JOURNAL OF ALGEBRA 143, 269–294 (1991)

Arithmetic of Normal Rees Algebras

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Communicated by Melvin Hochster

Received February 7, 1989

1. INTRODUCTION

Two reasons drive this study of the *arithmetical* properties of blow-up like algebras. First, one still lacks effective criteria for the Rees algebra of an ideal to be normal and, second, there are several new phenomena related to torsionfree symmetric algebras of modules—notably bundles—that need clarification.

As yet another related question, there is the description of the canonical modules of such algebras, a necessary step in determining their Gorenstein loci.

The main thread for these questions is the understanding of the divisor class group, essentially defined by a pair made up of a morphism and a free subgroup of finite rank. The behaviour of the mapping has puzzling

* Partially supported by CNPq, Brazil.
† Partially supported by the NSF.

0021-8693/91 $3.00

aspects. To deal with the subgroup, the notion of so-called **f**-number of a module (**f** for *Fitting*) is introduced.

The paper is organized as follows. The second section reviews earlier results and contains a novel characterization of normality in terms of the primary decomposition of the exceptional ideal. This constitutes a criterion which is no less and no more effective than are the means of calculating primary decompositions of ideals and symbolic powers of prime ideals. It is also established that, for a normal ideal I, the divisor class group of the Rees algebra $R[It]$ is an extension of that of the extended Rees algebra $R[It, t^{-1}]$ by a free group of rank one, and a condition is given, in terms of the format of the above primary decomposition, for this extension to be split. One solves in principle the question of when the aformentioned free subgroup of the divisor class group of the Rees algebra $R[It]$ is a direct summand. Based on this criterion, one recovers the main splitting result proved in [26]. It suggests that, in case I is the maximal ideal of a normal local domain R with torsion divisor class group, the free exceptional subgroup is a direct summand only if $\mathrm{Cl}(R)$ is trivial.

The third section stresses the role of modules in the theory, through the aforementioned **f**-number. Even in the case of ideals, to the best of our knowledge, computing this invariant is the only way to get the rank of the free subgroup. This computation is actualy effective, but may get stalled as it depends on calculating codimensions of determinantal ideals whose complexity build up very fast. Also, the usefulness of the **f**-number is restricted to modules of *analytic type*—a generalization of the situation in which the *residual* scheme coincides with the actual blow-up.

In the next section, one rounds up some of the results in an earlier work [13]. Here, the main tool is a theorem inspired by [5] that allows one to read the canonical module of the extended Rees algebra off that of the (ordinary) Rees algebra. As a byproduct, one can show that, for an ideal I of codimension at least two in a regular local ring R, such that $R[It]$ is Cohen–Macaulay and $gr_I R$ is reduced, the following statements are equivalent: (1) $gr_I R$ is Gorenstein; (2) I is unmixed. A testing ground for the theory, that of *small* Rees algebras (algebras over rings of dimension two or ideals generated by two or three elements) is considered last.

2. Divisor Class Group of Blow-ups

We assemble first some material on normal Rees algebras. Throughout rings are noetherian and unspecified modules are finitely generated. For general references on terminology and basic results, we shall use [21] and [25].

2.1. *Review of Normal Ideals*

An elementary but basic result in the theory of integral domains [21, Theorem 53] can be stated as follows. If A is a domain and $x \in A$ then

$$A = A_x \cap \left(\bigcap_P A_P \right),$$

where P runs through the associated primes of $A/(x)$. This is particularly useful in questions related to integral closure. Recall that the *integral closure* of an ideal $I \subset A$ is the. st of all elements $a \in A$ satisfying a monic equation

$$a^n + b_1 a^{n-1} + \cdots + b_n = 0,$$

with $b_i \in I^i$. The ideal I is said to be *integrally closed* provided it coincides with its integral closure. If all the powers of I are integrally closed, it is said to be *normal.*

LEMMA 2.1.1. *Let A be a domain and let $x \in A$ be such that A_x is integrally closed. The following conditions are equivalent*:

(i) *A is integrally closed.*

(ii) *A_P is integrally closed for every $P \in \operatorname{Ass} A/(x)$.*

(iii) *(x) is an integrally closed ideal.*

The proof is straightforward.

PROPOSITION 2.1.2. *Let R be a noetherian normal domain and let $I \subset R$ be an ideal. The following conditions are equivalent*:

(i) *The Rees algebra $R[It]$ is normal.*

(ii) *I is normal.*

(iii) *The ideal $IR[It] \subset R[It]$ is integrally closed.*

(iv) *The ideal $(t^{-1}) \subset R[It, t^{-1}]$ is integrally closed.*

(v) *$R[It, t^{-1}]_P$ is normal for every $P \in \operatorname{Ass} gr_I(R)$.*

(vi) *$R[It, t^{-1}]$ is normal.*

Proof. The implications (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), and (vi) $\Rightarrow$ (i) are easy or well known. The equvalence among conditions (iv), (v), and (vi) follows immediately from the preceding lemma. Finally, the implication (iii) $\Rightarrow$ (iv) is a direct calculation using the definition. ∎

The following result does not seem to have been noticed before.

PROPOSITION 2.1.3. *Let R be a normal domain and let $I \subset R$ be an ideal. The following conditions are equivalent:*

(i) *There is a primary decomposition*

$$IR[It] = P_1^{(l_1)} \cap \cdots \cap P_r^{(l_r)},$$

*where P_i is a height one prime ideal of $R[It]$ and $P_i^{(l_i)}$ stands for its l_i*th *symbolic power.*

(ii) *I is normal.*

Proof. Set $S := R[It]$ and $T := R[It, t^{-1}]$. Note the one-to-one correspondence between the height one primes of S containing I and the height one primes of T containing t^{-1} given by $P \mapsto (P, t^{-1})$ and $Q \mapsto Q \cap S$. In particular, for any such P, $S_P = T_{(P, t^{-1})}$. Moreover, if I admits a primary decomposition as above then $t^{-1}\,T$ admits a similar decomposition

$$t^{-1}T = (P_1, t^{-1})^{(l_1)} \cap \cdots \cap (P_r, t^{-1})^{(l_r)}.$$

It follows that $(P_i)_{P_i}^{l_i}$ is a principal ideal in the one-dimensional local ring S_{P_i}. By the theory of reduction, this forces $(P_i)_{P_i}$ to be principal. Now, for a height one prime $P \subset S$ not containing I, S_P is trivialy a DVR. We have thus proved that S satisfies the property (R_1).

To see that S satisfies the property (S_2), it suffices to show that $gr_I(R)$ satisfies (S_1) [4, Theorem 1.5]. But the associated primes of the latter are all minimal by assumption. ∎

2.2. *Exact Sequences of Divisor Class Group of Blow-ups*

Let R be a noetherian ring and $I \subset R$ an ideal for which the Rees algebra $R[It]$ is a normal domain. Then R and $R[It, t^{-1}]$ are normal and there are exact sequences

$$0 \to G \to \mathrm{Cl}(R[It]) \to \mathrm{Cl}(R) \to 0$$

and

$$0 \to G^{-1} \to \mathrm{Cl}(R[It, t^{-1}]) \to \mathrm{Cl}(R) \to 0,$$

where G is freely generated by the classes of height one prime ideals $P \subset R[It]$ containing $IR[It]$ and such that $P \cap R$ has height at least two, and G^{-1} has a similar description. If I has height at least two and, say, $\mathrm{Cl}(R) = 0$, then $\mathrm{Cl}(R[It]) \simeq \mathbf{Z}^r$ and $\mathrm{Cl}(R[It, t^{-1}]) \simeq \mathbf{Z}^r/\mathbf{Z}(l_1, ..., l_r)$ in a natural way, where $l_1, ..., l_r$ are the (symbolic) exponents in the primary decompositon of $IR[It]$. Consequently, G^{-1} is a free group if and only if $(l_1, ..., l_r)$ is unimodular. This will be the case if I is a radical, generically complete intersection ideal.

