Entropy of entangled states and SU(1,1) and SU(2) symmetries

A. E. Santana,[1,2] F. C. Khanna,[2,3] and M. Revzen[2,4]
[1]*Instituto de Física, Universidade Federal da Bahia, Campus de Ondina, 40210-340 Salvador, Bahia, Brazil*
[2]*Physics Department, Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2J1*
[3]*TRIUMF, 4004, Wesbrook Mall, Vancouver, BC, Canada V6T 2A3*
[4]*Department of Physics, Technion, Haifa 32000, Israel*
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Based on a recent definition of a measure for entanglement [Plenio and Vedral, Contemp. Phys. **39**, 431 (1998)], examples of maximum entangled states are presented. The construction of such states, which have symmetry SU(1,1) and SU(2), follows the guidance of thermofield dynamics formalism.

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I. INTRODUCTION

Due to the structure of the Hilbert space as well as the superposition principle, quantum mechanics gives rise to the notion of entangled states, which are states of two or more systems correlated with each other, but without a classical analog. In particular, this entanglement may have nonlocal features. Bell [1,2] was the first to present a systematic way to analyze such entangled states, by comparing such correlations to the classical correlated states, that are defined via classical probability distributions.

A renewed interest in entanglement arose because it can be used for teleporting quantum states, from one locus to another, which is a basic ingredient in quantum computers [3,4]. In order to progress with such a program of quantum communication, the measure for entanglement is a crucial aspect of the theory that should be fully developed. And this aspect has been approached in several different methods [5–7].

The conditions for teleporting require specific entangled states characterized by a maximum entanglement. In this sense, Barnett and Phoenix [8] and Plenio and Vedral [9], using the notion of thermal-like states with quantum correlations (see also the paper by Ekert and Knight [10] for a detailed account of two-mode squeezed states and thermal-like states), have suggested that a consistent measure of the entanglement of two subsystems, say A and B, described by a pure state $|\psi(A,B)\rangle$, is the entropy of the reduced density operator ρ_A, where

$$\rho_A = \mathrm{Tr}_B(|\psi(A,B)\rangle\langle\psi(A,B)|), \tag{1}$$

with Tr_B standing for the trace over the coordinates of the subsystem B. In other words, a measure of entanglement of the state $|\psi(A,B)\rangle$ is given by

$$S(\rho_A) = -\mathrm{Tr}\rho_A \ln \rho_A . \tag{2}$$

Here our goal is to set forth examples of states that maximize the entanglement. We use as a guide the formalism of thermofield dynamics (TFD) [11–14], in order to build such (maximum entangled) states that will be associated with the SU(1,1) and SU(2) symmetries.

Let us go a bit further with the Barnett and Phoenix [8] and Plenio and Vedral [9] ideas. Notice that $S(\rho_A)$ is a homogeneous function of first degree in its dependency on E_A, the energy of the subsystem A. Then since we wish to maximize $S(\rho_A)$ we require $\delta S(E_A)=0$, under the constraints

$$E_A = \langle H_A\rangle = \mathrm{Tr}\rho_A H_A , \quad \mathrm{Tr}\rho_A = 1, \tag{3}$$

where H_A is the energy operator of systems A. Following methods similar to those of statistical mechanics, we derive then a constraint equation for ρ_A, that is,

$$\alpha_o - 1 + \alpha_1 H_A + \ln \rho_A = 0, \tag{4}$$

where α_o and α_1 are the Lagrange multipliers attached to the given constraints. Using Eq. (4) we get a Gibbs-like density operator, that is,

$$\rho_A = \frac{1}{Z}\exp(\alpha_1 H_A), \tag{5}$$

where $Z=\exp(1-\alpha_o)$. Multiplying Eq. (4) by ρ_A, taking the trace and using Eqs. (3) and (5), we derive

