were not eliminated by our process. Then, we have the following.

PROPOSITION 2.13. *Let $T(x')$ be a maximal chain. If $(\tilde{x}, i)$ belongs in $T(x')$, it cannot have (a different) copy in any other maximal chain.*

Proof. Suppose that it has a copy $(\tilde{x}, j) \in T(x)$, $T(x)$ maximal too, and $i \neq j$. First, suppose that $i < 0$. Then, if $j < i$, $(x, j - i)$ would belong in $T(x')$, so $T(x) \prec T(x') \Rightarrow T(x)$ is not maximal. The same is valid if $j > i$, $j \neq 0$, exchanging the roles played by x and x'. If $i = 0$, then $j < 0$ (in order to have $(\tilde{x}, j) \neq (\tilde{x}, 0)$). Therefore, $f^{-j}(\tilde{x}) \subset x'$ has not reached its first hyperbolic time until j. So, $(\tilde{x}, 0)$ is the first hyperbolic time for $f^{-k}(\tilde{x}) \subset x$, $k > j$. Then, $(x, j - k)$ belongs in $T(x')$ and again we have $T(x) \prec T(x')$, which contradicts $T(x)$'s maximality. □

Finally, the following proposition assures that our maximal chains glue nicely.

PROPOSITION 2.14. *If $(x, 0)$ is a bottom in a maximal chain $T(x')$, it is also the top of another maximal chain.*

Proof. Suppose the contrary. Then, there will be a copy $(x, i), i \neq 0$, of x in another chain, for instance $T(x'')$. Then, if $f^{-j}(x) \subset x'$ $(j < 0)$, we must have $j < i$; otherwise $T(x') \prec T(x'')$, which cannot be since $T(x')$ is maximal.

Then, $(x'', j - i)$ belongs in $T(x')$, hence $T(x'') \prec T(x')$ By Proposition 2.12, each point in $T(x'')$ has a subscript greater than or equal to the subscript of its copy in $T(x')$ so $i \geq 0$, which is a contradiction. □

Definition 2.15. (Infinitely hyperbolic point) Given $x \in \Lambda$, where Λ is a partially hyperbolic attractor (with center-stable direction) and $0 < \varsigma < 1$, then we say that x is a *ς-infinitely hyperbolic point* if for some Riemannian metric $\|\cdot\|$ it is true that

$$\prod_{j=0}^{n-1} \|Df|_{E^{cs}}(f^j(x))\| \leq \varsigma^n, \quad \text{for all } n \in \mathbb{N}.$$

When ς is implicit in the context, we simply say that x is an *infinitely hyperbolic point* if the equation above holds.

From now on, we will call $\mathcal{A}$ the set of points in the 0th floor of the tower that have survived after we applied the redundance elimination algorithm. Note that due to Proposition 2.14, all points in $\mathcal{A}$ are ς-infinitely hyperbolic points. This fact, and the fact that the angle between the central direction and the strong unstable direction is bounded, imply that all points $x \in \mathcal{A}$ have a long central manifold, as we will see in the next section.

Let $\mathcal{S} \simeq \Lambda \setminus \mathcal{D}$ be the set of tower surviving points (that rested after we apply our elimination algorithm).

So far, we can lift f to $\mathcal{S}$ (we will use the same symbol f for the lift):

$$f : \mathcal{S} \to \mathcal{S}$$
$$f(x,l) = \begin{cases} (f(x), l+1), & \text{if } l < 0, \\ (f(x), j) & \text{otherwise} \end{cases}$$

where j is the only one such that $(f(x), j) \in \mathcal{S}$.

Except for the fact that we have discarded $\mathcal{D}$, we are back to a diffeomorphism on Λ. Indeed, in the following we always write Λ instead of $\mathcal{S}$. However, the reduced tower structure permits us to decompose almost every orbit into minimal segments, whose endpoints are the return times to the 0th floor $\mathcal{A}$. We are led to consider the *return map* induced by f, defining $F : \mathcal{A} \to \mathcal{A}$ as the first return map of $\mathcal{A}$. This is the same map that takes bottoms in $\mathcal{S}$ in their correspondent tops.

The map F is a bijection (onto its image) since it is a first return map of an invertible map. Since the bottoms correspond to the first hyperbolic time for their respective tops, if we take any $x \in \mathcal{A}$, then we have that

$$\|Df^n|E^{cs}(x)\| \le \prod_{j=0}^{n-1} \|Df|E^{cs}(f^j(x))\| < \varsigma^n, \quad \text{for all } n > 0,$$

which means that $x \in \mathcal{A}$ is an *infinitely hyperbolic point*.

In the next section, we use this property to construct stable manifolds for points in $\mathcal{A}$ and its iterates. We prove that such stable manifolds have their sizes bounded from below for infinitely hyperbolic points. This implies not only the fact that F restricted to any local stable leaf (intersected with $\mathcal{A}$) is a (uniform) contraction, but also that F has good hyperbolic properties (e.g. bounded distortion statements).

For the moment we have the following last important fact about $\mathcal{A}$.

PROPOSITION 2.16. *$\mathcal{A}$ is contained in R_0.*

Proof. In fact, suppose not. So, we can take $x \in \mathcal{A} \cap (\Lambda \setminus R_0)$. In such a case, in the calculations to build the tower, we used $(1+\epsilon_0)$ to bound the derivative $Df|_{E^{cs}}(x)$. If we consider the chain of $y = f(x)$, this means that $(x, -1) \in T(y)$. Therefore $T(x)$ is not a maximal chain, which implies $x \notin \mathcal{A}$, a contradiction. □

Remark 2.17. Note that each $x \in \Lambda$ (remaining from the elimination algorithm), corresponds to one unique point $(x, -i), i \ge 0$, in the tower. So, $f^i(x)$ belongs to $\mathcal{A}$. This implies that x has only negative Liapunov exponents in its center-stable space. Following [**4**, Theorem A] (see also [**12, 13**]), we can prove that f has a unique SRB measure μ_0.