The first of the two above sequences was named *fundamental exact sequence* in [26]. For later use, we record the definition of the above map

$$\mathrm{Cl}(R[It]) \to \mathrm{Cl}(R).$$

First one defines it on the level of divisors, to wit: let $\mathrm{Div}^R(R[It])$ stand for the subgroup of the divisor group $\mathrm{Div}(R[It])$ generated by the elements $[P]$ satisfying the property that $P \cap R \neq 0$. Now define a homomorphism

$$\mathrm{Div}^R(R[It]) \to \mathrm{Div}(R)$$

by the assignment

$$[P] \mapsto \begin{cases} [P \cap R] & \text{if } \ ht(P \cap R) = 1 \\ 0 & \text{if } \ ht(P \cap R) \geqslant 2. \end{cases}$$

There is an induced homomorphism at the level of divisor classes [26, Theorem 2.1] wich is the one we are looking for.

For handy reference, we quote the following two results.

THEOREM 2.2.1 [26]. *Let R be a normal domain which is a factor of a Gorenstein local ring and let $I \subset R$ be an ideal of height at least two such that:*

(i) *I is radical*

(ii) *I is generically a complete intersection*

(iii) *$gr_I(R)$ is torsionfree over R/I.*

Then $R[It]$ is normal and

$$\mathrm{cl}(\Omega) = \mathrm{cl}(\omega) - \sum_P (ht(PR) - 2)\, \mathrm{cl}(P),$$

where Ω (resp. ω) stands for the canonical module of $R[It]$ (resp. R) and P runs through the prime divisors of the exceptional ideal $IR[It]$.

THEOREM 2.2.2 [19]. *Let R be a regular local ring and let $I \subset R$ be an ideal of height at least two such that R/I is reduced. Then the following are equivalent:*

(a) *$R[It]$ is normal and $\mathrm{Cl}(R[It]) \simeq \mathbf{Z}^r$, where r is the number of associated primes of R/I.*

(b) *$gr_I(R)$ is (R/I)-torsionfree.*

(c) *$gr_I(R)$ is reduced.*

Another useful exact sequence is stated in the following result.

PROPOSITION 2.2.3. *Let R be a normal domain and let $I \subset R$ be an ideal of height at least two such that $R[It]$ is normal. Then the canonical inclusion $R[It] \subset R[It, t^{-1}]$ induces an exact sequence*

$$0 \to \mathbf{Z}\,\mathrm{cl}(IR[It]) \to \mathrm{Cl}(R[It]) \to \mathrm{Cl}(R[It, t^{-1}]) \to 0.$$

If the exponents in the primary decomposition of $IR[It]$ form a unimodular row, and the fundamental sequence splits, then the preceding sequence splits.

Proof.[1] Set $A := R[It]$ and $S := R[It, t^{-1}]$. Then S is the subintersection of A taken over all the discrete valuation rings of A except the one defined by ItA. Indeed, this subintersection coincides with the ideal transform $T(ItA, A)$, hence contains S. On the other hand, since $ht(I) \geqslant 2$ and S is normal, ItS has grade at least two and, therefore, $T(ItS, S) = S$. From the inclusions $S \subset T(ItA, A) \subset T(ItS, S)$ we obtain the claim.

The exact sequence now follows from Nagata's theorem [9, Theorem 7.1]. It may be worth observing that the obtained map $\mathrm{Cl}(A) \to \mathrm{Cl}(S)$ is explicitly given by the assignment $\mathrm{cl}(P) \mapsto \mathrm{cl}(P, t^{-1})$. The second statement follows easily from earlier remarks. ∎

2.3. *The Splitting of the Fundamental Sequence*

We consider the question as to when the fundamental sequence of divisor class group splits, a problem first considered in [15] and later, in [26]. As it turns out, the question depends on the existence of certain integer-valued homomorphisms from $\mathrm{Div}(R)$ with a special commuting property.

For a prime ideal Q in a noetherian ring A and an element $x \in A$, $x \neq 0$, $v_Q(x)$ will denote the unique integer n such that $x \in Q_Q^n \setminus Q_Q^{n+1}$. $\mathscr{P}(A)$ denotes the set of height one prime ideals of A.

THEOREM 2.3.1. *Let R be a normal domain, $I \subset R$ a normal ideal of height at least two, and $P_1, \ldots, P_r \subset R[It]$ the height one primes containing $IR[It]$. Set $\mathbf{p}_i = P_i \cap R$ for $1 \leqslant i \leqslant r$. Then the following conditions are equivalent*:

(a) *The fundamental sequence of divisor class groups splits.*

(b) *For every $i = 1, \ldots, r$ there exists a map $\pi_i\colon \mathscr{P}(R) \to \mathbf{Z}$ such that*

$$v_{P_i}(x) = \sum_{p \in \mathscr{P}(R)} v_p(x)\pi_i(p)$$

for all $0 \neq x \in R$. Suppose, moreover, that $gr_I(R)$ is reduced and that $\mathbf{p}_i \not\subset \mathbf{p}_j$ for all $i \neq j$. Then, in condition (b) *one can replace v_{P_i} by $v_{\mathbf{p}_i}$.*

[1] We thank the referee for pointing out an alternative argument to our original proof.

Proof. We first prove the equivalence of conditions (a) and (b). The argument will be split into two claims.

Claim 1. The subgroup $\mathrm{Div}^R(R[It]) \subset \mathrm{Div}(R[It])$ is freely generated by the divisors of prime ideals associated to $gr_I(R)$ and of prime ideals of the form $pR[It]$ for some $p \in \mathscr{P}(R)$.

By definition, $\mathrm{Div}^R(R[It])$ is freely generated by the divisors of primes associated to $gr_I(R)$ and of height one primes not containing I and having a nonzero contraction in R. On the other hand, for every $p \in \mathscr{P}(R)$, there exists a unique prime $\wp \subset R[It]$ lying over p and, moreover $\wp$ is unramified. It follows that $[pR[It]] = [\wp] + D$, where D is a linear combination of exceptional prime divisors. Therefore, we obtain the required basis by effecting an elementary transformation of the preceding one.

Claim 2. The fundamental sequence splits if and only if there exists a homomorphism

$$\sigma\colon \mathrm{Div}^R(R[It]) \to \sum_{i=1}^{r} \mathbf{Z}[P_i]$$

such that $\sigma([P_i]) = [P_i]$ for every i and $\sigma([xR[It]]) = 0$ for every $x \in R$.

To say that the fundamental sequence splits amounts to have a section of the inclusion

$$\sum_{i=1}^{r} \mathbf{Z}[P_i] \subsetneq \mathrm{Cl}(R[It]).$$

Since $\mathrm{Cl}(R[It])$ is generated by the image of $\mathrm{Div}^R(R[It])$ in $\mathrm{Div}(R[It])$, the assertion is clear. Now, given $0 \neq x \in R$, one has (Claim 1)

$$[xR[It]] = \left(\sum_{i=1}^{r} v_{P_i}(x)[P_i]\right) + \sum_{p \in \mathscr{P}(R)} v_p(x)[pR[It]].$$

The conditions in Claim 2 then translate into

$$0 = \sigma([xR[It]]) = \left(\sum_{i=1}^{r} v_{P_i}(x)[P_i]\right) + \left(\sum_{p \in \mathscr{P}(R)} v_p(x)\left(\sum_{i=1}^{r} (-\pi_i(p))\,[P_i]\right)\right)$$

which yields the required equations.