$$\ln Z + \alpha_1 E_A + S = 0.$$

For the sake of convenience, let us write $\alpha_1 = -1/\tau$, then we have $\tau \ln Z = E_A - \tau S$. The function $F(\tau)=\tau \ln Z$ describes the Legendre transform of S since we assume that $\tau = \partial E/\partial S$. Here τ is an intensive parameter describing the fact that the average $E_A=\langle H_A\rangle$, given by Eq. (3), is constant in the state described by ρ_A. Although the fluctuations can exist, these are not a result of any heat bath (or ensemble of states) but rather *a consequence of the entanglement* of the state A with B. Therefore, now we are in a position to look for an entangled state $|\psi(A,B)\rangle$ such that the corresponding reduced matrix as defined in Eq. (1) is explicitly given by Eq. (5), such that

$$Z = \mathrm{Tr}\exp(-\tau H_A). \tag{6}$$

In this way we show in the following that, using the scheme of TFD for the SU(1,1) and SU(2) symmetries, we can explicitly construct examples of maximum entanglement states, such that the measurement of the entanglement is given by Eq. (2).

II. MAXIMUM ENTANGLED STATES AND SU(1,1) SYMMETRY

Let us consider a two-mode bosonic system described by bosonic operators $a,a^\dagger,b$, and $b^\dagger$ satisfying the algebraic relation

$$[b,b^\dagger]=[a,a^\dagger]=1, \tag{7}$$

$$[a,b]=[a,b^\dagger]=[a^\dagger,b]=[a^\dagger,b^\dagger]=0. \tag{8}$$

Following the TFD approach [11,12], with these two-boson operator, we can construct a two-mode linear canonical transformation presenting an SU(1,1) symmetry. First, consider the following operators:

$$S_+=a^\dagger b^\dagger,$$

$$S_-=ab,$$

$$S_o=\frac{1}{2}(a^\dagger a+bb^\dagger),$$

which satisfy the su(1,1) algebra, namely, $[S_o,S_\pm]=S_\pm$, and $[S_+,S_-]=-S_o$. Introducing then a canonical transformation by

$$U(\gamma)=\exp[\gamma(S_+-S_-)],$$

with the vacuum $|0_a,0_b\rangle=|0_a\rangle\otimes|0_b\rangle$, such that $a|0_a\rangle=b|0_b\rangle=0$, we have

$$|0(\gamma)\rangle=\exp[\gamma(S_+-S_-)]|0_a,0_b\rangle, \tag{9}$$

where γ is a parameter to be specified later. The canonical nature of $U(\gamma)$ maintains the invariance of the algebra given by Eqs. (7) and (8) for the transformed operators given by

$$a(\gamma)=U(\gamma)aU(\gamma),\quad a^\dagger(\gamma)=U(\gamma)a^\dagger U(\gamma),$$

$$b(\gamma)=U(\gamma)bU(\gamma),\quad b^\dagger(\gamma)=U(\gamma)b^\dagger U(\gamma).$$

.

Here we use the notation $|0(\gamma)\rangle=|\psi(A,B)\rangle$ in order to emphasize that we have two bosonic systems, such that A describes the degrees of freedom of the bosonic operators a and $a^\dagger$, and B the operators b and $b^\dagger$. It is convenient to write Eq. (9) by

$$\begin{aligned}|\psi(A,B)\rangle&=\exp[\tanh\gamma a^\dagger b^\dagger]\exp[-\ln\cosh\gamma(bb^\dagger\\&\quad+a^\dagger a)]\exp[\tanh\gamma(-ba)]|0_a,0_b\rangle\\&=\exp(-\ln\cosh\gamma)\sum_m(-\tanh\gamma)^m\\&\quad\times\frac{1}{m!}(a^\dagger b^\dagger)^m|0_a,0_b\rangle.\end{aligned} \tag{10}$$

Following the scheme delineated in the introduction, we show that the state $|\psi(A,B)\rangle$ is a maximum entangled state.