2.3. *Stable manifolds for $\mathcal{A}$.* Now we provide center-stable manifolds passing through the points of set $\mathcal{A}$ and crossing R_1.

The first step is to define an adapted Finsler $|\cdot|^*$ over the set of $\mathrm{Sat}(\mathcal{A}) = \bigcup_{j\in\mathbb{Z}} f^j(\mathcal{A})$. There is no loss of generality in supposing that the expansion we have in the strong unstable direction is greater than ς^{-1}, where ς is the constant fixed in the introduction. Note that since $\mathrm{Sat}(\mathcal{A}) = \bigcup_{j\in\mathbb{Z}} f^j(\mathcal{A})$ exhausts Λ (except for a μ_0-measure zero set) and for $x \in \mathcal{A}$ we have

$$\|Df^n|_{E^{cs}}(x)\| \leq \varsigma^n, \quad \text{for all } n \in \mathbb{N},$$

the Lyapunov exponents are all negative (and less than or equal to $\log(\varsigma)$) in the central direction, for μ_0-a.e. point $z \in \Lambda$. This will imply (cf. [**12, 13**]) that the central manifolds we are going to construct will coincide μ_0-a.e. with Pesin stable manifolds.

Given any $z \cong (z, i)$, $i \in -\mathbb{N} \in \mathrm{Sat}(\mathcal{A})$, we define the inner product $\langle\cdot,\cdot\rangle^*$ on T_zM by

$$\langle v, v'\rangle^* = \omega_i(z, i)^2\langle v, v'\rangle \quad \text{if } v, v' \in E_z^{cs},$$
$$\langle v, v'\rangle^* = \langle v, v'\rangle \quad \text{if } v, v' \in E_z^{uu},$$
$$\langle v, v'\rangle^* = 0 \quad \text{if } v \in E_z^{uu}, v' \in E_z^{cs} \text{ or } \textit{vice versa}.$$

For simplicity, from now on we write $\omega(z) := \omega_i(z, i)$.

The following proposition is parallel to [**13**, Proposition 3.3].

PROPOSITION 2.18. (Adapted Finsler) *The induced Finsler* $|\cdot|^*$ *on* $T_{\mathrm{Sat}(\mathcal{A})}$ *is Borel and has the following properties.*

(a) *Given* $z \in \mathrm{Sat}(\mathcal{A})$*, then*

$$\sqrt{\tfrac{1}{2}}|w| \leq |w|^*, \quad w \in T_zM,\ z \in \mathrm{Sat}(\mathcal{A}).$$

If $y = f(z)$ *and* $z \neq (z, 0)$*, then*

$$\varsigma^{-1}\cdot(1+\epsilon_0) \geq \frac{\omega(y)}{\omega(z)} \geq \varsigma^{-1}\cdot\lambda_s.$$

In the case $z \cong (z, 0)$ *and* $y = f(z)$*, we have*

$$\lambda_s^{-1}\cdot\varsigma \geq \frac{\omega(y)}{\omega(z)} \geq 1.$$

(b) $\|Tzf\|^*$ *and* $\|Tzf^{-1}\|^*$ *are uniformly bounded for* $z \in \mathrm{Sat}(\mathcal{A})$.

(c) $T_{\mathrm{Sat}(\mathcal{A})}f$ *is uniformly hyperbolic respecting* $E^{uu} \oplus E^{cs}$ *equipped with the adapted Finsler* $|\cdot|^*$*; i.e. for all* $z \in \mathrm{Sat}(\mathcal{A})$,

$$\|T_z^s f\|^* < \varsigma < 1 < \varsigma^{-1} < \|T_z^u f^{-1}\|^{*-1}.$$

Proof. Note that we can restrict ourselves to the central subbundle, since the norm restricted to the unstable direction was not modified. Item (a) is a consequence of the definition of ω_i and the construction of the tower. In fact, $\omega = \omega_i \geq 1$, so given $w = v_s + v_u$ with $v_s \in E^{cs}(z)$, and $v_u \in E^{uu}(z)$, we obtain

$$|w| \leq |v_s| + |v_u| \leq |v_s|\cdot\omega(z) + |v_u| = |v_s|^* + |v_u|^* \leq \sqrt{2}\cdot|w|^*.$$

Furthermore, if $y = f(z)$ and $z \neq (z, 0)$, then by the definition of ω (see the tower construction in the last section) we have

$$\omega(z) = \omega(y)\cdot\kappa(z)\cdot\varsigma^{-1},$$

which implies

$$\varsigma^{-1} \cdot (1+\epsilon_0) \geq \frac{\omega(y)}{\omega(z)} \geq \varsigma^{-1} \cdot \lambda_s.$$

The case $y = f(z) \cong (y, 0)$ and $z \cong (z, 0)$ is trivial. Also by the tower construction, if $z = (z, 0)$ and $f(z) = y \cong (y, i)$, $i < 0$ we have that $\omega(y) > 1$, but $\omega(y) \cdot \kappa(z) \cdot \varsigma^{-1} \leq 1 = \omega(z)$. This implies that $z \in R_0$ and $\kappa(z) = \lambda_s$. Therefore, we obtain

$$\lambda_s^{-1} \cdot \varsigma \geq \frac{\omega(y)}{\omega(z)} \geq 1,$$

which concludes (a).

For item (c), let us consider two cases.