For the second part, it suffices to show that $v_{P_i}(x) = v_{\mathbf{p}_i}(x)$ for all i and all $x \in R$. This will follow if one proves the equalities

$$R_{\mathbf{p}_i} \cap P_i^n R[It]_{P_i} = \mathbf{p}^n_{\mathbf{i}_{\mathbf{p}_i}}$$

for all i and all $n \geqslant 0$. Now, the assumptions imply that $I = \bigcap_{i=1}^{r} \mathbf{p}_i$, an irredundant decomposition. From this, $IR_{\mathbf{p}_i} = \mathbf{p}_i R_{\mathbf{p}_i}$ for all i. But $R[It]_{P_i}$ is a localization of the ring of fractions $R[It]_{R \setminus \mathbf{p}_i}$ and the latter is simply the Rees algebra of the maximal ideal of the local ring $R_{\mathbf{p}_i}$. Thus, one may assume that R is local with maximal ideal I. Next, the assumptions also imply that there is only one prime ideal P of height one in $R[It]$ contracting to I. Since $gr_I(R)$ is reduced, one must have $P = IR[It]$, so actually $gr_I(R)$ is a domain.

To conclude, let $x \in P^n R[It]_P \cap R$. Then, there exists a homogeneous element $g \in R[It] \setminus P$ such that $xg \in P^n$. Write $g = bt^i$, $b \in I^i \setminus I^{i+1}$, and let $x \in I^j \setminus I^{j+1}$. Then, since $gr_I(R)$ is a domain, $xb \in I^{i+j} \setminus I^{i+j+1}$, hence $xg \in I^{i+j}t^i \setminus I^{i+j+1}t^i$. It follows that $n \leqslant j$ and so, $x \in I^n$, as required. ∎

On the way to proving the theorem, we have obtained the following special case which deserves being singled out.

COROLLARY 2.3.2. *If R is normal and $gr_I(R)$ is a domain, then the fundamental sequence splits if and only if there exists a (single) map $\pi: \mathscr{P}(R) \to \mathbf{Z}$ such that*

$$v_I(x) = \sum_{p \in \mathscr{P}(R)} v_p(x)\, \pi(p)$$

for all $0 \neq x \in R$.

An important case where splitting occurs is the following.

COROLLARY 2.3.3 [26]. *Let R be a normal domain and let $I \subset R$ be a normal ideal of height at least two such that:*

(i) *I is a radical generically complete intersection ideal*

(ii) *$gr_I(R)$ is torsionfree.*

Then $R[It]$ is normal and $\mathrm{Cl}(R[It]) \simeq \mathrm{Cl}(R) \oplus \mathbf{Z}^r$, *where r is the number of minimal primes of R/I.*

Proof. The assumptions imply that $gr_I(R)$ is contained in a polynomial ring over a reduced base ring, hence must be reduced. Moreover, there is an induced bijective map $\mathrm{Ass}(gr_I(R)) \to \mathrm{Ass}(R/I)$. Thus, by the preceding corollary, it is enough to define a set of maps as above indexed by $\mathrm{Ass}(R/I)$. Set

$$\pi_{\mathbf{p}}: \mathscr{P}(R) \to \mathbf{Z},$$

one for each $\mathbf{p} \in \mathrm{Ass}(R/I)$, by assigning

$$\pi_{\mathbf{p}}(p) = \begin{cases} 0 & \text{if } p \not\subset \mathbf{p} \\ v_{\mathbf{p}}(y) & \text{otherwise,} \end{cases}$$

where y is a uniformizing parameter at p. ∎

EXAMPLE 2.3.4. Let $R := k[X, Y, Z]/(X^2 - YZ)$ and let $I := (x, y, z)$ (small letters denote residues modulo $(X^2 - YZ)$). Consider the height one prime $p := (x, z) \subset R$. Clearly, $p^{(2)} = (z)$, hence $v_p(z) = 2$. The obstruction condition yields here $1 = v_I(z) = v_p(z)\, \pi_I(p) = 2\pi_I(p)$, which is impossible. Therefore, the fundamental map $\mathrm{Cl}(R[It]) \to \mathrm{Cl}(R)$ does not split. Actually, it is easy to see that the fundamental sequence in this example is isomorphic to the exact sequence

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\cdot 2} \mathbf{Z} \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

EXAMPLE 2.3.5. Let $R := k[[X, Y, Z]]/(X^2 - FG)$, where F and G are distinct polynomials in Y, Z, of degree two and irreducible. Let $I := (x, y, z) \subset R$.

It is easy to see that R is a normal domain. Moreover, $(I, xt) \subset R[It]$ is the only exceptional prime divisor, hence, by [4], $R[It]$ is normal as well. Consider the height one primes $p := (x, f)$ and $q := (x, g)$ of R. Clearly, $p^{(2)} = (f)$, $q^{(2)} = (g)$, and $p \cap q = (x)$. The obstruction equations yield

$$2 = v_I(f) = v_p(f)\, \pi_I(p) = 2\pi_I(p)$$
$$2 = v_I(g) = v_q(g)\, \pi_I(q) = 2\pi_I(q)$$
$$1 = v_I(x) = v_p(x)\, \pi_I(p) + v_q(x)\pi_I(q).$$

As these are clearly incompatible, the fundamental map does not split.

One can derive similar obstruction equations for the splitting of the fundamental sequence in the case of a height one ideal. We consider one further instance.

EXAMPLE 2.3.6. Let again $R := k[X, Y, Z]/(X^2 - YZ)$, but this time consider the ideal $I := (x, z)$.

A straightforward computation shows that $gr_I(R)$ is reduced—so $R[It]$ is normal—and the exceptional prime divisors are $P := (I, zt)$ and $Q := (x, y, z)\, R[It]$. Since $P^{(2)} = (z, zt)$, it follows that $\mathrm{cl}(Q) = -2\, \mathrm{cl}(P)$ as elements of $\mathrm{Cl}(R[It])$. From this, one gets that $\mathrm{Cl}(R[It]) = \mathbf{Z}\, \mathrm{cl}(P) \simeq \mathbf{Z}$, hence the fundamental map cannot split. We note *en passant* that I is not normally torsionfree.

The obstruction equations impose rather strong conditions, not only on the ideal I, but on $\mathrm{Cl}(R)$ too, in order that the fundamental exact sequence split. Even more so in the case R is a local ring and $I := \mathbf{m}$, the maximal ideal of R. One is tempted to set forward the

Question. Let $R := k[[X_1, ..., X_n]]/J$ be a normal domain such that $R[\mathbf{m}t]$ is normal, where $\mathbf{m} := (X_1, ..., X_n)$. If $\mathrm{Cl}(R)$ is torsion and the fundamental map $\mathrm{Cl}(R[\mathbf{m}t]) \to \mathrm{Cl}(R)$ splits, is $\mathrm{Cl}(R) = 0$?

3. Modules of Linear Type

In this section, we consider the class of modules whose corresponding symmetric algebras are torsionfree. These modules are the natural analogue of ideals of linear type and, as it turns out, their number of generators is reasonably under control [14].

3.1. *The* **f**-*Number*

First recall a notion introduced in [14]: an R-module E with a rank is said to satisfy condition $\mathscr{F}_1$ if

$$\mu(E_{\mathbf{p}}) \leqslant ht(\mathbf{p}) + rk(E) - 1,$$

for every prime $\mathbf{p}$ such that $E_{\mathbf{p}}$ is not free. This condition can be further interpreted in terms of Fitting ideals, namely, let

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \longrightarrow 0$$

be a presentation of E. Then E satisfies $\mathscr{F}_1$ if and only if

$$ht(I_t(\varphi)) \geqslant rk(\varphi) - t + 2$$

for $1 \leqslant t \leqslant rk(\varphi)$.