Consider $\rho(A,B)=|\psi(A,B)\rangle\langle\psi(A,B)|$ and take the trace in the B variables, that is,

$$\begin{aligned}\rho_A&=\mathrm{Tr}_B[|\psi(A,B)\rangle\langle\psi(A,B)|]\\&=\frac{1}{(\cosh\gamma)^2}\sum_{m,n}\sum_l\frac{1}{m!n!}\\&\quad\times(-\tanh\gamma)^{m+n}(a^+)^n|0_a\rangle\langle 0_a|\\&\quad\times a^m\langle l|(b^\dagger)^n|0_b\rangle\langle 0_b|b^m|0_b\rangle\\&=\frac{1}{(\cosh\gamma)^2}\sum_m(-\tanh\gamma)^{2m}|m\rangle\langle m|.\end{aligned}$$

Define

$$\cosh\gamma(\tau)=\frac{1}{(1-e^{-\tau w})^{1/2}},$$

$$\tanh\gamma(\tau)=e^{-\tau w/2},$$

such that

$$\rho_A=(1-e^{-\tau w})\sum_m e^{-\tau wm}|m\rangle\langle m|.$$

This expression can be written in the canonical Gibbs ensemble form by defining $H_A=wa^+a$ and $Z(\tau)=\mathrm{Tr}e^{-\tau H_A}=(1-e^{-\tau w})^{-1}$. Then we have

$$\rho_A=\frac{1}{Z(\tau)}e^{-\tau H_A},$$

showing that the state $|\psi(A,B)\rangle=\exp[\gamma(S_+-S_-)]|0_a,0_b\rangle$ is a maximum entangled state with symmetry SU(1,1). In the following section we discuss a situation of an entangled state with SU(2) symmetry.

III. MAXIMUM ENTANGLED STATES AND SU(2) SYMMETRY

In order to construct a state of two systems A and B of maximum entanglement with SU(2) symmetry, we use the two-boson Schwinger representation for the su(2) Lie algebra, given [14,15]

$$S_+=a_1^\dagger a_2, \tag{11}$$

$$S_-=a_2^\dagger a_1, \tag{12}$$

$$S_o=\frac{1}{2}(a_1^\dagger a_1-a_2^\dagger a_2), \tag{13}$$

where a_1 and a_2 are two bosonic operators. Then we have

$$[S_o,S_\pm]=\pm S_\pm, \tag{14}$$

$$[S_+,S_-]=2S_o. \tag{15}$$

With these two bosons [fulfilling the same algebra as that given in Eqs. (7) and (8)] the number operators

$$N_1=a_1^\dagger a_1,\quad N_2=a_2^\dagger a_2, \tag{16}$$

have the spectrum

$$N_1|n_1,n_2\rangle=n_1|n_1,n_2\rangle,$$

$$N_2|n_1,n_2\rangle=n_2|n_1,n_2\rangle,$$

where

$$|n_1,n_2\rangle=\frac{1}{(n_1!n_2!)^{1/2}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0,0\rangle.$$

Therefore, the operators defined in Eqs. (11) and (12) are such that

$$S_+|n_1,n_2\rangle=[n_2(n_1+1)]^{1/2}|n_1+1,n_2-1\rangle, \tag{17}$$

$$S_-|n_1,n_2\rangle=[n_1(n_2+1)]^{1/2}|n_1-1,n_2+1\rangle, \tag{18}$$

$$S_o|n_1,n_2\rangle=\frac{1}{2}(n_1-n_2)|n_1,n_2\rangle. \tag{19}$$

Observe that

$$S_-|n_1,n_2\rangle=S_-|n_1=0,n_2=1\rangle=0, \tag{20}$$

that is, the state $|n_1=0,n_2=1\rangle=|0_a\rangle$ is the vacuum state for S_-. The connection with the original su(2) algebra and the value of spin is obtained if we define

$$n_1=s+m, \tag{21}$$

$$n_2=s-m, \tag{22}$$

where s and m are related to the usual results

$$\mathbf{s}^2|s,m\rangle=s(s+1)|s,m\rangle,$$

$$s_o|s,m\rangle=m|s,m\rangle.$$

Let us study now the particular case of $s=1/2$ and $m=1/2,\ -1/2$. Or in terms of the two-boson spectrum $n_1=0,1$ and $n_2=0,1$. Therefore, the action of S_+ and S_- on such states is