Case $z \cong (z, 0)$. Take $v \in E^{cs}(z)$. Then (recall that $z \in \mathcal{A} \subset R_0$)

$$\begin{aligned}|T_z f(v)|^* &\leq |T_z f(v)|\omega(f(z)) \leq \lambda_s \cdot |v| \cdot \omega(f(z)) = \lambda_s \cdot |v|^* \cdot \omega(f(z))\\ &\leq \lambda_s \cdot \lambda_s^{-1} \cdot \varsigma \cdot |v|^* = \varsigma \cdot |v|^*.\end{aligned}$$

Case $z \cong (z, i)$, $i < 0$. Take $v \in E^{cs}(z)$. As above

$$\begin{aligned}|T_z f(v)|^* &\leq |T_z f(v)|\omega(f(z)) \leq \kappa(z) \cdot |v| \cdot \omega(f(z)) = \kappa(z) \cdot |v|^* \cdot \omega(f(z))/\omega(z)\\ &= \kappa(z) \cdot \kappa(z)^{-1} \cdot \varsigma \cdot |v|^* = \varsigma \cdot |v|^*.\end{aligned}$$

In order to see that $\|Tf\|^*$ and $\|Tf^{-1}\|$ are bounded, let us again take $v \in E^{cs}$, then

$$\begin{aligned}|T_z f^{-1}(v)|^* &\leq |T_z f^{-1}(v)|\omega(f^{-1}(z)) \leq \|T_z f^{-1}\| \cdot |v|^* \cdot \omega(f^{-1}(z))/\omega(z)\\ &\leq \lambda_s^{-1} \cdot \varsigma^{-1} \cdot \|T_z f^{-1}(v)\| \cdot |v|^*.\end{aligned}$$

For $v \in E^{uu}$, we have that

$$|T_z f^{-1} \cdot v| = |T_z f^{-1} \cdot v|^* \leq \lambda_u \cdot |v| = \lambda_u \cdot |v|^*.$$

For general vectors in $T_z M$, just combine the bounds above. The bounds for $\|T_z f\|^*$ are equally easy to prove. □

A relevant ingredient for the proof of Pesin's stable manifold (cf. [**13**, Proposition 3.4]) is analogous to the following proposition.

PROPOSITION 2.19. (C^1 uniformity of $\overline{f}$) *Suppose that the diffeomorphism $f : M \to M$ is of class C^2. Let $\overline{f}$ be the expression of f in exponential charts and let $\nu > 0$ be given. Then there exists $0 < r < 1$ such that*

$$\|(D\overline{f}_z)_v - T_z f\|^* \leq \nu, \quad \textit{for all } z \in \mathrm{Sat}(\mathcal{A}) \textit{ and all } v \in T_z M \textit{ with } |v|^* \leq r.$$

Proof. Given $\nu > 0$, just take

$$r = \frac{\lambda_s \cdot \varsigma^{-1} \cdot \nu}{2c}.$$

Then, we have

$$\begin{aligned}\|(D\overline{f}_z)_v - T_z f\|^* &= \sup_{w \neq 0} \frac{|(D\overline{f}_z)_v(w) - T_z f(w)|^*}{|w|^*} \leq \|(D\overline{f}_z)_v - T_z f\| \cdot \lambda_s^{-1} \cdot \varsigma\\ &\leq c|v| \cdot \lambda_s^{-1} \cdot \varsigma \leq 2c|v|^* \cdot \lambda_s^{-1} \varsigma \leq \nu.\end{aligned}$$

□

Remark 2.20. We note that if we take $\nu = 4cr_0 \cdot \lambda_s^{-1} \cdot \varsigma$ then, for $z \in \mathrm{Sat}(\mathcal{A})$ and $v \in T_zM$ such that $|v|^* \leq 2r_0$, we obtain

$$\|(D\overline{f}_z)_v - T_z f\|^* < (1-\varsigma)/2.$$

We also recall that over $T_{\mathcal{A}}M$, we have $|v| = |v|^*$, for all $v \in T_xM$, $x \in \mathcal{A}$.

We recall the notion of a stable set from [**13**].

Definition 2.21. (Stable set of a point $p \in M$) The points $p, y \in M$ are *exponentially forward asymptotic* if, for some $C > 0$ and some λ, $0 < \lambda < 1$,

$$d(f^n(y), f^n(p)) \leq C\lambda^n, \quad \text{for all } n \geq 0.$$

The *stable set* of p is

$$W^s(p) \colon = \{y \in M : y \text{ is exponentially forward asymptotic with } p\}.$$

Finally,

$$\mathcal{W}^s \colon = \{W^s(p) : p \in M\}$$

is called the *stable partition of M*.

The next theorem is a clone of Theorem 3.8 in [**13**].

THEOREM 2.22. (Stable manifolds for $\mathrm{Sat}(\mathcal{A})$) *Let $r = 2r_0$ be the radius supplied by Proposition 2.19, and which corresponds to the value of constant ν in Remark 2.20. Given $z \in \mathrm{Sat}(\mathcal{A})$, call $E^{cs}_z(r)$ the ball in $E^{cs}(z)$ of radius r with respect to the adapted Finsler $|\cdot|^*$. An analogous definition holds for E^{uu}. For each $z \in \mathrm{Sat}(\mathcal{A})$, the stable set $W^s(z)$ is locally the graph of a C^1 map $g^s : E^{cs}_z(r) \to E^{uu}_z(r)$, exponentiated into M, and $x \mapsto W^s(x,\cdot)$ is C^1-continuous respecting $x \in \overline{\mathcal{A}}$. We call this local stable manifold $W^s_z(r)$ and at z it is tangent to $E^{cs}(z)$. Under f^{-1}, $\mathcal{W}^s(r) \colon = \{W^s_z(r)\}$ overflows in the sense that $f^{-1}(W^s_z(r)) \supset W^s_{f^{-1}(z)}(r)$.*

Proof. The proof is very analogous to the proof of [**13**, Theorem 3.8]. We copy part of that proof here, adapting the necessary to our context.