Definition 3.1.1. Let E be a finitely generated module with a rank satisfying $\mathscr{F}_1$, with presentation as above. Let h_t denote the number of minimal primes $\mathbf{p}$ of $I_t(\varphi)$ such that $ht(\mathbf{p}) = rk(\varphi) - t + 2$. The **f**-*number* of E is the sum

$$\sum_{1}^{rk(\varphi)} h_t.$$

It is possible to show directly that the **f**-number does not depend on the chosen presentation. It is more useful, however, to recover it directly in terms of divisorial primes of the corresponding symmetric algebra.

The starting point is to identify some privileged height one primes in $S(E)$. The $\mathbf{p}$-*torsion* of $S(E)$, for a given prime $\mathbf{p} \subset R$ is the prime ideal

$$T(\mathbf{p}) := \ker(S_R(E) \to S_{R/\mathbf{p}R}(E/\mathbf{p}E)_{\mathbf{p}}).$$

Note that if, moreover, $S(E)$ is a domain then $T(\mathbf{p})$ is nothing but the contraction $\mathbf{p}S(E)_{\mathbf{p}} \cap S(E)$, so $ht(T(\mathbf{p})) = ht(\mathbf{p}S(E))$.

Assume, for the rest of this section, that R is universally catenarian in order to avoid dwelling into technicalities. In this case, if in addition $S(E)$ is equidimensional, one sees that

$$ht(T(\mathbf{p})) = 1 \qquad \text{if and only if} \quad \mu(E_{\mathbf{p}}) = ht(\mathbf{p}) + rk(E) - 1.$$

We also observe, for further considerations, that unless $\mathbf{p}$ is a height one prime itself, E is not free if $T(\mathbf{p})$ is to have height one.

PROPOSITION 3.1.2. *Let R be universally catenarian noetherian ring and let E be a finitely generated module such $S(E)$ is a domain. Then the set*

$$\{T(\mathbf{p}) \mid ht(\mathbf{p}) \geqslant 2 \text{ and } ht(T(\mathbf{p})) = 1\}$$

is finite. More precisely, for any presentation $G \xrightarrow{\varphi} F \to E \to 0$, this set is in bijection with

$$\{\mathbf{p} \subset R \mid E_{\mathbf{p}} \text{ not free}, \mathbf{p} \in \mathrm{Min}(R/I_t(\varphi)) \text{ and } ht(\mathbf{p}) = rk(\varphi) - t + 2\},$$

where $1 \leqslant t \leqslant rk(\varphi)$.

Proof. Given a prime $\mathbf{p} \subset R$, set $t := rk(F) - \mu(E_{\mathbf{p}})$. If $ht(\mathbf{p}) \geqslant 2$ and if $ht(T(\mathbf{p})) = 1$ then, since $rk(\varphi) = rk(F) - rk(E)$, by the preceding remarks we have

$$ht(\mathbf{p}) = rk(\varphi) - t + 2 \qquad \text{and} \qquad \mathbf{p} \supset I_t(\varphi) \backslash I_{t-1}(\varphi).$$

On the other hand, since $S(E)$ is a domain, E satisfies $\mathscr{F}_1$ [14]. Therefore, one has $ht(I_t(\varphi)) \geqslant rk(\varphi) - t + 2$. It follows that $\mathbf{p} \in \mathrm{Min}(R/I_t(\varphi))$.
The converse is similar. ∎

3.2. *Divisor Class Group of Symmetric Algebras*

Our main result expresses the divisor class group of a normal integral symmetric algebra as an extension of a finitely generated free (abelian) group and gives a condition for this extension to be split.

THEOREM 3.2.1. *Let R be a universally catenarian ring and let E be a finitely generated module such that $S(E)$ is a normal domain. Then:*

(i) *There is an exact sequence*

$$0 \to F \to \mathrm{Cl}(S(E)) \to \mathrm{Cl}(R) \to 0,$$

where F is a free (abelian) group of rank equal to the $\mathbf{f}$*-number of E.*

(ii) *The exact sequence in* (i) *splits if and only if there exist maps $\pi_{\mathbf{p}}: \mathscr{P}(R) \to \mathbf{Z}$, one for each $\mathbf{p} \in R$ with $ht(\mathbf{p}) \geqslant 2$ and $ht(T(\mathbf{p})) = 1$, such that*

$$v_{\mathbf{p}}(x) = \sum_{p \in \mathscr{P}(R)} v_p(x)\, \pi_{\mathbf{p}}(p)$$

for every $0 \neq x \in R$.

Proof. (i) We first observe that R is automatically normal. We use a well-known result on the generation of $\mathrm{Cl}(S(E))$ by height one primes whose contraction to R is nonzero [9, Proposition 10.2]. Thus, let $P \subset S(E)$ be such a prime and set $\mathbf{p} := P \cap R$. Then $\mathbf{p}S(E_{\mathbf{p}})$ is a nonzero prime of $S(E_{\mathbf{p}})$ contained in $PS(E_{\mathbf{p}})$, hence $\mathbf{p}S(E_{\mathbf{p}}) = PS(E_{\mathbf{p}})$. By definition, also $T(\mathbf{p})_{\mathbf{p}} = \mathbf{p}S(E_{\mathbf{p}})$. Therefore, $P = T(\mathbf{p})$.

Now, to deal with height one primes $\mathbf{p}$ obtained in this process, one defines a convenient map from $\mathrm{Cl}(S(E))$ to $\mathrm{Cl}(R)$. First we define a map on the level of prime divisors of $S(E)$ whose contraction to R is nonzero. We set, namely,

$$[P] \overset{\psi}{\longmapsto} \begin{cases} [P \cap R] & \text{if } ht(P \cap R) = 1 \\ 0 & \text{if } ht(P \cap R) \geqslant 2. \end{cases}$$

Embed $S(E)$ in a polyomial ring $R[\mathbf{t}]$ in a suitable set of indeterminates $\mathbf{t}$ in such a way that the fraction field of $S(E)$ becomes isomorphic with $K(\mathbf{t})$, where K is the fraction field of R. We must show that a principal divisor in $\mathrm{Div}^R(S(E)) := \langle [P] \mid P \cap R \neq 0 \rangle$ is mapped by ψ to a principal divisor in $\mathrm{Div}(R)$. Thus, let $g \in S(E)$ be such that $[g]_{S(E)} = \sum_{P \cap R \neq 0} v_P(g)[P]$. Then $\psi([g]_{S(E)}) = \sum_{ht(P \cap R) = 1} v_P(g)[P \cap R]$. Now, since $v_P(g) = 0$ for all height one primes P such that $P \cap R = 0$, we must have $v_Q(g) = 0$ for every height one prime $Q \subset K[\mathbf{t}]$. Therefore, $g \in K$ and so, actually, $g \in R = K \cap S(E)$. Next, for any height one prime $P \subset S(E)$ such that $P \cap R$ has height one, set $p = P \cap R$. Since E_p is R_p-free, one has $PS(E)_P = pR_p[t]_{pR_p[t]}$ and since R_p is a discrete valuation ring, certainly $p^n R_p[t]_{pR_p[t]} \cap R = p^n$ for all $n \geqslant 0$. Therefore, $v_P(g) = v_p(g)$ and so, $\psi([g]_{S(E)}) = [g]_R$, as required. Therefore, we have obtained a homomorphism $\mathrm{Cl}(S(E)) \to \mathrm{Cl}(R)$, which is clearly surjective since E is locally free at the height one primes of R. The kernel of this homomorphism is generated by the classes $\mathrm{cl}(T(\mathbf{p}))$, where $ht(\mathbf{p}) \geqslant 2$. It is actually freely generated by the classes $\mathrm{cl}(T(\mathbf{p}))$ such that $ht(\mathbf{p}) \geqslant 2$ and $ht(T(\mathbf{p})) = 1$. To see this, one argues as in the proof of [26, Theorem 2.1] or [31, Theorem 2.1.4]. By Proposition 3.1.2, the rank of the kernel is the $\mathbf{f}$-number of E.