$$S_+|s,m\rangle=S_+|n_1,n_2\rangle=S_+\frac{1}{(n_1!n_2!)^{1/2}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0\rangle|0\rangle, \tag{23}$$

$$S_-|s,m\rangle=S_-|n_1,n_2\rangle=S_-\frac{1}{(n_1!n_2!)^{1/2}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}|0\rangle|0\rangle, \tag{24}$$

such that, according to Eqs. (17) and (20), we have only two possibilities,

$$n_1=0,n_2=1\Rightarrow m=-1/2, \tag{25}$$

$$n_1=1,n_2=0\Rightarrow m=1/2. \tag{26}$$

Or in another way,

$$S_+|s=1/2,m=-1/2\rangle\equiv S_+|n_1=0,n_2=1\rangle=|1,0\rangle, \tag{27}$$

$$S_+|s=1/2,m=1/2\rangle\equiv S_+|n_1=1,n_2=0\rangle=0, \tag{28}$$

$$S_-|s=1/2,m=-1/2\rangle\equiv S_-|n_1=0,n_2=1\rangle=0, \tag{29}$$

$$S_-|s=1/2,m=1/2\rangle=S_-|n_1=1,n_2=0\rangle=|0,1\rangle, \tag{30}$$

where we have used Eqs. (17), (20), (23), and (24). Then, as we have observed before, the vacuum state for the spin system is $|0\rangle\equiv|0,1\rangle$.

In order to construct the entangled state following the TFD procedure, we analyze another system of two bosons, denoted by b_1 and b_2, such that these operators commute among themselves and also with the boson operator a_1 and a_2, giving rise to the doubling of the su(2) algebra. That is,

$$[S_o,S_\pm]=\pm S_\pm, \tag{31}$$

$$[S_+,S_-]=2S_o, \tag{32}$$

$$[\tilde{S}_o,\tilde{S}_\pm]=\pm\tilde{S}_\pm, \tag{33}$$

$$[\tilde{S}_+,\tilde{S}_-]=2\tilde{S}_o, \tag{34}$$

such that the tilde operators $\tilde{S}_-,\tilde{S}_+$, and $\tilde{S}_o$ commute with the nontilde operators and are now given by

$$\tilde{S}_+=b_1^\dagger b_2, \tag{35}$$

$$\tilde{S}_-=b_2^\dagger b_1, \tag{36}$$

$$\tilde{S}_o=\frac{1}{2}(b_1^\dagger b_1-b_2^\dagger b_2). \tag{37}$$

Consider the state $|\psi(A,B)\rangle$ given by

$$\begin{aligned}|\psi(A,B)\rangle&=\exp[\gamma(S_+\tilde{S}_++S_-\tilde{S}_-)]|0_a,0_b\rangle,\\&=\exp[\gamma(a_1^\dagger b_1^\dagger a_2b_2-a_2^\dagger b_2\dagger a_1b_1)]|0_a,0_b\rangle\\&=(\cos\gamma+\sin\gamma a_1^\dagger b_1^\dagger a_2b_2)|0_a,0_b\rangle,\end{aligned} \tag{38}$$

where A represents the degrees of freedom described by the variables S,B represents the other system described by the variables $\tilde{S}$, and $|0_a,0_b\rangle=|0_a\rangle\otimes|0_b\rangle=|0,1\rangle_a\otimes|0,1\rangle_b\equiv|0\rangle_{a_1}|1\rangle_{a_2}|0\rangle_{b_1}|1\rangle_{b_2}$. The quantity γ is an arbitrary constant to be specified. Define $\rho(A,B)=|\psi(A,B)\rangle\langle\psi(A,B)|$, and take the trace in the B variables, that is,

$$\rho_A = \mathrm{Tr}_B|\psi(A,B)\rangle\langle\psi(A,B)|$$

$$= \sum_{m,n} {}^{(b)}\langle m|_{b_1}\langle n|_{b_1}(\cos\gamma + \sin\gamma a_1^\dagger b_1^\dagger a_2 b_2)$$