We start by considering the partition of Λ in the sets of the tower, namely $\Lambda \cong \bigcup_{i\leq 0} T_i$, where $T_i := \{(z,i) \in \mathcal{T}\}$ (we identify here each point (z,i) in the tower with its respective point z in Λ). We subdivide each T_i into a finite number of subsets $T_{i,j}$ such that the Finsler cocycle $\omega(\cdot)$ is constant when restricted to each $T_{i,j}$. Note that $\mathcal{A}$ need not be subdivided.

As in [**13**], we define $H = \bigcup \overline{T}_{i,j}$ to be the disjoint union of pre-images of subsets of $\overline{\mathcal{A}}$, equipped with the metric

$$d(z,z') = \begin{cases} d_M(z,z'), & \text{if } z, z' \in \overline{T}_{i,j}, \quad \text{for some } i, j, \\ \mathrm{diam}(M), & \text{if } z \in \overline{T}_{i,j},\, z' \in \overline{T}_{i',j'}, \quad \text{for some } (i,j) \neq (i',j'). \end{cases}$$

By definition, $\overline{T}_{i,j} \cap \overline{T}_{i',j'} = \emptyset$ even though $f^i(\overline{\mathcal{A}}) \cap f^{i'}(\overline{\mathcal{A}})$ may be non-empty. The restriction of $|\cdot|^*$ to each $\overline{T}_{i,j}$ is constant; in particular, it is continuous. We rescale $|\cdot|^*$ on $T_HM = \bigcup T_{\overline{T}_{i,j}}M$ by setting

$$|v|^{**} = \frac{|v|^*}{r}, \quad \text{if } v \in T_zM,\ z \in H.$$

We recall that we are writing $\overline{f}$ for the lift of f to TM via the smooth exponential associated with the original Riemannian structure. Respecting $|\cdot|^{**}$, $\overline{f}_z$ is a C^1-uniformly hyperbolic embedding of the unit ball U_z. Indeed,

$$\|(D\overline{f}_z)_v - T_z f\|^{**} = \sup_{w\neq 0} \frac{|(D\overline{f}_z)_v \cdot w - T_z f \cdot w|^{**}}{|w|^{**}} =$$

(recall that r is constant)

$$\|(D\overline{f}_z)_v - T_z f\|^{*} \leq \nu < (1-\varsigma)/2.$$

This shows that $\overline{f}_z$ is C^1-uniformly approximated by $T_z f$ on U_z and $T_z f$ is uniformly hyperbolic because, respecting $|\cdot|^{**}$, it expands E^{uu} at least as sharply as ς^{-1} and contracts E^{cs} at least as sharply as ς. The uniformity refers to z varying over H. We then apply the standard graph transform construction of unstable manifolds (in this case, we are particularly interested in stable manifolds, so we take f^{-1} in the place of f) in [**10**] or [**14**], and the result follows as in [**13**]. The only subtle point is that we take each trial disk (D_z), $z \in H$, which we start to iterate by the graph transform (defined for f^{-1}, to provide Pesin stable manifolds for f) tangent to the center-stable cones $(C^{cs}_a(z))$, $z \in H$. This can be done, for example, by choosing D_z as the exponential of disks in E^{cs}. By the f^{-1}-invariance of these cones, this implies that the local Pesin stable manifolds we obtain in the limit are also tangent to the center-stable cones. This will be important in the following arguments. □

The next proposition is similar to Lemma 2.7 in [**2**].

PROPOSITION 2.23. (Contraction with respect to the original Riemannian structure) *Let D be a disk contained in a neighborhood $V \supset \Lambda$ such that $f^j(D)$ is tangent to the center-stable cone field, for all $j \geq 0$. If $x \in \mathcal{A} \cap D$, then for all $k \geq 1$ we have (with respect to the original Riemannian structure)*

$$\operatorname{dist}_{f^k(D)}(f^k(x), f^k(y)) \leq \varsigma^{k/2} \operatorname{dist}_D(x,y), \quad \textit{for all } y \in D \textit{ such that } \operatorname{dist}_D(x,y) \leq \rho_0.$$

Proof. Let us prove the proposition by induction. For $k = 1$, by our choice of ρ_0, we have that the derivative $Df|_{T_z D}$ is contractive for all $z \in D$ such that $\operatorname{dist}_M(x,z) \leq \rho_0$:

$$|Df|_{T_z D}| \leq \frac{1}{\sqrt{\varsigma}} |Df|_{E^{cs}}(x)| \leq \frac{\varsigma}{\sqrt{\varsigma}} = \varsigma^{1/2}.$$

So, if we take a curve η_0 of minimal length in D connecting x and y, the length of $\eta_1 := f(\eta_0)$ will be multiplied by $\varsigma^{1/2}$. In particular, the distance

$$\operatorname{dist}_{f(D)}(f(x), f(y)) \leq \varsigma^{1/2} \operatorname{dist}_D(x,y) \leq \varsigma^{1/2} \cdot \rho_0 < \rho_0.$$

Now write $\eta_j = f^j(\eta_0)$ and we assume that

$$\operatorname{length}(\eta_j) \leq \rho_0, \quad \text{for all } 0 \leq j < k.$$

If we denote by $\dot{\eta}_0(z)$ the tangent vector of the curve η_0 at the point z we have

$$|Df^k(z) \cdot \dot{\eta}_0(z)| \leq \left(\frac{1}{\sqrt{\varsigma}}\right)^k \cdot \prod_{j=0}^{k-1} |Df|_{E^{cs}}(f^j(x))| \cdot |\dot{\eta}_0(z)| \leq \varsigma^{k/2} |\dot{\eta}_0(z)|.$$

