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Left Serial Rings and Their Factor Skew Fields

DAVID A. HILL

Instituto de Matematica, Universidade Federal da Bahia, Salvador, Bahia, Brazil

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A module is *uniserial* if its submodules are linearly ordered by inclusion. An Artinian ring is *left* (*right*) *serial* if it is a direct sum of uniserial modules as a left (right) module, and it is *left* (*right*) *uniserial* if it is uniserial as a left (right) module. It is *serial* (*uniserial*) if it is serial (uniserial) as both a left and a right module. Let R be an Artinian ring, J its Jacobson radical, and e a primitive idempotent. Then eRe/eJe is a factor skew field of R. The object of this paper is to study left serial rings using their factor skew fields.

The first two sections are devoted to a study of skew fields. In Section 1 skew fields of type 1 are introduced (skew field is of type 1 if its non-zero endomorphisms are all automorphisms), and in Section 2 the concept of an inner basis for a skew field over a subskew field is studied. These two concepts are brought together in Section 3, where they are used to prove a number of results about left serial rings. Among the most important of these results is the following theorem: If R/J is finitely generated over its center then R is left serial if and only if every indecomposable right injective module is uniserial.

All rings considered are associative with identity and all modules are unital. For a ring R, $J(R)$ or simply J when R is understood denotes the Jacobson radical of R. Let M be an R-module. We use $c(M)$ to denote the length of a composition series of M when such a series exists.

For an Artinian ring R a set $\{e_i\}_{i=1}^{n}$ of primitive orthogonal idempotents is called *basic* in the case where $Re_i \not\cong Re_j (i \neq j)$ and for any primitive idempotent e there exists an e_i such that $Re \cong Re_i$. The set $\{Re_i\}$ associated with the e_i's is called a basic set of principal indecomposable left ideals. Set $e = e_1 + \cdots + e_n$. The subring eRe is called the basic subring of R associated with the set $\{e_i\}$. The basic subring has the important property that each eRe_i/eJe_i is isomorphic as a left eRe-module to the skew field e_iRe_i/e_iJe_i.

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1. Skew Fields of Type 1 and 2

In the study of serial rings skew fields play an important role. The properties of the factor skew fields of a given serial ring often determine what sort of module theoretic properties the ring has. Here we examine the behavior of some skew fields with respect to their endomorphisms. As we see, when the factor skew fields of a left serial ring have well-behaved endomorphism rings, the ring often has well-behaved module theoretic properties. We make the following definition.

Definition. Let D be a skew field. Then D is said to be a skew field of *type* 1 if for all skew field endomorphisms φ such that $[D : \varphi(D)]$ finite implies that $D = \varphi(D)$. A skew field is said to be of *type* 2 if it is not of type 1.

It is clear that a skew field D is of type 2 if and only if there is a proper subskew field D_1 of D such that $D_1 \cong D$ and $\dim_{D_1}(D)$ is finite.

Any finite extension of a field whose only non-zero endomorphism is the identity must be a field of type 1. With this observation in mind it is immediate that the following fields are of type 1: finite extensions of the rational numbers, finite fields, real and complex numbers. Using these facts it is an easy exercise to show that the field of algebraic numbers is also of type 1.

Here are some examples of fields of type 2.

1. Let $F(t)$ be the field of rational functions over a field F with variable t and $p(t)$ a polynomial with $\deg(p) \geqslant 2$. Then the map that carries an element $r(t) \in F(t)$ to $r(p(t))$ yields an endomorphism $\varphi: F(t) \to F(p(t))$ such that $F(p(t)) \neq F(t)$ and $[F(t): F(p(t))]$ is finite.

2. Let $F(t)$ be the field of rational functions over the finite field F with p^n elements and K a finite extension of $F(t)$. Then K is of type 2, for if $K = F(t)(\beta)$ is a simple algebraic extension, the endomorphism $\varphi: K \to K$ that carries

$$t \to t^{p^n}$$

$$\beta \to \beta^{p^n}$$

yields an endomorphism such that $F(t^{p^n})(\beta^{p^n}) \cong F(t)(\beta)$ and $[F(t)(\beta) : F(t^{p^n})(\beta^{p^n})]$ is finite. Since any finite extension is a succession of a finite number of simple algebraic extensions, the result is clear.

3. Let $F(t)$ be the field of rational functions over a field F with $\mathrm{char}(F) = 0$ and $F(t)(\beta)$ a finite extension of $F(t)$, where β has minimum polynomial $s(Y) = Y^n \pm t$ $(n > 1)$. Then the endomorphism that carries

$$\beta \to t$$

$$t \to \pm t^n$$

yields an endomorphism such that $F(t) \cong F(t, \beta)$ and $[F(t, \beta) : F(t)]$ is finite.

4. Let $F(t)$ be as in (3) and consider the finite simple extension $F(t)(\beta)$, where β has minimum polynomial $s(Y) = Y^2 + p(t)\,Y + q(t)$ with $p(t), q(t)$ polynomials in $F(t)$. A map $\varphi : F(t)(\beta) \to F(t)(\beta)$ defined on generators t, β as

$$\varphi(\beta) = a_1(t)\,\beta + a_0(t)$$
$$\varphi(t) = p_1(t)\,\beta + p_0(t)$$

is an endomorphism if and only if $\varphi(s(\beta)) = 0$.