(ii) The proof of this statement is analogous to that of Theorem 2.3.1, taking into account the required adaptations. ∎

EXAMPLE 3.2.2. Let I be the ideal generated by the six square-free monomials

$$(x_1 x_2, x_2 x_3, x_2 x_5, x_3 x_4, x_3 x_5, x_5 x_6\}.$$

A calculation with *Macaulay* [2], shows that I is an ideal of linear type, and that the Rees algebra $R[It]$ is normal. From its presentation, we obtain

$$h_1 = 1, \qquad h_2 = h_3 = h_4 = 0, \qquad h_5 = 4.$$

Its $\mathbf{f}$-number is 5, and therefore the divisor class group of $R[It]$ is $\mathbf{Z}^5$.

In full analogy with the notion of ideals of *analytic type*, we have:

DEFINITION 3.2.3. Let R be an integral domain and let E be a finitely generated R-module. E is said to be of *analytic type* if $\operatorname{Spec}(S(E))$ is irreducible.

The advantage of this definition (cf. [3]) is that the condition $\mathscr{F}_1$ still holds for E and in many cases it is much easier to prove irreducibility than integrality. Note that both imply $\mathscr{F}_1$ for E, and the former is (in the presence of this condition) characterized by equi-dimensionality (cf. [3]). It would be of interest to express this in ideal-theoretic terms.

Stretching a bit the above argument, one has:

THEOREM 3.2.4. *Let R be a universally catenarian factorial domain and let E be a module such that $S(E)_{red}$ is a normal domain. Then the divisor class group of $S(E)_{red}$ is a free abelian group of rank equal to the* $\mathfrak{f}$-*number of E.*

We next introduce a formalism that may be used to view from another angle the symmetric algebras of modules with a linear presentation.

Given an R-module E with a presentation

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \longrightarrow 0, \qquad \varphi = (a_{ij}),$$

the ideal of definition of its symmetric algebra can be written as a matrix product

$$J = [f_1, ..., f_m] = \mathbf{T} \cdot \varphi, \qquad \mathbf{T} = [T_1, ..., T_n],$$

in an essentially unique manner.

Assume that R is a polynomial ring $k[\mathbf{x}] = k[x_1, ..., x_d]$ over some base ring k, and that the entries of φ are k-linear forms in the variables x_i. We can write the ideal J as

$$J = \mathbf{x} \cdot B, \qquad B = (b_{lj}),$$

where B is a $d \times m$ matrix of k-linear forms in the variables T_l.

Let F be the $k[\mathbf{T}]$-module defined by the matrix B:

$$k[\mathbf{T}]^m \xrightarrow{B} k[\mathbf{T}]^d \longrightarrow F \longrightarrow 0.$$

For reference we note the following

PROPOSITION 3.2.5. $S = S_{k[\mathbf{x}]}(E) = S_{k[\mathbf{T}]}(F)$.

This permits toggling between representations. Thus a question over a d-dimensional ring turns into the same question on a d-generated module over the other ring. For instance, if S is an integral domain (with $k=$ field), its Krull dimension is $d+r_{\mathrm{x}}=n+r_{\mathrm{T}}$, where r_{x} and r_{T} are the ranks of E and F, respectively.

EXAMPLE 3.2.6. *The Trautman–Vetter Bundle.* Let R be a polynomial ring of dimension n over a field. The module of global sections of this bundle is a torsionfree module E of rank $n-2$. Its resolution is given by

$$0 \longrightarrow R \longrightarrow R^n \xrightarrow{\varphi} R^{2n-3} \longrightarrow E \longrightarrow 0.$$

For $n=5$, it can be shown that $S(E)$ is a Cohen–Macaulay domain. To check normality, we use the criterion of [28]. It suffices to check the Jacobian condition. It is easy to write down the required matrix because the matrix of φ has linear entries. We get, namely,

$$\begin{bmatrix} 0 & T_1 & T_2 & T_3 & T_4 \\ -T_1 & 0 & -T_3 & -T_4 & T_5 \\ -T_2 & T_3 & 0 & T_5 & T_6 \\ -T_3 & T_4 & -T_5 & 0 & T_7 \\ -T_4 & -T_5 & -T_6 & -T_7 & 0 \end{bmatrix}.$$

Since the rank of this matrix is four, the criterion yields that $S(E)$ is normal. It is easy to see that the f-number of E is one. Actually, $\mathrm{Cl}(S(E))$ is freely generated by the class of $T(\mathbf{m})$, $\mathbf{m}$ standing for the maximal homogeneous ideal of R.

We apply Proposition 3.2.5 to obtain that the module F associated to E (which has rank 3), has rank 1. The matrix B is skew-symmetric and it is easy to see that F can be identified to the ideal generated by the Pfaffians of B. It will follow that $S(E)$ is Cohen–Macaulay.

3.3. *Submaximal Minors*

Let X be a $n\times n$ matrix of one of the following kind: a (completely) generic matrix, a generic symmetric matrix, or a generic skew symmetric matrix (all considered over a field). We will round out the picture regarding the normality, the divisor class group, and the canonical module of the associated Rees algebras of certain determinantal ideals attached to X, through the mechanism of f-numbers.

EXAMPLE 3.3.1. *Generic Matrices.*

THEOREM 3.3.2. *Let X be a generic $n\times n$ matrix, $n\geqslant 2$, and let I be the ideal $I_{n-1}(X)$. I is a normal ideal and the divisor class group of $S=R[It]$ is a free module of rank $n-1$.*

Proof. We shall make use of certain elements of the proof by Huneke [16] that I is of linear type. Noteworthy is the following fact: $S/(\Delta t)$ is a reduced ring, where Δ is the determinant of X.

LEMMA 3.3.3. *I is a normal ideal.*

Proof. We are gong to argue by induction on n, the case $n=2$ being clear. From general facts, we know that $S=\bigcap_P S_P$, P running over the associated primes of principal ideals. We claim that each such localization is a discrete valuation domain. By the remark above, this is certainly clear for any such prime containing Δt—in particular for the ideal MS generated by the entries of X. Note also that any ideal properly containing MS has grade at least 2. Therefore any other prime of S of grade 1 must intersect R properly inside of M, so that the induction hypothesis takes over (see [16] for details). ∎

Now we get to the determination of the divisor class group $\mathrm{Cl}(S)$ of S; it will appeal to the methods of [15, 13, 31] and particularly [26, Theorem 1.1].

LEMMA 3.3.4. *$\mathrm{Cl}(S)$ is free of rank $n-1$.*

Proof. We quote Section 2.1 to get that $\mathrm{Cl}(S)$ is a free group of rank equal to the cardinality of the set of minimal primes of IS.

In case $n=2$, IS itself is prime, since I is a prime ideal generated by a regular sequence. Henceforth we assume $n>2$. Let P be a minimal prime of IS; put $\mathbf{p}=P\cap R$. Because S is normal, IS is a divisorial ideal and each P has height one. Furthermore, if $\mathbf{p}=M=$ irrelevant maximal ideal of R, then $P=MS$, since by the previous lemma any prime properly containing MS must have grade at least two.

We are now in a position to describe all the minimal primes of IS. Because I is of linear type, the prime ideals we want are those $T(\mathbf{p})$ of height one.

For each integer $1\leqslant t\leqslant n-1$, denote by $\mathbf{p}_t$ the prime ideal generated by the minors of order t of the matrix X. We claim that the $T(\mathbf{p}_t)$'s are the desired primes. Note each of these is a minimal prime of Δt. This is clear for $\mathbf{p}_1=M$ and $\mathbf{p}_{n-1}=I$. Because $M\not\subset \mathbf{p}_t$ for $t\geqslant 2$, we can localize at the powers of some entry x_{ij} of X and reduce the matrix to a smaller matrix of indeterminates X' such that $I_t(X)_{x_{ij}}=I_{t-1}(X')_{x_{ij}}$, thus showing that $T(\mathbf{p}_t)$, for $t>1$, may be defined in terms of the matrix X'. Since Δ belongs to each of the $\mathbf{p}_t$'s, the assertion now follows. ∎

Remark 3.3.5. In characteristic zero, Bruns [6] has proved the normality and computed the divisor class group of the determinantal ideals of arbitrary size of a generic $m\times n$ matrix.