This implies that

$$\text{length}(\eta_k) \leq \varsigma^{k/2} \cdot \text{length}(\eta_0) = \varsigma^{k/2} \operatorname{dist}_D(x, y),$$

which concludes the proof. □

Note that the last proposition would have an even simpler proof if we had that D and all its forward iterates have d_M-diameters less than ρ_0. However, this is true by construction if we take D as a local stable manifold of a point $x \in \mathcal{A}$ constructed in Theorem 2.22. This is because the manifolds in $\text{Sat}(\mathcal{A})$ have size r_0 with respect to the Finsler $|\cdot|^*$. This yields a size less than ρ_0 with respect to the Riemannian structure. We also recall that the manifolds we constructed in Theorem 2.22 are graphs of C^2 maps $g^s : E^{cs} \to E^{uu}$ exponentiated over M, such that $\text{Lip}(g^s) \leq 1$ with respect to the Finsler $|\cdot|^*$. As the Finsler restricted to the points of $\mathcal{A}$ is equal to the Riemannian structure restricted to the same points, we conclude that the distance induced by $|\cdot|$ in Pesin manifolds over $\mathcal{A}$ is uniformly equivalent to the ambient distance. This implies the following corollary.

COROLLARY 2.24. *There exists $C' > 0$ such that for any local stable manifold $\gamma(x)$ passing by $x \in \mathcal{A}$ we have*

$$d_M(f^n(x), f^n(y)) < C' \cdot \varsigma^{n/2}, \quad \textit{for all } y \in \gamma(x).$$

Proof. This is an immediate consequence of Proposition 2.23 and the comments above. □

Due to the last corollary, from now on we call any Pesin stable manifold constructed in Theorem 2.22 passing through a point in $\mathcal{A}$ a *local stable leaf* or a *local stable manifold*. We use the lower case Greek letter γ to denote any such manifold. We recall (see Remark 2.20) that such manifolds cross a ball (in the usual Riemannian metric of M) with a radius of at least $2r_0$. In particular, due to Proposition 2.16, such manifolds cross R_1.

A similar terminology will be used with respect to the unstable direction: we call each (not necessarily especially) small strong unstable manifold passing through a point in $\mathcal{A}$ a *local unstable leaf* or a *local unstable manifold*. Local unstable leaves will be represented by the upper case Greek letter Γ.

PROPOSITION 2.25. *Let $f : M \to M$ be a diffeomorphism exhibiting an attractor Λ such that conditions 1–5 hold. Then any unstable manifold $\Gamma \subset \Lambda$ is dense in Λ.*

Proof. Using conditions 1–5 we proved the existence of a set $\mathcal{A} \subset R_0$ such that we have the following.

(a) $\bigcup_{j\in\mathbb{N}} f^j(\mathcal{A})$ is equal to Λ, except for a zero set with respect to the conditional Lebesgue measure of any strong unstable manifold Γ.
(b) Any point in $\mathcal{A}$ returns to $\mathcal{A}$ in an infinite number of both positive and negative iterates.
(c) Each point $x \in \mathcal{A}$ admits a center-stable disk $\gamma(x)$ of radius r_0 (Theorem 2.22).
(d) There exists a constant $C' > 0$, independent of $x \in \mathcal{A}$ such that

$$d(f^n(x), f^n(y)) < C' \cdot \varsigma^{n/2}, \quad \text{for all } y \in \gamma(x),$$

where d is the distance in M (Corollary 2.24).

Let $O \subset \Lambda$ be an open subset for the induced topology and let Γ be an unstable manifold contained in Λ. Now we prove that Γ intersects O.

By (a), there exists $y \in \mathcal{A}$ and $j \in \mathbb{N}$ such that $f^j(y) \in O$. Suppose that the ball (in the induced topology) $B(f^j(y), \epsilon) \subset O$ for some $\epsilon > 0$.

By (b), we can take $x \in \mathcal{A}$ such that $f^k(x) = y$ for some $k \in \mathbb{N}$ such that $C' \cdot \varsigma^{(k+j)/2} < \epsilon$.

By (c) and the fact (a consequence of condition 4(iii)) that any unstable manifold crosses R_0, there exists $z \in \Gamma \cap \gamma(x)$.

By (d) and our choice of x, we conclude that $f^{j+k}(z)$ belongs in $O \cap \Gamma$, and this finishes the proof of the proposition. □

2.4. *Completing $\mathcal{A}$.* The following notions are borrowed from [**16**].

Definition 2.26. (Hyperbolic product structure) A set $\Lambda' \subset M$ has *hyperbolic product structure*, if there exists a continuous family of unstable disks $\mathcal{F}^u = \{\Gamma\}$ and a continuous family of stable disks $\mathcal{F}^s = \{\gamma\}$ such that:

(i) $\dim \Gamma + \dim \gamma = \dim M$;
(ii) the Γ-disks are transversal to the γ-disks with the angles between them bounded away from zero;
(iii) each Γ-disk meets each γ-disk in exactly one point; and
(iv) $\Lambda' = \bigcup \Gamma \cap \bigcup \gamma$.

Definition 2.27. (s-subsets and u-subsets) We say that a subset S' of a hyperbolic product structure set S is an *s-subset* if it also has a hyperbolic product structure and its defining families can be chosen to be the same unstable family $\mathcal{F}^u$ of S and a subset $\mathcal{G}^s \subset \mathcal{F}^s$ of the stable family of S. We have an analogous definition for *u-subsets*.

By construction, given $x \in \mathcal{A}$, we have that

$$\|Df|^i_{E^{cs}}(x)\| < \varsigma^i, \quad \text{for all } i \in \mathbb{N}.$$

By continuity, the same expression is valid for any y in the closure $\overline{\mathcal{A}}$ of $\mathcal{A}$.