$$\times|0\rangle_{a_1}|1\rangle_{a_2}|0\rangle_{b_1}|1\rangle_{b_2}\langle 1|_{b_2}\langle 0|_{b_1}\langle 0|_{a_2}\langle 1|_{a_1}$$

$$\times(\cos\gamma + \sin\gamma a_1^\dagger b_1^\dagger a_2 b_2)^\dagger|n\rangle_{b_1}|m\rangle_{b_2},$$

where the indices in the states as b_2 in $|1\rangle_{b_2}$ or $\langle 1|_{b_2}$, are used to specify the action of the different operators, so that $|1\rangle_{b_2} = b_2^\dagger|0\rangle_{b_2}$, and so on. With some algebric manipulation, we get,

$$\rho_A = \cos^2\gamma|0\rangle_{a_1}|1\rangle_{a_2}\langle 1|_{a_2}\langle 0|_{a_1} + \sin^2\gamma|1\rangle_{a_1}|0\rangle_{a_2}\langle 0|_{a_2}\langle 1|_{a_1}. \tag{39}$$

In the case of spin 1/2 we have $|0\rangle_{a_1}|1\rangle_{a_2} = |s=1/2,m=-1/2\rangle \equiv |-1/2\rangle$, and $|1\rangle_{a_1}|0\rangle_{a_2} = |s=1/2,m=1/2\rangle \equiv |1/2\rangle$. Defining

$$\cos\gamma = \frac{1}{\sqrt{1+e^{-\tau\omega}}}, \quad \sin\gamma = \frac{e^{-\tau\omega/2}}{\sqrt{1+e^{-\tau\omega}}},$$

Eq. (39) can be written as

$$\rho_A = \frac{1}{Z}e^{-\tau\omega S_o}\left|\frac{1}{2}\right\rangle\left\langle\frac{1}{2}\right| + \frac{1}{Z}e^{-\tau\omega S_o}\left|-\frac{1}{2}\right\rangle\left\langle-\frac{1}{2}\right|,$$

since the eigenvalue of S_o is $\pm\frac{1}{2}$. As $\mathrm{Tr}\rho_A = 1$ then $Z = e^{-\tau\omega/2} + e^{\tau\omega/2}$. Or still

$$\rho_A = \frac{1}{Z}\sum_{m=1/2,-1/2} e^{-\tau\omega S_o}|m\rangle\langle m|$$

$$= \frac{1}{Z}e^{-\tau\omega S_o}\sum_{m=1/2,-1/2}|m\rangle\langle m|$$

$$= \frac{1}{Z}e^{-\tau H_A},$$

where $H_A = \omega S_o$. Therefore, the state given by Eq. (38) is a maximally entangled state. The generalization for any value of spin is straightforward.

IV. ENTANGLEMENT OF SYSTEM WITH FIXED SPIN: SCHWINGER METHOD REVISITED

In the preceding section $|\psi(A=S,B=\tilde{S};\gamma)\rangle$ was used to describe a maximally entangled state. If we consider an arbitrary spin (arbitrary values for the number operators n_1 and n_2) value, $|\psi(A=S,B=\tilde{S};\gamma)\rangle$ is a maximum entangled state of two systems, each one with two bosons. However, for the system of two (defined) spin 1/2, for instance, we have the eigenvalues of the number operator as $n_1 = 0,1$ and $n_2 = 0,1$, which are no longer the spectrum of bosonic number operators, but rather fermioniclike operators. In such a situation the bosonic algebra does not describe physical bosons but works as auxiliary variables to treat the entanglement of two spin systems. Accordingly, since we define a fixed (not arbitrary) value for the spin, we have to analyze more closely the consequences of that, over the Schwinger representation that is usually introduced for arbitrary spin.