EXAMPLE 3.3.6. *Symmetric Matrices.* Until recently a similar theory for generic symmetric matrices was missing. In the characteristic zero case, however, the condition that $I = I_{n-1}(X)$ be of linear type can be bypassed. The normality would follow from the decomposition results of [1], as has been pointed out in [6]. The rank formula could be derived as above; that is, the f-number of I is $n-1$.

The situation has become much neater in view of the recent proof of Kotsev [22] that such ideals are of linear type. He proves also the normality of the Rees algebras and computes the divisor class group.

EXAMPLE 3.3.7. *Pfaffians.* There are two cases to consider, according to whether n is odd or even. The first of these is understood best: I is the ideal generated by the Pfaffians of order $n-1$; it is of linear type, its Rees algebra $S = R[It]$ is normal with free divisor class group, generated by the class of the canonical module $\omega_S \simeq IS$.

For n even, $n = 2p$, I is generated by the Pfaffians of order $2p-2$. It follows directly from [20] that I is locally generated by analytically independent elements and that its f-number is $p-1$. It is not known however whether I is of linear type.

Remark 3.3.8. A remaining unresolved issue is which of these ideals are generated by d-sequences. It is known that for the generic matrices the answer is negative (for $n \geqslant 3$), for the Pfaffians it is affirmative if n is odd, and, in the symmetric case it is positive if $n = 3$.

4. CANONICAL MODULES OF BLOW-UPS

This section deals with the computation of the canonical module of a Rees algebra. Previous results were obtained in [15, 13, 26].

We assume that all relevant rings are residue rings of Gorenstein rings. By and large, the canonical module of a ring A will be denoted ω_A. We refer to [12] for the theory of canonical modules.

4.1. *Canonical Module of Extended Rees Algebras*

Bruns [5] proved that, under sutable conditions, the canonical module of $gr_I(R)$ is extended from $R[It]$ by *killing torsion.* Under the same conditions, we next prove that the canonical module of $R[It, t^{-1}]$ is extended from $R[It]$.

THEOREM 4.1.1. *Let R be a Cohen–Macauley, generically Gorenstein, local ring and let $I \subset R$ be an ideal of height at least two such that*:

(i) *$R[It]$ is Cohen–Macaulay*

(ii) *$R[It]_P$ is Gorenstein for every prime ideal $P \subset R[It]$ associated to $gr_I(R)$.*

Then there is a suitable embedding $\omega_{R[It]} \simeq \Gamma \subset R[It]$ such that the extended ideal $\Gamma R[It, t^{-1}]$ is a canonical module of $R[It, t^{-1}]$.

Proof. Set $S := R[It]$ and $T := R[It, t^{-1}]$. For convenience, we recall, in its general lines, the portion of the argument in [5] needed here. First, under the given hypotheses on R and I, S turns out to be generically Gorenstein and locally Gorenstein at the minimal primes of S_+. The crucial point, at this stage, is to be able to choose an embedding of the canonical module of S into S whose image is not contained in any of the prime ideals belonging to the set of minimal primes of S, S/S_+, and S/IS. This is possible thanks to condition (ii). Having done so, denote ω as the image of one such embedding. The next step brings up an argument of *prime avoidance*, by which one selects an element $x \in I \setminus I^2$ which is not a zerodivisor on either S or S/ω and such that xt is not a zerodivisor on either S/ω or $S/IS \simeq gr_I(R)$ [5].

We next fix the above choices and introduce the adaptations required to handle the present situation.

First, consider the presentation of T as an S-algebra

$$S[U]/J \simeq T,$$

where U is an indeterminate over S and $U \mapsto t^{-1}$. It is easy to see that the presentation ideal J is generated by the polynomials $ytU - y$, $y \in I$. Using this presentation and the assumption to the effect that S is Cohen–Macaulay, we know [12] that

$$\omega_T \simeq \mathrm{Ext}^1_{S[U]}(T, \omega_{S[U]}).$$

We now list the properties that intervene substantially in the computation of the latter module:

(a) $\omega_{S[U]} \simeq \omega \otimes_S S[U] \simeq \omega[U]$

(b) $xtU - x$ as chosen above is a nonzerodivisor on $(S/\omega)[U]$

(c) $(xtU - x)\, S[U] :_{S[U]} J = (xS :_S IS) S[U]$

(d) $xS :_S IS = (x, xt) S$

(e) $(x, xt) S \cap \omega = (x\omega, xt\omega) S$.

As for the proofs of these assertions, (a) is well known, (b) follows easily from the choice of x as a nonzerodivisor on S/ω, (c) is a straightforward calculation using the form of the generators of the ideal J as mentioned earlier, (d) is [17, Lemma 1.1], and (e) is proved in [5].

Gathering the above information, one has

$$\omega_T \simeq \operatorname{Hom}_{S[U]}(S[U]/J, (\omega, xtU - x)S[U]/(xtU - x)S[U])$$

by (a) and (b)

$$\simeq (((xtU - x)S[U] : J) \cap (\omega, xtU - x)S[U])/(xtU - x)S[U]$$

$$= ((x, xt)S[U] \cap (\omega, xtU - x)S[U])/(xtU - x)S[U] \qquad \text{by (c) and (d)}$$

$$= ((x, xt)S[U] \cap \omega[U], (xtU - x)S[U])/(xtU - x)S[U]$$

$$= ((x\omega, xt\omega)S[U], (xtU - x))/(xtU - x)S[U] \qquad \text{by (e)}$$

$$= (xt\omega, xtU - x)S[U]/(xtU - x)S[U]$$

$$\simeq (xt\omega)T$$

$$\simeq \omega T \qquad \text{as } xt \text{ is a nonzerodivisor on } T. \blacksquare$$

COROLLARY 4.1.2 [5]. *Under the same hypotheses and conditions as in Proposition* 4.1.1, *letting* $G := gr_I(R)$, *one has*

$$\omega_G \simeq (\omega_S, IS)/IS.$$

Proof. Let $\omega \simeq \omega_S$ be an embedding as chosen in Proposition 4.1.1, so ωT is a canonical module of T and, moreover, t^{-1} is a nonzerodivisor on $T/\omega T$. It follows that

$$\begin{aligned} \omega_G &\simeq \omega T/t^{-1}\omega T \\ &= \omega T/t^{-1}T \cap \omega T \\ &\simeq (\omega, t^{-1})T/t^{-1}T \\ &\simeq (\omega, IS)/IS. \quad \blacksquare \end{aligned}$$

4.2. *Gorenstein Normal Cones*

We now address the question as to when $gr_I(R)$ is a Gorenstein ring. The actual move here is to state sufficient conditions for it to be *quasi-Gorenstein*, so the question of Cohen–Macaulayness is left untouched.

First, we focus on a result obtained in [13] only from a slightly different viewpoint.