If $a_1(t) = p_1(t) = 0$, this implies that

$$0 = a_0^2(t) + p(p_0(t))\,a_0(t) + q(p_0(t)).$$

Thus $a_0(t)$ must be a solution to the polynomial (over $F(t)$)

$$u(Y) = Y^2 + p(p_0(t))\,Y + q(p_0(t)).$$

This is true if and only if there exists a polynomial $r(t)$ such that

$$p^2(p_0(t)) - 4q(p_0(t)) = r^2(t). \tag{1}$$

A slightly more general question is the following: Given a polynomial $Q(t)$ do there exist polynomials $P(t)$ and $R(t)$ such that

$$Q(P(t)) = R^2(t)? \tag{2}$$

It seems unlikely that (2) has a solution for arbitrary polynomials. However, if (2) has a solution for a class of polynomials, (1) might also have a solution for a similar class and thus, for certain values of p and q, $a_0(t)$ and $p_0(t)$ could be found that would yield proper endomorphisms of $F(t)(\beta)$.

Equation (2) is satisfied whenever $\deg(Q) = 1$. Let

$$Q(t) = b_1 t + b_0 \qquad (b_1 \neq 0)$$
$$P(t) = c_2 t^2 + c_1 t + c_0$$
$$R(t) = r_1 t + r_0.$$

Assuming that the $P(t)$, $Q(t)$, and $R(t)$ with the values above satisfy (2) yields the following set of equations:

$$\begin{aligned} r_1^2 &= b_1 c_2 \\ 2r_0 r_1 &= b_1 c_1 \\ r_0^2 &= b_1 c_0 + b_0. \end{aligned} \tag{I}$$

(I) is a system of three equations in five unknowns (assuming b_0, b_1 fixed). Fixing r_1 and c_1 different from zero yields a non-trivial solution to (I) and hence to (2). Now assume that F is algebraically closed. If $p(t)$ and $q(t)$ considered as coefficients of $s(Y)$ satisfy $\deg(p^2-4q)\leqslant 1$ then $F(t)(\beta)$ is a field of type 2, and in fact by what we have just shown $F(t)(\beta)$ is isomorphic to a subfield of $F(t)$.

The question of whether a finite extension of a skew field of type 1 (type 2) is of type 1 (type 2) appears to be very difficult even in the case of $K=F(t)$, where a solution to the problem involves solving systems of non-linear equations. The rest of this section is devoted to a study of this problem for skew fields finitely generated over their centers.

LEMMA 1.1. *Let D be a skew field of type 2 finitely generated over its center K and D_1 a subskew field isomorphic to D with $[D:D_1]$ finite. The following conditions hold*:

(1) *There exist subskew fields S and S_1 of D and an isomorphism $\varphi: D\to S$ such that the image of the restriction of φ to D_1 is S_1.*

(2) *The center of S_1 is properly contained in the center of S.*

Proof. Set $D=D_0$ and suppose D is of type 2. Then there exists a strictly decreasing sequence of skew fields,

$$D_0\supset D_1\supset\cdots\supset D_s\supset\cdots,$$

and an isomorphism $\varphi_0: D_0\to D_1$ and such that for each $s\geqslant 1$, $D_s=\operatorname{Im}(\varphi_0^s(D_0))$.

Observe that $D_s\cong D_t$ for all pairs of integers s and t and that $[D_s:D_{s+1}]$ is finite and greater than 1 for all values of s. Let C_s be the centralizer of D_s in D. By observing that $C_0=K$ we obtain an increasing sequence of K-vector spaces,

$$K\subset C_1\subset\cdots\subset C_s\subset\cdots.$$

Since D is finitely generated over K, the chain terminates. So there exists a positive integer t such that $C_{t+k}=C_t$ for all $k\geqslant 0$. Set $C=C_t$. Clearly the center of $D_n\,(n\geqslant t)$ is $D_n\cap C=K_n$. All the D_n being isomorphic yields $K_n\cong K_{n+1}\cong K$. Since D_{n+1} is properly contained in D_n and $[D_{n+1}:K_{n+1}]=[D_n:K_n]$ we obtain

$$K_{n+1}=D_{n+1}\cap C\subset D_n\cap C=K_n$$

for all $n\geqslant t$. That is K_{n+1} is properly contained in K_n. Statement (2) now follows easily. Statement (1) is an immediate consequence of our construction of the D_s's.

LEMMA 1.1 *yields the following corollary.*

COROLLARY 1.2. *Let D be a skew field that is finitely generated over its center K. Suppose K is of type* 1. *Then D is of type* 1.

EXAMPLE. Let K be a field with $\mathrm{char}(K) \neq 2$ and D the algebra of quaternions over K with basis $\{1, i, j, k\}$ such that

$$i^2 = j^2 = k^2 = -1$$

$$ij = k, \qquad jk = i, \qquad ki = j.$$

Then D is of type 1 if and only if K is of type 1.

The if part follows from Corollary 1.2, and for the only if part we need only observe that if φ is an isomorphism of K onto a proper subfield F of K, there exists a natural map of D onto the quaternions over F which fixes the basis of D over K and maps K onto F. Thus D is of type 2.

COROLLARY 1.3. *Let R be local and suppose that R/J is finitely generated over its center K. Assume that K is a field of type* 1. *Then R is left (right) uniserial if and only if R is uniserial.*

Proof. Assume R is left uniserial. Apply Corollary 1.2 and [3, Lemma 5.1]. The other direction is trivial.

2. SUBSKEW FIELDS OF A GIVEN SKEW FIELD

The purpose of this section is to pave the way for the study of modules over certain classes of left serial rings, a topic to be taken up in Section 3. The questions of what the subskew fields look like inside a given skew field and which ones, if any, are isomorphic to the original skew field often determine the module theoretic properties of the left uniserial rings whose factor skew field modulo the radical is isomorphic to the original skew field. The following definition plays an important role for the work to be done in this section and in Section 3.