Returning to the Schwinger bosonic representation as presented in Sec. II, imposing the conditions on the spectrum of n_1 and n_2 as is the case of Eqs. (25) and (26) (and so defining s with a fixed value), then we are led to a situation of redefining the algebra of the auxiliary operators a_1 and a_2. Summarizing first our results, originally the operator a_1 and a_2 satisfy the bosonic algebra, say

$$[a_1, a_1^\dagger] = [a_2, a_2^\dagger] = 1, \tag{40}$$

$$[a_1, a_2] = [a_1, a_2^\dagger] = [a_1^\dagger, a_2] = [a_1^\dagger, a_2^\dagger] = 0. \tag{41}$$

In addition, for the spin 1/2, we have the subsidirary conditions (allowing the fixed value for the spin),

$$[a_1, a_1^\dagger]_+ = [a_2, a_2^\dagger]_+ = 1, \tag{42}$$

$$[a_1, a_2]_+ = [a_1, a_2^\dagger]_+ = [a_1^\dagger, a_2]_+ = [a_1^\dagger, a_2^\dagger]_+ = 0, \tag{43}$$

where $[,]_+$ stands for the anticommutator. A solution that fulfills all these conditions, Eqs. (40)–(43), is found by assuming the algebra for the operators a_i and $a_i^\dagger (i=1,2)$ to be

$$a_i a_i^\dagger = 1, \tag{44}$$

$$[N_i, a_i^\dagger] = a_i^\dagger, \tag{45}$$

$$[N_i, a_i] = -a_i, \tag{46}$$

with $a_i a_i = a_i^\dagger a_i^\dagger = 0$ and $n_i = a_i^\dagger a_i$. Indeed it is a simple matter to show that in this case $N_i = 0,1$, where $i = 1,2$.

For the case of spin 1, we consider the basic algebra given by Eqs. (44)–(46) with N_i given by

$$N_i = a_i^\dagger a_i + a_i^\dagger a_i^\dagger a_i a_i,$$

such that $a_i^\dagger a_i^\dagger a_i^\dagger = a_i a_i a_i = 0$, that is, three and higher monomials of $a_i^\dagger$ and a_i are zero. In this case we derive $n_i = 0,1,2$, and with Eqs. (21) and (22) we find $s=1$ and $m = -1,0,1$.

The above procedure can be generalized for an arbitrary but fixed value of spin. That is, for a spin s, such that $n_i = 2s$, we consider Eqs. (44)–(46) supplemented by a proper definition of N_i, which reads

$$N_i = \sum_{j=1}^{2s} (a_i^\dagger)^j (a_i)^j. \tag{47}$$

For the particular situation in which $s \to \infty$, we derive the approach of infinite statistics proposed by Greenberg [16], and Chung [17]. In the general situation presented here, it is worthy of noting that the algebra given in Eqs. (44)–(46) also satisfies a Hopf structure. Let us investigate this aspect closely.

Defining $a_1 = a \otimes I$, $a_2 = I \otimes a$, we can see that writing

$$\Delta^r(a)=1\otimes a,$$

$$\Delta^r(a^\dagger)=1\otimes a^\dagger,$$

$$\Delta^r(N)=N\otimes 1+1\otimes N,$$

$$\Delta^r(I)=I\otimes I,$$

where Δ^r provides a representation for the algebra given by Eqs. (44)–(46) for any value of s in Eq. (47). Actually this fact has been shown in the particular situation in which $s\to\infty$, in the context of the Greenberg operators [16,17]. A Hopf structure is introduced by using Δ^r as the coproduct and defining the counit ϵ and the antipode s by

$$\epsilon(I)=1,\quad \epsilon(a)=a,\quad \epsilon(a^\dagger)=a^\dagger,\quad \epsilon(N)=0, \tag{48}$$

$$s(I)=1,\quad s(a)=a,\quad s(a^\dagger)=a^\dagger,\quad s(N)=-N. \tag{49}$$

Another Hopf algebra is introduced, however, for the operators $a_1=a\otimes I$ and $a_1^\dagger=a^\dagger\otimes I$, defining the coproduct, say Δ^l, by