PROPOSITION 4.2.1. *Let R be a Cohen–Macauley normal domain* (*which is a residue ring of a Gorenstein local ring*) *and let $I \subset R$ be a normal ideal of height at least two. If $R[It]$ is Cohen–Macaulay, the following conditions are equivalent*:

(i) *$gr_I(R)$ is Gorenstein.*

(ii) *$\omega_{R[It]} \simeq IR[It]^{(m)}$ for some integer m.*

Proof. As before, set $S := R[It]$, $G := gr_I(R)$, and $T := R[It, t^{-1}]$. Observe that, under the present conditions, G is Gorenstein if and only if T is Gorenstein. Indeed, if G is Gorenstein then R must be Gorenstein (cf. [13]). Let M be a maximal ideal of T. If $t^{-1} \in M$ then $G_M \simeq T_M/t^{-1}T_M$ is Gorenstein, hence so is T_M. If $t^{-1} \notin M$ then $T_M = R[t, t^{-1}]_{MR[t,t^{-1}]}$ is Gorenstein as R is. Now, by Proposition 4.1.1, we may choose an embedding $\omega_S \subset S$ in such a way as to have $\mathrm{cl}(\omega_S)$ mapped to $\mathrm{cl}(\omega_T)$ via the map $\mathrm{Cl}(S) \to \mathrm{Cl}(T)$ in the exact sequence of Proposition 2.2.3. From this it follows easily that $\mathrm{cl}(\omega_T) = 0$ if and only if $\mathrm{cl}(\omega_S) \in \mathbf{Z}\,\mathrm{cl}(IR[It])$, which was our contention. ∎

COROLLARY 4.2.2. *Let R be a Cohen–Macaulay normal domain, residue of a Gorenstein local ring, and let $I \subset R$ be an ideal of height at least two satisfying the following conditions:*

(i) *I is radical and generically a complete intersection*

(ii) *$gr_I(R)$ is torsionfree over R/I*

(iii) *$R[It]$ is Cohen–Macaulay.*

Then the following conditions are equivalent:

(1) *$gr_I(R)$ is a Gorenstein ring.*

(2) *I is an unmixed ideal.*

Proof. It immediately follows from the previous corollary and Theorem 2.2.1. ∎

COROLLARY 4.2.3. *Let R be a regular local ring and let $I \subset R$ be an ideal of height at least two such that $R[It]$ is Cohen–Macaulay. If $gr_I(R)$ is reduced then it is Gorenstein if and only if I is unmixed.*

Proof. It follows from the preceding corollary and Theorem 2.2.2. ∎

EXAMPLE 4.2.4. Let

$$\begin{aligned} I &:= (X_1, \ldots, X_n)(Y_1, \ldots, Y_m) \\ &\subset R := k[X_1, \ldots, X_n; Y_1, \ldots, Y_m]_{(X_1, \ldots, X_n;\, Y_1, \ldots, Y_m)}. \end{aligned}$$

It can be shown that $gr_I(R)$ is reduced and Cohen–Macaulay [7]. Therefore, by the previous corollary, $gr_I(R)$ is Gorenstein if and only $n = m$. Special cases of this example appear elsewhere, e.g., [8, Corollary 3.7].

4.3. *Two-Generated Ideals and Associated Divisors*

Let R be a Cohen–Macaulay normal domain and let I be a divisorial ideal generated by two elements. We discuss here some objects derived from I: its dual, the symmetric and symbolic squares, and its Rees algebra. The peculiarity of divisors attached to two-generated ideals has been observed also by Griffith [10].

The following proposition is a special case of [4, (2.6)], but we choose to give here a simple direct proof.

PROPOSITION 4.3.1. *Let I be a divisorial ideal generated by two elements. Then its Rees algebra satisfies Serre's condition S_2.*

Proof. A presentation of $I=(a, b)$ can be described as

$$0 \to I^{-1}(be_1 - ae_2) \to Re_1 \oplus Re_2 \to I \to 0,$$

from which it follows easily (recall that I is of linear type) that

$$R[It] = R[x, y]/I^{-1}(bx-ay),$$

where $R[x, y]$ is a polynomial ring.

To prove the claim, let P be a prime of $R[It]$ of height at least 2. Write $\mathfrak{p} = P \cap R$. If height$(\mathfrak{p}) = 1$ or 2, $I_\mathfrak{p}$ is strogly Cohen–Macaulay, and there is no difficulty. If height $(\mathfrak{p}) \geqslant 3$, since I has no embedded prime, the depth of I^{-1} is at least three and therefore the depth of $R[It]_P$ is at least two, as desired. ∎

One can similarly decide when the associated graded ring is torsionfree. For this, set $B := R[x, y]$ and recall the *approximation complex* of [14]:

$$0 \to H_1 \otimes B_{t-1} \to B_t \to I^t/I^{t+1} \to 0.$$

One claims that

$$\operatorname{depth}(H_1)_\mathfrak{p} \geqslant \begin{cases} 2 & \text{if} \quad \operatorname{depth}(R_\mathfrak{p}) \geqslant 3 \\ 1 & \text{if} \quad \operatorname{depth}(R_\mathfrak{p}) = 2. \end{cases}$$

Indeed, we have an exact sequence

$$0 \to R \to I^{-1}(be_1 - ae_2) \to H_1 \to 0$$

from which the argument as above applies.

It follows that the associated prime ideals of I^t/I^{t+1} have height at most two. Once can then state:

COROLLARY 4.3.2. *The associated graded ring of I is R/I-torsionfree if and only if height $(I \cdot I^{-1}) \geqslant 3$.*

Proof. The last condition can also be stated by saying that I is principal at height two primes, and that it certainly implies normal torsionfreeness.

Conversely, assume R is a two-dimensional ring and I is not principal. Then $I^t/I^{t+1} = S_t(I/I^2)$ are rank one Cohen–Macaulay modules over the one-dimensional local ring R/I, with $t+1$ generators. Eventually this number wil exceed the multiplicity of R/I, which bounds all such numbers of generators. ▮

Finally we can state the normality condition (cf. [4]):

COROLLARY 4.3.3. *The Rees algebra $R[It]$ is normal if and only if for each prime ideal $\mathbf{p}$ of height two containing $J = I \cdot I^{-1}$, $(\mathbf{p}/\mathbf{p}^2 + J)_P$ is a cyclic module.*

For two simple examples:

EXAMPLE 4.3.4. Let $R = k[x, y, z]/(x^2 - y^2 + z^2)$ and $I = (z, x - y)$. Then I is normal. If the ring is now the hypersurface $x^3 - y^3 + z^3 = 0$, the similar ideal is not normal.

Remark 4.3.5. A point to elucidate here is the form of the canonical module. One has from the presentation of $S = R[It]$ (with ω = canonical module of R and $B = R[x, y]$) the exact sequence

$$0 \to \omega B \to \operatorname{Hom}_B(I^{-1}(bx - ay), \omega B) \to \omega_S = \operatorname{Ext}^1_B(S, \omega B) \to 0.$$

Since

$$\begin{aligned} \operatorname{Hom}(\operatorname{Hom}(I, R), \omega) &= \operatorname{Hom}(\operatorname{Hom}(I \otimes \omega, \omega), \omega) \\ &= (I \otimes \omega)^{**} = (I\omega)^{**}, \end{aligned}$$

the sequence becomes

$$0 \longrightarrow \omega B \xrightarrow{f} (bx - ay)^{-1} (I\omega)^{**} \longrightarrow \omega_S \longrightarrow 0,$$

thus yielding an explicit presentation of ω_S.

4.4. *Rings of Type Two*

We shall look at the case where I is the canonical module ω of a ring of type 2. We begin with the following formula for the number of generators $\mu(\omega^*)$ of the dual ω^* of ω.

PROPOSITION 4.4.1. *Let R be a local Cohen–Macaulay ring of dimension d and type 2. Then $\mu(\omega^*) = \dim_k \operatorname{Ext}^{d+1}_R(k, R)$.*

Observe that this number is the $(d+1)$st Bass number of R.

Proof. Associated to a two-generated faithful module, there exists a canonical exact sequence

$$0 \to R \to \omega \oplus \omega \to S_2(\omega) \to 0.$$

For ω itself the symmetric square $S_2(\omega)$ is a Cohen–Macaulay module [30, Proposition 3.1]. Note that ω^* is the ω-dual of $S_2(\omega)$, so that the number of generators of one module is the type of the other module.