DEFINITION. Let D be a skew field and D' a subskew field. We say that D' is *inner* in D in the case where there exists a basis $\{v_i\}_{i \in I}$ for D over D' such that $v_i D' v_i^{-1} = D'$ for all $i \in I$. The basis $\{v_i\}_{i \in I}$ is called an *inner basis* for D over D'.

This says that there exists a basis for D over D' such that D' is invariant under the inner automorphisms given by the basis elements. Of course each basis element then induces an automorphism of D'. Examples of this situation are quite common.

1. Let D be a skew field and K a subfield contained in the center of D. Then K is inner in D.

2. Suppose D is a skew field with center K and $\dim_K(D) = p^2$, where p is a prime. Let D' be a subskew field of D such that K is not contained in D'. Then D has a basis over D' consisting of elements of K. Clearly D' is then inner in D. In fact, whenever a basis consisting of elements of the center exists, this always implies that D' is inner in D.

3. Let D be a skew field with center K and D_1 a subskew field of D with $\dim_{D_1}(D) = p$, p a prime. Suppose K is not contained in D_1. Then D has a basis over D_1 consisting of elements of K, so that D_1 is inner in D.

4. Let D be the generalized quaternions over a field K with basis $1, i, j, k$ such that

$$i^2 = \alpha, \qquad j^2 = \beta, \qquad ij = -ji = k, \qquad \alpha, \beta \in K.$$

Then $\{1, j\}$ is an inner basis for D over $K[i]$ since $jij = -\beta i$.

Examples are by no means restricted to skew fields finitely generated over their centers. We construct a class of non-trivial examples of skew fields D and D' with D' inner in D and D not finitely generated over its center.

Consider the following situation: Let F be a skew field and σ an endomorphism of F. If t is an indeterminate we can form the skew polynomials in t with multiplication given by

$$ta = \sigma(a)\, t \qquad \text{for all} \quad a \in F.$$

Then $F[t, \sigma]$ is a left principal ideal domain and therefore satisfies the left Ore condition. Thus we can form the skew field of left quotients $F(t, \sigma)$ of $F[t, \sigma]$.

Let $D' \subseteq F(t, \sigma)$ be a division subring of $F(t, \sigma)$. Call D' a *division subalgebra* of $F(t, \sigma)$ if D' is a vector space over F such that for $a \in F$, $d \in D'$ the scaler multiplication is induced by an endomorphism φ of F, that is

$$a \cdot d = \varphi(a)\, d,$$

where $a \cdot d$ represents scalar multiplication and $\varphi(a)\, d$ is multiplication in D'. Observe that our definition implies that D' contains a subskew field of F isomorphic to F, namely $\varphi(F)$. By an algebra isomorphism between $F(t, \sigma)$ and a subskew field D', we mean a ring isomorphism which preserves scalar multiplication. Thus any such isomorphism when restricted to

F is just the endomorphism φ and therefore maps F onto the subskew field $\varphi(F)$. Throughout the rest of this section isomorphism always means algebra isomorphism unless otherwise stated.

We turn our attention to the question of what the division subalgebras of $F(t, \sigma)$ look like inside $F(t, \sigma)$, what their structure is like, which are isomorphic to $F(t, \sigma)$, and which possess an inner basis. The following propositions deal with these problems. First some notation: Let m denote the order of σ. If m is finite let $C \subseteq F(t, \sigma)$ be the subdivision algebra whose elements consists of those rational functions in $F(t, \sigma)$ which are generated by polynomials of the form

$$p(t) = \sum_{i=0}^{n} c_i t^{k_i m}, \qquad c_i \in F. \tag{1}$$

It is clear that C consists of elements $q^{-1}p$, where p and q have the form given in (1). If F is a field then C also is a field. If m is infinite we take C to be the subskew field F.

Proposition 2.1. *Let F be a field and R a subring of $F[t, \sigma]$ with $F \subseteq R$.*

(1) *Suppose m is finite. Then the skew field of quotients of R is generated by a set of polynomials of the form $p = qt^s$, where $q \in C$ and $s \geqslant 0$.*

(2) *If m is infinite the skew field of quotients is generated by t^n for some $n > 0$.*

Proof. Let D be the skew field of quotients of R and $p(t)$ a polynomial in R. To prove the first statement observe that $p(t)$ can be written in the form

$$(1) \quad p(t) = p_1(t)\, t^{k_1} + \cdots + p_n(t)\, t^{k_n},$$

where $p_i(t) \in C$ with k_i's positive integers such that $k_i \neq k_j \bmod(m)$ for $i \neq j$. We show that each $p_i t^{k_i} \in D$. We may assume that σ is not the identity. Thus there exists an $\alpha \in F$ such that $\sigma^{k_1}(\alpha) \neq \sigma^{k_2}(\alpha)$. Consider the non-zero polynomial in D

$$(2) \quad p(t)\, \alpha - \sigma^{k_2}(\alpha)\, p(t) = (\sigma^{k_1} - \sigma^{k_2})(\alpha)\, p_1(t)\, t^{k_1} + \sum_{i=3}^{n} (\sigma^{k_i} - \sigma^{k_2})(\alpha)\, p_i t^{k_i}.$$

The term involving $p_2(t)$ drops out and by the choice of α, p_1 is multiplied by a non-zero constant. Multiply the polynomial in (2) by the inverse of $(\sigma^{k_1} - \sigma^{k_2})(\alpha)$ to get

$$q_2(t) = p_1(t)\, t^{k_1} + \sum_{i=3}^{n} c_i\, p_i(t)\, t^{k_i} \text{ for certain } c_i \in F.$$

Now apply the same argument as before to get rid of the term involving $p_3(t)$ if it is non-zero in the above expression. Applying this argument at most $n-1$ times yields $p_1(t)\,t^{k_1} \in D$. A similar argument for each $p_i t^{k_i}$ in place of $p_1 t^{k_1}$ shows that $p_i t^{k_i} \in D$. Thus the first statement follows since each $p_i \in C$.