At this point, if $\overline{\mathcal{A}}$ had hyperbolic product structure, it would be quite easy to prove our Theorems A and B. Since, in general, this is not the case, we need to add points to $\mathcal{A}$, completing it to a set $\mathcal{A}'$ with hyperbolic product structure, while keeping some of the good properties of $\mathcal{A}$.

By Pesin theory, given any point x in $\mathcal{A}$, there exists an embedded smooth Pesin stable manifold $\gamma(x)$ of radius (at least) $2r_0$. Due to our hypotheses, we also have an embedded smooth strong unstable manifold $\Gamma(x)$ crossing the r_0-neighborhood R_1 of R_0. Given $x, y \in \mathcal{A}$, by transversality there is exactly one point z in R_1 such that $\gamma(x)$ and $\Gamma(y)$ intersect each other in z. In other words, the bracket map $[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \to R_1$ is well defined. So we add to $\mathcal{A}$ the points in the image of the bracket map, and we define

$$\mathcal{A}' := \left(\bigcup_{x\in\mathcal{A}} \gamma(x)\right) \cap \left(\bigcup_{x\in\mathcal{A}} \Gamma(x)\right) = \bigcup_{(x,y)\in\mathcal{A}\times\mathcal{A}} (\Gamma(x) \cap \gamma(y)).$$

Since the central diameter of R_0 is less than r_0, the equation above implies that $\mathcal{A}'$ has hyperbolic product structure.

Given a point $y \in \mathcal{A}'$, it belongs to some Pesin stable manifold of a point $x \in \mathcal{A}$.

3. *Decay of correlations and the central limit theorem*

At this point, we can deduce the decay of correlations and the central limit theorem from the properties of $\mathcal{A}'$ that we have obtained above and the framework of [**16**].

We begin by noting that since we assumed conditions 1–5 os §1.1 and proved that all global unstable leaves are dense in the attractor, as a consequence of [**4**], each positive iterate of the diffeomorphism f has a unique ergodic SRB measure μ_0 associated with the attractor Λ.

We have the following facts about $\mathcal{A}$.

(P1) $\mathcal{A}'$ has *hyperbolic product structure*. Moreover, $m_\Gamma(\mathcal{A} \cap \Gamma) > 0$, where m_Γ is the conditional Lebesgue measure over a local unstable manifold Γ.

In fact, the local unstable and local center-stable leaves form two families of disks as in the definition above. Since

$$\mu_0\bigg(\bigcup_{j\in\mathbb{Z}} f^j(\mathcal{A}')\bigg) = \mu_0(\Lambda) = 1,$$

we have $\mu_0(\mathcal{A}') > 0$. From the fact that μ_0 is a uu-Gibbs measure, this implies that there exists some local unstable manifold $\tilde{\Gamma}$ such that $m_{\tilde{\Gamma}}(\mathcal{A}\cap\tilde{\Gamma}) > 0$. Since the holonomy along Pesin stable manifolds is absolutely continuous and $\mathcal{A}'$ has hyperbolic product structure, this implies that $\mu_\Gamma(\mathcal{A}' \cap \Gamma) > 0$.

(P2) There is a countable number of *s-subsets* of $\mathcal{A}'$ (the union of local stable leaves restricted to $\mathcal{A}'$) , say $\Lambda_1, \Lambda_2, \dots$, such that:

- on each Γ-disk (contained in an unstable manifold), $m_\Gamma\{\mathcal{A}' - \cup\Lambda_i\} = 0$;
- for each i, there exist $t_i \in \mathbb{N}^+$ such that $f^{t_i}(\Lambda_i)$ is a *u-subset* of $\mathcal{A}$ (we require in fact that, for all $x \in \Lambda_i$, $f^{t_i}(\gamma(x)) \subset \gamma(f^{t_i}(x))$ and $f^{t_i}(\Gamma(x)) \supset \Gamma(f^{t_j}(x))$);
- for each n, there are at most finitely many i with $t_i = n$;
- $\min\{t_i\} \geq$ some t_0 depending only on f.

Let us define Λ_i and t_i in our context. First, we consider the partition $\mathcal{R}$: $\{R_0, \Lambda \setminus R_0\}$. For $j > 0$, $j \in \mathbb{N}$, we say that two local stable manifolds $\gamma, \tilde{\gamma}$ belong to the same $\mathcal{A}' - j$-cylinder if they have some points $x \in \gamma \cap \mathcal{A}$ and $y \in \tilde{\gamma} \cap \mathcal{A}$ that stay together (visit the same elements of the partition $\mathcal{R}$) for j iterates. Sometimes in this case, we say that γ and $\tilde{\gamma}$ *stay together* for j iterates.

For each local stable manifold $\gamma \in \mathcal{F}^s$, we define its *(first) return time*, or simply its *return time* to be the first positive time $j_\gamma \in \mathbb{N}$ such that some $x \in \gamma \cap \mathcal{A}'$ reaches (or returns to) $\mathcal{A}$. The *kth return time* of γ is defined in the same way. Note that if j_γ is the return time of γ, then $f^{j_\gamma}(\gamma)$ is contained in R_1.

We then divide $\mathcal{A}'$ into an infinite partition

$$\mathcal{A}' = \sum_{j\in\mathbb{N}} \mathcal{A}'_j$$

of s-subsets $\mathcal{A}'_j$. Each $\mathcal{A}'_j$ is simply the union of the local stable manifolds (intersected with $\mathcal{A}'$) whose first return time is j.