$$\Delta^l(a)=a\otimes 1,$$

$$\Delta^l(a^\dagger)=a^\dagger\otimes 1,$$

$$\Delta^l(N)=\Delta^r(N),$$

$$\Delta^l(I)=\Delta^r(I).$$

The counit ϵ and the antipode s are the same as those given in Eqs. (48) and (49). It is then a simple matter to verify all the Hopf-algebra axioms. Indeed, for the coproduct axiom we have $(i\otimes\Delta^l)\Delta^l(a)=(i\otimes\Delta^l)(a\otimes 1)=a\otimes 1\otimes 1=(\Delta^l\otimes i)(a\otimes 1)$, and so $(i\otimes\Delta^l)\Delta^l=(\Delta^l\otimes i)\Delta^l$. Following the same procedure we can verify the counit axiom $(\epsilon\otimes\Delta^l)\Delta^l=(\Delta^l\otimes\epsilon)\Delta^l$ as well as the antipode axiom $(i\otimes s)\Delta^l=(s\otimes i)\Delta^l$.

Closing this section let us emphasize, first, that a motivation for studying the Hopf-algebra structure attached to the entangled states with SU(2) symmetry is for the sake of experiments. Actually, in the way as the algebra given in Eqs. (44)–(46) has emerged in our formalism, the Hopf-algebra induced by that can be used to introduce a deformation parameter, say q, associated with SU(2) [18,19]. This q parameter in turn can be useful for fitting experimental results. Second any representation of the su(2) algebra can be decomposed in terms of operators a and $a^\dagger$ satisfying Eqs. (44)–(46) with N given in Eq. (47) for some fixed s, such a formalism can be used to describe not only fixed spin but also, for instance, isospin. In the case of pion, isospin 1, it would be thought as being composed of (at least, auxiliary) objects described by the operators $a_1, a_1^\dagger, a_2$, and $a_2^\dagger$. In this way we could find a physical interpretation for the Greenberg infinite-statistics approach [18].

V. CONCLUDING REMARKS

In this paper we have improved the Barnett and Phoenix [8] and Plenio and Vedral [9] definition for a measure of entanglement among states, based on a Liouville-von Neumann entropy. We then explore the similarity with the usual definition of entropy in statistical mechanics to construct states maximally entangled using the approach of thermofield dynamics. As TFD is a thermal formalism essentially founded on algebraic bases (duplication of the usual Hilbert space and Bogoliubov transformations), it has been used as a compass to give the proper direction to build maximum entangled states with a well-defined symmetry. We have studied the case of two bosons with SU(1,1) symmetry and four bosons (actually two systems, each one with two bosons) with symmetry SU(2). Still in the case of SU(2) symmetry, we have studied the entanglement of two systems with fixed value of spin, using a modified version of the two-boson Schwinger representation for the su(2) Lie algebra.

The usual way to describe a fixed value of spin using boson operators was proposed by Holstein and Primakof [20]. However, such a method works if we are interested in describing a system with spin via one bosonic operator. But this has not been the case here, since it has needed two operators associated to each spin to introduce the state of maximum entanglement through TFD. Obviously that for a finite spectrum, the couple of Schwinger operators loose the bosonic characteristic (up to now, an aspect not investigated in the literature, to the best of our knowledge), giving rise to a new algebra. What we have shown were some properties of the new operators.

With these results we can set forth some final conclusions: (i) the TDF states, seen as pure states, are naturally maximum entangled states; (ii) the SU(1,1) symmetry could be written in terms of a doubled bosonic algebra, such that it can be derived from elements of a coproduct given by $\Delta(a)=a\otimes 1+1\otimes a=(\Delta^l+\Delta^r)a$ and $\Delta(a^\dagger)=a^\dagger\otimes 1+1\otimes a^\dagger=(\Delta^l+\Delta^r)a^\dagger$ associated with the Weyl-Heisenberg algebra, induced from the bosonic algebra. This is similar to the case of the SU(2) symmetry. As a consequence, the structure of Hopf algebra is at the bottom of all the construction, since the left (Δ^l) and the right (Δ^r) parts of coproducts have been used to find specific representations for the SU(2) and SU(1,1) symmetries and also to the TDF states (which are here synonymous with the maximum entangled states).

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