For statement (2), observe that p can be written in the form given in Eq. (1) with each p_i a constant polynomial and $k_i \neq k_j$ for $i \neq j$. Using the same argument as in the proof of (1), each polynomial in D is an F linear combination of powers of t which belong to R. Among the set of $t^i \in D$ there must exist one whose exponent is minimal, say t^n, and it is easily seen that t^n generates D.

We now determine which subalgebras are isomorphic to $F(t, \sigma)$. Also we use $C(F)$ to denote the center of F, and for $a \in F(t, \sigma)$, ρ_a denotes the inner automorphism given by $\rho_a(x) = axa^{-1}$ for $x \in F(t, \sigma)$. This notation is standard for the rest of the paper.

LEMMA 2.2. *Let p and q be polynomials of $F(t, q)$ with no factors of t and whose coefficients belong to $C(F)$. Suppose the restriction of the inner automorphism $\rho_{q^{-1}p}$ to F induces an automorphism of F. Then $p, q \in C$.*

Proof. Consider $0 \neq a \in F$. Then $q^{-1}pap^{-1}q = b^{-1} \in F$. Thus

$$q^{-1}p(t)\,ap^{-1}q(t)\,b = 1.$$

Write $p(t) = \sum p_i t^i$ and $q(t) = \sum q_j t^j$. By observing that $p^{-1}q(t)\,b = (p(t)\,a)^{-1}\,q$ we may apply the left Ore condition to get for certain $0 \neq u_1, u_2 \in F(t, \sigma)$

$$\begin{aligned} u_2\,p(t) &= u_1\,p(t)\,a \\ u_2 q(t)\,b &= u_1 q(t). \end{aligned} \tag{I}$$

Now suppose that $p_i \neq 0$ for some $i > 0$. Using that $p_0 \neq 0$ and (I) yields

$$u_2\,p_0 = u_1\,p_0 a \qquad \text{and} \qquad u_2\,p_i = u_1\,p_i \sigma^i(a).$$

Simplifying and combining the above expressions yields $a = \sigma^i(a)$. Since a varies over all elements of F, we obtain $\sigma^i = I$. If the order m of σ is finite, this means that $i = km$. If the order is infinite, p is a constant. Thus in either case $p \in C$. Using that $\rho_{p^{-1}q}$ is onto, a similar argument also yields that $q \in C$.

THEOREM 2.3. *Consider the skew field $F(t, \sigma)$, where σ is an automorphism of F. Let $F(s(t), \sigma)$ be the subalgebra of $F(t, \sigma)$ generated by F and $s(t) = q^{-1}p(t)\,t^k$ or $s(t) = t^{-k}q^{-1}p(t)$. Assume that the coefficients of*

p and q are in $C(F)$. Then $F(s(t), \sigma)$ is isomorphic to $F(t, \sigma)$ by an isomorphism φ if and only if

(1) *$p, q \in C$ and*

(2) *$\varphi\sigma = \sigma^k\varphi$ or $\varphi\sigma = \sigma^{-k}\varphi$.*

Proof. We prove the theorem for the case $s(t) = q^{-1}p(t)\,t^k$ since the proof of the other case is similar. Write $p(t) = \sum p_i t^i$ and $q(t) = \sum q_j t^j$. We may assume that φ maps t to $s(t)$. Let $a \in F$ and look at

$$s(t)\,\phi(a) = \varphi(ta) = \varphi(\sigma(a)\,t) = \varphi\sigma(a)\,s(t).$$

The above equation may be re-written in the form

$$q^{-1}p\sigma^k\varphi(a) = \varphi\sigma(a)\,q^{-1}\,p \tag{1}$$

$$\rho_{q^{-1}p}(\sigma^k\varphi(a)) = \varphi\sigma(a).$$

Since φ(when restricted to F) and σ are both automorphisms of F apply Lemma 2.2 to get $p, q \in C$.

To prove statement (2) use Eq. (1) and the fact that p and q have coefficients in $C(F)$ to get that $\varphi\sigma = \sigma^k\varphi$.

It is a straightforward exercise to show that a division subalgebra satisfying the hypothesis of Theorem 2.3 and conditions (1) and (2) is isomorphic to $F(t, \sigma)$.

Remarks. (1) If m is finite there always exist proper division subalgebras isomorphic to $F(t, \sigma)$. Define $\varphi(t) = t^{km+1}$, where $0 \neq k \in Z$ and $\varphi(a) = a$ for $a \in F$. Then φ defines an isomorphism between $F(t, \sigma)$ and $F(t^{km+1}, \sigma)$.

(2) Let F be a field and m infinite. Thus $C = F$ so the only possibilities for division subalgebras isomorphic to $F(t, \sigma)$ are $F(t^k, \sigma)$, where the isomorphism $\varphi: F(t, \sigma) \to F(t^k, \sigma)$ satisfies $\sigma^k\varphi = \varphi\sigma$ when φ is restricted to F. Clearly the existence of such division subalgebras depends on the structure of $\mathrm{End}(F)$.