Note that the expansion in the unstable direction and the contraction in the center-stable direction for points in $\mathcal{A}'$ (see Proposition 2.23) will guarantee $f^j(\gamma(x)) \subset \gamma(f^j(x))$ and $f^j(\Gamma(x)) \supset \Gamma(f^j(x))$ for $x \in \mathcal{A}'_j$ and $j \geq t_0$, for some t_0. In fact, if the lower bound t_0

was equal to one, we would just take the collection $\{\Lambda_i\}$ as a re-indexing of the collection $\{\mathcal{A}'_j \cap C_{j,m}\colon C_{j,m}$ is an $\mathcal{A}' - j$-cylinder$\}$. In this case, if Λ_i is contained in some $\mathcal{A}'_j$, we just put $t_i = j$. If $t_0 > 1$, for each $j < t_0$ we subdivide each $\mathcal{A}'_j$ into s-subsets $\mathcal{A}'_{j,k}$. Each $\mathcal{A}'_{j,k}$ consists of points $x \in \mathcal{A}'_{j,k}$ such that for some $l > 0$ minimal, the lth return time of $\gamma(x)$ is $k \geq t_0$. In this case, if $\Lambda_i \subset \mathcal{A}'_{j,k}$, we put $t_i = k$.

So, it suffices to take the collection $\{\Lambda_i\}$ to be a renumbering of

$$\{\mathcal{A}'_j \cap C_{j,m} : j \geq t_0, C_{j,m} \text{ is an } \mathcal{A}' - j\text{-cylinder}\} \\ \cup \{\mathcal{A}'_{j,k} \cap C_{k,m} : j < t_0, C_{k,m} \text{ is an } \mathcal{A}' - k\text{-cylinder}\}.$$

(P3) There is a partition $\mathcal{P}$ of $\cup f^n(\mathcal{A}')$ to which we can define the *separation time* between two points in $\mathcal{A}'$; that is, the time their images stay together in the same rectangle of such a partition. Such a *separation time* $s_0(\cdot,\cdot)$ has the following properties:

(i) $s_0(\cdot,\cdot) \geq 0$ and depends only on γ-disks containing the two points;

(ii) the number of 'distinguishable' n-orbits starting from $\mathcal{A}$ is finite for each n;

(iii) for $x, y \in \Lambda_i$, $s_0(x, y) \geq t_i + s_0(f^{t_i}(x), f^{t_i}(y))$.

In fact, just take $\mathcal{P}$ as the restriction of $\mathcal{R}$ to Λ. Then, given $x, y \in \mathcal{A}'$, $s_0(x, y)$ will just be the time that the two manifolds $\gamma(x)$, $\tilde{\gamma}(x)$ stay together (see the definition in the last item) with respect to $\mathcal{R}$ (or $\mathcal{P}$) and it is, by definition, non-negative. As points in the same γ stay together forever, s_0 is constant in γ. This confirms property (i). If we fix n, the number of 'distinguishable' n-orbits starting from $\mathcal{A}$ is just the number of n-cylinders that intersect $\mathcal{A}$. Property (iii) is a trivial consequence of the definition of Λ_i.

We saw that there exist $C' > 0$, $0 < \varsigma < 1$, $0 < \lambda_u < 1$ such that the following hold for all $x, y \in \mathcal{A}$.

(P4) We have contraction along γ-disks. For $y \in \gamma(x)$ and for some $0 < \tilde{\varsigma} < 1$, the equation

$$d(f^n(x), f^n(y)) \leq C'\tilde{\varsigma}^n, \quad \text{for all } n \geq 0,$$

holds.

We have this by the construction of $\mathcal{A}$ and the tower. By Proposition 2.6, we have $\|Df^n(z)|E^{cs}\| \leq \varsigma^n$, for $z \in \mathcal{A}$ and for all $n > 0$. Corollary 2.24 gives exactly the relation in (P4), with $\tilde{\varsigma} := \sqrt{\varsigma}$.

(P5) Backward contraction and distortion along Γ. For $y \in \Gamma(x)$ and $0 \leq k \leq n < s_0(x, y)$, we have:

(a) $d(f^n(x), f^n(y)) \leq C \cdot \lambda_u^{s_0(x,y)-n}$;

(b)

$$\log \prod_{i=k}^{n} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C \cdot \lambda_u^{s_0(x,y)-n}.$$

Item (a) is due the fact that x, y belong to the same $\mathcal{A}' - s_0(x, y)$-cylinder, and the same unstable leaf. The proof of (b) is similar to the proof of Lemma 2.4.

(P6) Convergence of $D(f^i|\Gamma)$ and absolute continuity of Γ.

(a) There exist $0 < \alpha < 1$ and $C'' > 0$ such that for $y \in \gamma(x)$,

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C'' \cdot \alpha^n, \quad \text{for all } n \geq 0.$$

(b) For $\Gamma, \Gamma' \in \mathcal{F}^u$, if $\Theta : \Gamma \cap \mathcal{A} \to \Gamma' \cap \mathcal{A}$ is defined by $\Theta(x) = \gamma(x) \cap \Gamma'$, then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1} m'_\Gamma)}{dm_\Gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta(x)))}.$$

Again, the proof of (P6)(a) looks like the proof of Lemma 2.4. Taking x and y in the same γ, their distance is (uniformly) bounded by the supremum of the diameters of the center-stable leaves, which we will call d_s. So we have

$$\begin{aligned}
&\log \prod_{i=n}^{\infty} \frac{|\det(Df^u)(f^i(x))|}{|\det(Df^u)(f^i(y))|} \\
&\le \sum_{i=n}^{\infty} \big|\log|\det(Df^u(f^i(x)))| - \log|\det(Df^u)(f^i(y))|\big| \\
&\le \sum_{i=n}^{\infty} K_u \cdot d(f^i(x), f^i(y)) \le K_u \cdot C' \cdot \sum_{i=n}^{\infty} d_s \cdot \varsigma^i \le K_u \cdot C' \cdot d_s \cdot \alpha^n, \quad \text{(by (P4) above)}
\end{aligned}$$

where K_u is a uniform bound for the Lipschitz constant of Df^u. Property (P6)(b), in our context, is a consequence of (P3)–(P6)(a).