To count the number of generators we may reduce to the zero-dimensional case $S = R/(x)$. Applying the functor $\mathrm{Hom}_S(k, \cdot)$ we obtain the exact sequence

$$\begin{aligned}\mathrm{Hom}_S(k, S) &\to \mathrm{Hom}_S(k, \omega \oplus \omega) \to \mathrm{Hom}_S(k, S_2(\omega)) \\ &\to \mathrm{Ext}^1_S(k, S) \to \mathrm{Ext}^1_S(k, \omega \oplus \omega) = 0.\end{aligned}$$

The conclusion follows because the first mapping is an isomorphism. ∎

Another way to derive the equality above follows from the exact sequence

$$0 \to \omega^* \to R^2 \to \omega \to 0,$$

and the fact that

$$\mathrm{Tor}_i^R(k, \omega) \simeq \mathrm{Ext}_R^{d+i}(k, R).$$

EXAMPLE 4.4.2. Let I be a Cohen–Macaulay ideal of codimension 3 that admits a pure resolution

$$0 \to R^e(-d-a-b) \to R^f(-d-a) \to R^g(-d) \to I \to 0.$$

Suppose $e = 2$. According to [23] we may assume that $a + b \leqslant d$ (otherwise I is Gorenstein). We claim that $\mu(\omega^*) = f$.

Set $A := R/I$. From the resolution we have the presentation of ω

$$A^f \xrightarrow{\ \varphi\ } A^2 \longrightarrow \omega \longrightarrow 0.$$

It suffices to show that the number of relations of ω $(= \mu(\varphi(A^f)))$ is f. This is clear because $\sum r_i e_i \in R^f$ with r_i of degree 0 cannot map to 0 since $d > a$.

THEOREM 4.4.3. *Let R be a local Cohen–Macaulay ring of type 2 with canonical module ω and let I be Cohen–Macaulay ideal of height one, generated by two elements. If, for a given $t \geqslant 2$, $S_t(I)$ is Cohen–Macaulay, then $t = 2$ and $I \simeq \omega$.*

LEMMA 4.4.4. *Let R be a Cohen–Macaulay ring with canonical module ω, and let I be a Cohen–Macaulay ideal of height one. If $I/(x)I \simeq \omega/(x)\omega$, for some system of parameters (x), then $I \simeq \omega$.*

Proof. There is no harm in assuming that (x) consists of a single regular element. From

$$0 \longrightarrow \omega \xrightarrow{\ \cdot x\ } \omega \longrightarrow \omega/x\omega \longrightarrow 0,$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}(I, \omega) \to \mathrm{Hom}(I, \omega) \to \mathrm{Hom}(I, \omega/x\omega)$$
$$= \mathrm{Hom}(I/xI, \omega/x\omega) \to 0.$$

But $\omega/x\omega$ is the canonical module of $R/(x)$, so that if $I/xI \simeq \omega/x\omega$, the right-most term in the last exact sequence is cyclic, say, $(R/(x))\,g$. Let $f \in \mathrm{Hom}(I, \omega)$ correspond to g. In the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow I \xrightarrow{f} \omega \longrightarrow \mathrm{coker}(f) \longrightarrow 0$$

we have $\mathrm{coker}(f) = 0$ by Nakayama's lemma. Furthermore, for each minimal prime $\wp$ of R, $I_\wp = R_\wp$ and since $\omega_\wp$ has the same length as $R_\wp$, $\ker(f)_\wp = 0$; this shows $\ker(f) = 0$. ∎

Proof of Theorem. We deform to the zero-dimensional case. Reduction of the exact sequence (cf. [29])

$$0 \to R^{t-1} \to I^{\oplus t} \to S_t(I) \to 0$$

implies that $l(\bar{I}) = l(\bar{R})$ and that $\bar{I}$ is faithful since $\bar{R}^{t-1}$ embeds into $\bar{I}^{\oplus t}$.

The assertion that I is isomorphic to either S or ω now follows from [11, Theorem 1]. That $S_t(\omega)$ cannot be Cohen–Macaulay for $t > 2$ is proved in [29, Proposition 2.8]. ∎

Finally, suppose R is a Cohen–Macaulay domain of type 2 and consider the Rees algebra of its canonical ideal ω. One has:

THEOREM 4.4.5. *Let R be a normal Cohen–Macaulay ring of type 2. Then $R[\omega t]$ is Cohen–Macaulay of type 3.*

Proof. That $R[\omega t]$ is Cohen–Macaulay follows since ω^* is Cohen–Macaulay. Using Remark 4.3.5 and the fact that ω^2 is divisorial, $\omega_{R[\omega t]}$ turns out to be just $(\omega^2/(ax - by)\omega)B$, and is therefore of type 3. ∎

4.5. *Two-Dimensional Rings*

Let R be a two-dimensional regular local ring and let I be an ideal generated by 3 elements. We make a syzygetic study of the normality of $S = R[It]$.

Assume that I is a primary ideal, and let

$$0 \longrightarrow R^2 \xrightarrow{\varphi} R^3 \longrightarrow I \longrightarrow 0$$

be a minimal resolution. If S is normal, it follows by [24, Proposition 5.5] (see also [18, Theorem 3.1]) that for any minimal reduction (a_1, a_2)

of I, $I^2=(a_1, a_2)I$. Then by [27, Propositon 3.1], S must be Cohen–Macaulay.

To get the equations of S, map a polynomial ring $R[T_1, T_2, T_3]$ onto S, with kernel J. Two of the generators of J are obtained from the rows of φ:

$$f=a_1T_1+a_2T_2+a_3T_3$$
$$g=b_1T_1+b_2T_2+b_3T_3.$$

In addition, to take into account the reduction equation, there must exist in J a quadratic form h whose coefficients generate the unit ideal.

Since J must be given by the maximal minors of an $r\times(r+1)$ matrix of forms in the T-variables, degree considerations say that J is generated by the 2×2 minors of a matrix

$$\begin{bmatrix} a & A_1 & A_2 \\ b & B_1 & B_2 \end{bmatrix},$$

where a and b are elements in the maximal ideal of R and the other elements are linear forms. Furthermore, f and g may be identified to two of these minors.

We claim that $(a, b)=I_1(\varphi)$. It is clear that the right-hand side is contained in (a, b). On the other hand, if we write

$$f=aA_1-bB_1$$
$$g=aA_2-bB_2,$$

passing to the faithfully flat extension $R\to R[T_1, T_2, T_3]_{mR[T_1,T_2,T_3]}$, the two ideals are equal since $A_1B_2-A_2B_1$ is now invertible.

Note that this affords a method to find h, and thus J (it had been alluded to as an heuristic in [4, 2.9]).

We denote by $h(0)$ the image of h in $R/m[T_1, T_2, T_3]$.

PROPOSITION 4.5.1. *Let I be a normal ideal. Then $I_1(\varphi)=(a, b)\not\subset m^2$. Conversely, if in addition $h(0)$ is square-free, then I is normal.*

Proof. We have to show the last assertion. For that we recall the criterion of [4, Proposition 2.1]): To test the condition (R_1) of Serre, let P be a prime ideal of $R[T_1, T_2, T_3]$, of height 3. Because we may restrict to the case $m\subset P$, $P=(m, p)$, p is mod $m=(x, y)$ a prime divisor of $h(0)$. In particular there are at most two minimal primes of $IR[It]$, corresponding to the divisors of $h(0)$.

Writing f, g, h as a linear combination of x, y, p, (R_1) is equivalent to the condition that the rank of the coefficient matrix, evaluated at P, be equal to 2. This leads immediately to the assertion. ∎

EXAMPLE 4.5.2. Here is a simple example showing that $h(0)$ may be a square yet the ideal I is normal. Let $I = (x^2, y^3, xy^2)$, so $h = yT_1T_2 - T_3^2$. It is easy to see that I is normal.

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