(3) As an example of how to construct non-trivial examples of division subalgebras isomorphic to $F(t, \sigma)$ not of the form in (1) take $F = Q(a)$, where Q is the rational numbers and F is an algebraic extension such that $\mathrm{Aut}_Q(F) = D_4$, where D_4 is the dihedral group of order 8. Let φ and σ be the generators of D_4 with $\varphi\sigma = \sigma^3\varphi$. Then $F(t, \sigma) \cong F(t^3, \sigma)$.

(4) Let F be the quaternions over the real numbers R and consider $F(t, \sigma)$. Since σ is a non-zero endomorphism of F, σ must fix R. Now apply the Skolem–Noether theorem: σ must be an inner automorphism. Thus using [2, 3.1 p. 295], we may assume that σ is the identity. Now apply [Theorem 2.3]. The subskew fields $F(q^{-1}p)$, where p and q have real coefficients, are all isomorphic to $F(t)$.

An interesting case is when F is a field of type 1. Observe that any ring isomorphism φ between $F(t, \sigma)$ and a subskew field of itself when restricted to F yields an automorphism of F. This follows from F being a field of type 1, and the fact that any homomorphism of F that mapped an element of F to an element of $F(t, \sigma)$ involving polynomials of degree greater than zero would imply that the homomorphic image of F is a field of type 2, clearly impossible.

COROLLARY 2.4. *Let F be a field of type 1 and D a subskew field of $F(t, \sigma)$ with $[F(t, \sigma) : D]$ finite. Then D is isomorphic to $F(t, \sigma)$ as a ring if and only if*

(1) $D = F(q^{-1}p(t)\, t^k, \sigma)$ *where* $k \in Z$ *and* $p, q \in C$.

(2) *There exists a* $\varphi \in \mathrm{Aut}(F)$ *such that* $\sigma^k \varphi = \varphi\sigma$.

It is of interest to know when a skew field is of type 1. In the context of Corollary 2.4, if m is infinite and $\mathrm{Aut}(F)$ abelian then $k = 1$ and $p, q \in F$. Thus $F(t, \sigma)$ must be of type 1. This yields:

COROLLARY 2.5. *Let F be a field of type 1 and σ an automorphism of infinite order with* $\mathrm{Aut}(F)$ *abelian. Then $F(t, \sigma)$ is of type 1.*

EXAMPLES. (1) As an illustration of Corollary 2.5 let F be the algebraic closure of Z_p, p a prime. Then it is well known that $\mathrm{Aut}(F)$ is the projective limit of the automorphism groups of $GF(p^n)$ and is therefore abelian. Thus for any $\sigma \in \mathrm{Aut}(F)$ whose order is infinite (as, for example, $\sigma(x) = x^{p^s}$ for $x \in F$) yields $F(t, \sigma)$ a skew field of type 1.

We now turn our attention to the important question of when $F(t, \sigma)$ has an inner basis over a fixed subskew field. Note that when σ is an automorphism $F[t, \sigma]$ satisfies both Ore conditions, and $F(t, q)$ is both the left and the right skew field of quotients of $F[t, \sigma]$.

LEMMA 2.6. *Let $D = F(t, \sigma)$, where σ is an automorphism of F and $D_1 = F(c_2^{-1} c_1 t^s, \sigma)$, $c_1, c_2 \in C$. Then $\dim_{D_1}(D) = k$ is finite and $\beta = \{1, t, ..., t^{k-1}\}$ is a basis.*

Proof. The proof is algorithmic in character so we omit some of the details.

First consider the case when $D_1 = F(t^s, \sigma)$ and $s > 0$. For a polynomial p let $d(p)$ denote the highest power of t with a non-zero coefficient of p when p is written as a vector involving powers of t with coefficients in D_1. It is clear that any polynomial can be written as a vector with coefficients in D_1 by taking $\{t^i\}_{i=0}^{s-1}$ as a set of generators. It is enough to show that inverses of polynomials can be expressed in terms of the above set of generators. To prove this use induction on $d(p)$. It is clearly true when $d(p)=0$. Assume true for inverses of polynomials p with $d(p)<n$.

Let q be a polynomial with $d(q)=n$. We may assume that $0<n<s$. Thus there exist non-zero polynomials $f_k, f_n \in D_1$ with $k<n$ and polynomials q_1, q_1' satisfying $d(q_1')<k$ and $q_1=f_k t^k+q_1'$ such that

$$q=f_n t^n+f_k t^k+q_1'=f_n t^n+q_1. \tag{1}$$

Multiply q on the left by $t^{s-n}f_k^{-1}=f_n' t^{s-n}$ to obtain

$$f_n' t^{s-n}q=f_n' t^{s-n}f_n t^n+t^{s-n+k}+f_n' t^{s-n}q_1'=t^{s-n+k}+q_2', \tag{2}$$

where $d(q_2')<s-n+k$.

Case 1. Suppose $s-n+k\leqslant n$. Multiply (2) on the left by $f_n t^{2n-s-k}$ to get

$$p_1 q=f_n t^n+q_2, \tag{3}$$

where p_1 and q_2 are expressed in terms of t^i $(i<s)$ over D_1 and $d(q_2)<n$. Thus using Eqs. (1) and (3) and $q_1=f_k t^k+q_1'$ yields

$$p_1 q=f_n t^n+q_2=f_n t^n+f_k t^k+q_1'-(f_k t^k+q_1')+q_2=q+q_2-q_1.$$

Since $d(q_2-q_1)<n$, $q^{-1}=(q_2-q_1)^{-1}(p_1-1)$ can be written as a D_1 linear combination of the t^i $(i<s)$.