(P7) There exists $C_0' > 0$ and $\theta_0 < 1$ such that for some $\Gamma \in \mathcal{F}^u$,

$$m_\Gamma\{x \in \Gamma \cap \mathcal{A}' : t_i(x) > n\} < C_0' \theta_0^n, \quad n \ge 0.$$

The proof of this fact is a consequence of the proof of Proposition 2.7 and Corollary 2.8. By that proposition (and that corollary), we know that the set of points of Λ (and so $\mathcal{A}'$) that need a lot of time to reach $\mathcal{A}$ is an exponentially small (conditional) Lebesgue measure set. The is due to the fact that the set of points in Λ that need a lot of time to reach $\mathcal{A}$ is contained in the floors in the tower with very small (negative) subscripts and Proposition 2.7 tells us that such floors have exponentially small (conditional) Lebesgue measure. Therefore, proceeding as in (P2), we have that the property (P7) is valid for the subset $\mathcal{S}_1$ of points x in $\mathcal{A}'$ such that $t_i(x) > n$ is the *first* return time of the stable manifold $\gamma(x)$ to $\mathcal{A}$. It is easy to see that the same property is valid for the subset $\mathcal{S}_2$ of points x such that $t_i(x) > n$ is the *second* return time and so on, until, at most, $\mathcal{S}_{t_0}$. Since the number of sets $\mathcal{S}_-$ we have constructed is finite, we have that their union satisfies (P7), as we wanted to prove.

Finally, we have the following property.

(P8) μ_0 is the unique SRB measure of f (in Λ) and (f^n, μ_0) is ergodic for all $n \ge 1$.

This is an immediate consequence of Proposition 2.25 and [**4**], as we have recalled in the beginning of this section.

Having proved all these facts, Theorem A can be easily obtained from the following result of Young.

THEOREM 3.1. (Exponential decay of correlations [**16**]) *Let $f : M \to M$ be a C^2-diffeomorphism on a compact manifold M, admitting a subset $\mathcal{A} \subset M$ with properties (P1)–(P8). Then (f, μ_0) has an exponential decay of correlations in the space of ν-Holder continuous functions. That is, for $\nu > 0$, if we define*

$$\mathcal{H}_\nu := \{\varphi : M \to \mathbb{R} \mid \exists C \text{ s.t. } |\varphi(x) - \varphi(y)| \le C \cdot d(x, y)^\nu, \ \forall x,\ y \in M\},$$

then, for functions φ, $\psi \in \mathcal{H}_\nu$, there exist $K = K(\varphi, \psi)$ and $\tau = \tau(\nu) > 0$ such that

$$\left| \int \varphi(\psi \circ f^n)\, d\mu_0 - \int \psi\, d\mu_0 \int \varphi\, d\mu_0 \right| \leq K \cdot \tau^n.$$

Similarly, Theorem B follows from the following.

THEOREM 3.2. (Central limit theorem [**16**]) *Under the same hypotheses of the last theorem, every $\varphi \in \mathcal{H}_\nu$ with $\int \varphi\, d\mu_0$ satisfies the central limit theorem with respect to (f, μ_0), with $\sigma = 0$ if and only if $\varphi = \psi \circ f - \psi$, for some $\psi \in L^2(\mu_0)$.*

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REFERENCES

[1] J. F. Alves. SRB measures for nonhyperbolic systems with multidimensional expansion. *PhD Thesis*, IMPA, 1997.

[2] J. F. Alves, C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly expanding. *Invent. Math.* **140** (2000), 351–398.

[3] J. F. Alves, S. Luzzatto and V. Pinheiro. Markov structures and decay of correlations for non-uniformly expanding dynamical systems. Preprint, 2002.

[4] C. Bonatti and M. Viana. SRB measures for partially hyperbolic systems whose central direction is mostly contracting. *Israel J. Math.* **115** (2000) 157–193.

[5] M. Brin and Ya. Pesin. Partially hyperbolic dynamical systems. *Izv. Acad. Nauk. SSSR* **1** (1974), 177–212.

[6] M. Carvalho. Sinai-Ruelle-Bowen measures for n-dimensional derived from Anosov diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **13** (1993), 21–44.

[7] A. A. Castro. Backward inducing and exponential decay of correlations for partially hyperbolic attractors with mostly contracting central direction. *PhD Thesis*, IMPA, 1998.

[8] A. A. Castro. Backward inducing and exponential decay of correlations for partially hyperbolic attractors. *Israel J. Math.* **130** (2002), 29–75.

[9] D. Dolgopyat. On dynamics of mostly contracting diffeomorphisms. *Comm. Math. Phys.* **213** (2000), 181–201.

[10] M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, Berlin, 1977.

[11] R. Mañé. *Ergodic Theory and Differentiable Dynamics*. Springer, Berlin, 1987.

[12] Ya. Pesin and Ya. Sinai. Gibbs measures for partially hyperbolic attractors. *Ergod. Th. & Dynam. Sys.* **2** (1982), 417–438.

[13] C. Pugh and M. Shub. Ergodic attractors. *Trans. Amer. Math. Soc.* **312** (1989), 1–54.

[14] M. Shub. *Global Stability of Dynamical Systems*. Springer, Berlin, 1987.

[15] M. Viana. *Stochastic Dynamics of Deterministic Systems (Lecture Notes XXI Braz. Math. Colloq.)*. IMPA, Rio de Janeiro, 1997.

[16] L.-S. Young. Statistical properties of dynamical systems with some hyperbolicity. *Ann. Math.* **147** (1998), 585–650.