Case 2. Suppose $s-n+k>n$. Set $\alpha=s+k-2n$. Then (2) can be written as

$$f_n' t^{s-n}q=t^\alpha t^n+q_2'. \tag{4}$$

Let $c\in D_1$ such that $ct^\alpha=t^\alpha f_n$. Multiplying (4) on the left by c and using $f_n t^n=q-q_1$ yields

$$pq=t^\alpha f_n t^n+q_3=t^\alpha q+(-t^\alpha q_1+q_3), \tag{5}$$

where $p=cf_n' t^{s-n}$ and $q_3=cq_2'$. Observe that $d(-t^\alpha q_1+q_3)<\alpha+n$. If $d(t^\alpha q_1+q_3)<n$ use Case 1. Otherwise note that if we write (5) as

$$(p-t^\alpha)\,q=-t^\alpha q_1+q_3$$

the degree $d(-t^\alpha q_1+q_3)$ is less than $d(t^\alpha t^n+q_2')$ of (4). Now continue the same argument until Case 1 is reached.

For the general case assume that m is finite and consider $u=c_2^{-1}c_1 t^s$ with $(c_1, c_2)=1$ and both c_1 and c_2 having no factor of t. Observe that

$F(u, \sigma)$ and $F(t^m, \sigma)$ contain the subskew field generated by $v = (c_2^{-1} c_1 t^s)^m$. It is enough to show that $F(t^m, \sigma)$ is finitely generated as a vector space over $F(v, \sigma)$. For this observe that t^m is a root of a polynomial with coefficients in $F(v, c)$. The fact that t^m commutes with elements of $F(v, \sigma)$ implies that t^m is a root of an irreducible polynomial with coefficients in $F(v, \sigma)$. Now the rest of the proof is like that used for fields.

THEOREM 2.7. *Consider $F(t, \sigma)$, where σ is an automorphism of F. Let $D_1 = F(c_2^{-1} c_1 t^s, \sigma)$ be the subskew field determined by $c_2^{-1} c_1 t^s$, where $c_1, c_2 \in C$. Then D has an inner basis over D_1 consisting of a finite number of powers of t whenever the coefficients of c_1 and c_2 lie in the fixed skew field of σ.*

Proof. In applying Lemma 2.6 there exists an n such that $\{1, t, ..., t^{n-1}\}$ is a basis for $F(t, \sigma)$ over D_1. Since c_1 and c_2 have coefficients in the fixed skew field of σ, the powers of t commute with $s(t) = c_2^{-1} c_1 t^s$. Thus if $p_1(s(t))$ is a polynomial in $s(t)$,

$$t^i p_1(s(t)) = p_2(s(t))\, t^i.$$

This implies that D_1 is inner in $F(t, \sigma)$.

Remark. The subdivision algebras isomorphic to $F(t, \sigma)$ are exactly those which satisfy Theorem 2.3. If they also satisfy Theorem 2.7 they have an inner basis consisting of powers of t. Thus if the order of σ is infinite all subdivision algebras containing F and isomorphic to $F(t, \sigma)$ are inner in $F(t, \sigma)$. It is not difficult to construct non-trivial examples of division subalgebras isomorphic to $F(t, \sigma)$ and inner in $F(t, \sigma)$. Thus it is easy to construct many examples of skew fields not finite dimensional over their centers which have isomorphic subskew fields admitting an inner basis. The significance of these results is seen Section 3, where we examine the modules over right uniserial rings.

3. SPECIAL TYPES OF RIGHT SERIAL RINGS

In this section we use the concepts developed in Sections 1 and 2 to study modules over right serial rings. Let I be a left ideal of a ring R. Define the idealizer ring of I to be

$$S_I = \{r \in R \mid Ir \subseteq I\}.$$

S_I is a subring of R and is in fact the largest subring in which I is a two-sided ideal. Note that $S_I/I \cong \mathrm{End}_R(R/I)$ canonically. If $I = Rw$ denote S_{Rw} by S_w. The idealizer ring of a right ideal is similarly defined and the same notational conventions hold. We use this notation throughout the rest of the paper.

We study the problem when for an arbitrary Artinian ring R the indecomposable left injective modules being uniserial is equivalent to R being right serial. Some results along this line are given in [4]. In the final part of this section some examples of right uniserial rings satisfying the above conditions and whose factor skew fields are not finitely generated over their centers are constructed.

Let D' and D be skew fields and ${}_{D'}V_D$ a bi-vector space over D' and D. Then ${}_{D'}V_D$ is said to be *simple* in case ${}_{D'}V_D$ contains no proper non-zero $D'-D$ bi-vector subspaces. A special case of ther following result appears in [4].

LEMMA 3.1. *Let D_1, D, D' be skew fields with D_1 a subskew field of D such that D_1 is linear in D and $\dim_{D_1}(D)$ is finite. Suppose ${}_{D'}V_D$ and ${}_{D'}W_{D_1}$ are $D'-D$ and $D'-D_1$ bi-vector spaces such that ${}_{D'}W_{D_1} \leqslant {}_{D'}V_{D_1}$. If ${}_{D'}V_D$ is simple and ${}_{D'}V_D$ and ${}_{D'}W_{D_1}$ satisfy*

$$\dim(W_{D_1}) = \dim(V_{D_1}) - 1 < \infty$$

$$\dim({}_{D'}W) = \dim({}_{D'}V) - 1 < \infty$$

then $\dim(V_D) = 1$.

Proof. Select an inner basis for D over D_1 and use the proof of [4, Lemma 3.1].

The following result is needed and appears in [4].

LEMMA 3.2. *Let R be an Artinian ring. Then R has every indecomposable left injective module uniserial if and only if R/J^2 does.*

This allows us to restrict our attention to rings with $J^2 = 0$. With this in mind suppose that e and f are primitive idempotents satisfying $fJe \neq 0$. Let I be a maximal proper semi-simple left ideal contained in Je with the property that the complement of I in Je is isomorphic to a copy of Rf/Jf. That is $I \oplus Rx = Je$ with $Rx \cong Rf/Jf$. Thus eS_Ie is a subring of eRe and $eJe \subseteq eS_Ie$. Therefore eS_Ie/eJe is a division subring of eRe/eJe. Also it is clear that fJe is a left fRf/fJf right eRe/eJe bi-vector space and fIe is a left fRf/fJf right eS_Ie/eJe subspace of fJe. If R has every indecomposable left injective module uniserial then clearly Re/Ie is the injective hull of Rf/Jf.

LEMMA 3.3. *Let R be a basic Artinian ring with $J^2 = 0$ and e, f primitive idempotents satisfying $fJe \neq 0$. If R has every indecomposable left injective module uniserial and eS_Ie/eJe is inner in eRe/eJe, then fR is uniserial.*

Proof. The proof given here is similar to that found in [4, Theorem 3.2].

Apply [4, Lemma 1.3] to get that $fJe = fJ$. Set $D' = fRf/fJf$, $D = eRe/eJe$, and $D_1 = eS_I e/eJe$. Now apply 1.4, 2.2, and 2.3 of [4] to get that ${}_{D'}(fJe)_D$ and ${}_{D'}(fIe)_{D_1}$ satisfy the hypotheses of 3.1. Thus

$$\dim_D (fJe) = c(fJ) = 1.$$

If R is local, set $D = R/J$. If $J^2 = 0$, set $D_1 = S_I/J \subseteq D$ for $I \subseteq J$. Clearly D_1 is a subskew field of D.

LEMMA 3.4. *Suppose $J^2 = 0$ and R is right uniserial with $J = wR$. Then $\beta = \{1, \tau_1, ..., \tau_{n-1}\}$ is a basis of D as a left vector space over S_w/J if and only if*

$$J = Rw \oplus Rw\tau_1 \oplus \cdots \oplus Rw\tau_{n-1}.$$

Thus $c({}_RJ) = \dim_{S_w/J}(D)$.

Proof. It is enough to observe that β is a basis for D over S_w/J if and only if $\{w, w\tau_1, ..., w\tau_{n-1}\}$ is a basis for ${}_RJ$ over D.

LEMMA 3.5. *Let R be local with $J^2 = 0$. Suppose the indecomposable left injective module is uniserial and let I be a maximal left ideal of J. The following conditions hold:*

(1) *There is a Morita duality between the finitely generated left R-modules and the finitely generated right S_I/I modules.*

(2) *$S_I/J \cong R/J$ as rings.*

(3) $\dim(R/J)_{S_I/J} = c({}_RJ)$, *where R/J is considered as a right S_I/J vector space.*

Proof. To prove (1) use $S_I/I \cong \mathrm{End}_R(R/I)$ and that R/I is an injective cogenerator along with Morita's characterization of duality [7, Theorem 6.3].

To prove (2) first observe that $J = I \oplus Rw = I \oplus wS_I$ using the injectivity of R/I. Define $\beta: S_I/J \to R/J$ by $\beta(\varphi + J) = r + J$, where $r \in R$ satisfies $w\varphi = x + rw (x \in I)$. The map β is a well-defined bijection by the above equation. It is easily verified that β is a ring isomorphism.

For (3) use (1) to show that

$$c({}_RJ) = \dim(R/I/J/I)_{S_I/J} = \dim(R/J)_{S_I/J}.$$

Applying Lemma 3.5 and [3, Lemma 5.1] yields the following corollary.

COROLLARY 3.6. *Let R be local with R/J a skew field of type* 1. *The following conditions are equivalent*:

(1) *R is uniserial.*

(2) *R is left(right) uniserial.*

(3) *R has its indecomposable right(left) injective module uniserial.*

(4) *R has both its left and right indecomposable injective modules uniserial.*

Remark. Using Corollary 1.2 it is clear that Corollary 3.6 holds for any R such that R/J is finitely generated over its center whenever its center is of type 1.

LEMMA 3.7. *Let R be a local ring with $J^2=0$. Suppose there exists $0 \neq w \in J$ such that $\beta = \{1, \tau_1, ..., \tau_{n-1}\}$ is an inner basis for D over S_w/J. Let $I = Rw \oplus Rw\tau_1 \oplus \cdots \oplus Rw\tau_{n-2}$. Then $S_w = S_{w\tau_k}$ for $k = 1, ..., n-1$ and $S_w \subseteq S_I$.*

Proof. The last statement clearly follows from the first statement. To prove the first statement observe that $S_{w\tau_k} = \tau_k^{-1} S_w \tau_k$. Since β is an inner basis we obtain $S_w = S_{w\tau_k}$.

THEOREM 3.8. *The following conditions are equivalent*:

(1) *R is right uniserial and there exists $0 \neq w \in J/J^2$ such that S_w/J is inner in D.*

(2) *There exists a non-zero $t \in J/J^2$ and maximal left ideal I of J/J^2 such that $J/J^2 = I \oplus tS_t$ and $S_I = S_t$ with S_I/J inner in D.*

(3) *The indecomposable left injective module is uniserial and there exists a maximal left ideal I of J/J^2 such that S_I/J is inner in D.*

Proof. In applying Lemma 3.2 and a theorem of Nakayama [8], it suffices to consider the case when $J^2=0$.

(1) implies (2). Let $\{1, \tau_1, ..., \tau_{n-1}\}$ be an inner basis for D over S_w. Using $\tau_i S_w = S_w \tau_i$ and setting $I = Rw \oplus \cdots \oplus Rw\tau_{n-2}$ yields $J = wR = w(S_w + \tau_1 S_w + \cdots + \tau_{n-1} S_w + J) = Rw + Rw\tau_1 + \cdots + Rw\tau_{n-2} + w\tau_{n-1} S_w = I \oplus tS_w (t = w\tau_{n-1})$. Applying 3.7 yields $S_t = S_w \subseteq S_I$ and so $J = I + tS_I$. It is easily verified that the above sum is direct. Since $S_t \subseteq S_I$ and $I \oplus tS_I = I \oplus tS_t$ we obtain $S_t = S_I$.

(2) implies (3). It suffices to show that R/I is left injective. Let $f\colon J/I \to R/I$ be defined as $f(t+I) = \alpha t + I$ for some $\alpha \in R$, where $J/I = (Rt+I)/I$. Therefore, $\alpha t = h + ts$ $(h \in I, s \in S_I)$. Thus f can be extended to R/I using right multiplication by s modulo I. This means that R/I is quasi-injective and since R/I is faithful, R/I is injective.

(3) implies (1). Apply Lemma 3.3 to get that R is right uniserial. Let Rw be a simple direct summand of I. Applying Lemma 3.7 yields $S_w \subseteq S_I$. Now apply Lemmas 3.4 as 3.5 to get that

$$\dim_{S_w/J}(D) = \dim(D)_{S_I/J} = c({}_RJ).$$

Since S_I/J is inner in D the above equation holds with D considered as a left vector space over S_I/J. This implies that $S_w = S_I$.

The rest of this Section is devoted to some applications of Lemma 3.3 and Theorem 3.8. First we show that a ring R with R/J finitely generated over its center is right serial if and only if the left indecomposable injective modules are serial. Second we construct some examples satisfying Theorem 3.8 when R/J is not finitely generated over its center.

LEMMA 3.9. *Let D be a skew field finitely generated over its center K and D_1 a subskew field of D. Then D has an inner basis over D_1.*

Proof. Set $D' = KD_1$ and apply [6, Proposition 2, p. 158] to get that D' is a finite dimensional central subskew field of D. Now apply [6, Theorem 2, p. 118] to get that $D = C_D(D') \cdot D'$, where $C_D(D')$ is the centralizer of D' in D. Thus

$$D = C_D(D') \cdot KD_1.$$

Therefore, a set of generators for D over D_1 can be chosen from $C_D(D')\,K$ and this yields an inner basis.

THEOREM 3.10. *Let R be an Artinian ring with R/J finitely generated over its center. Then every indecomposable injective left R-module is uniserial if and only if R is right serial.*

Proof. By Lemma 3.2 and a theorem of Nakayama [8], it suffices to consider the case when $J^2 = 0$.

Suppose every indecomposable injective left R-module is uniserial. Applying [4, Lemma 1.2] we may assume that R is basic. Let f be a primitive idempotent such that $fJ \neq 0$. Therefore there exists a primitive idempotent e such that $fJe \neq 0$. Letting I be as in Lemma 3.3, apply Lemma 3.9 to get that eS_Ie/eJe is inner in eRe/eJe. Now an application of Lemma 3.3 shows that fR is uniserial. Since $fJ = 0$ implies that fR is simple, we find that R is right serial.

Suppose R is right serial. Since right serial is preserved under Morita equivalence, we may assume that R is basic. Let e and f be primitive idempotents such that $eJf \neq 0$. Set $D' = fRf/fJf$ and $D = eRe/eJe$, $W = eJf$ and $E = \mathrm{End}_D(W)$. It is clear that $eJ = eJf$ and that ${}_DW_{D'}$ is a $D - D'$ bivector space with left and right dimensions finite. Consider the canonical map $\Lambda : D' \to E$ defined as $\Lambda(d')(x) = xd'$ $(x \in W, d' \in D')$. Thus $T = \Lambda(D')$ is a subring of E. We show that the left and right dimensions of E over T are the same.

Let $0 \neq w \in W$. Therefore, $W = wD'$. Now consider

$$S'_w = \{d' \in D' \mid wd' \in Dw\}.$$

Thus S'_w is a subskew field of D' and $S'_w \cong D$. Therefore, S'_w is finitely generated over its center. We now show that Λ maps S'_w into the subring $\mathrm{Diag}(E)$ of diagonal elements of E.

Observe that $\{1, \tau_1, ..., \tau_{n-1}\}$ is a basis for D' over S'_w if and only if $\{w_1, w\tau_1, ..., w\tau_{n-1}\}$ is a basis for W over D. Apply Lemma 3.9 to get an inner basis $\{1, \tau_1, ..., \tau_{n-1}\}$ of D' over S'_w.

Now consider $s \in S'_w$ and a basis element $w\tau_i$ of W over D. Thus for each i there exists $d \in D$ and $s' \in S'_w$ such that

$$dw = ws' \qquad \text{and} \qquad s'\tau_i = \tau_i s. \tag{1}$$

Therefore,

$$\Lambda(s)(w\tau_i) = w\tau_i s = ws'\tau_i = dw\tau_i. \tag{2}$$

Thus S'_w is mapped into $\mathrm{Diag}(E)$. Let $S = \Lambda(S'_w)$ and consider $d \in D$. Using that $Dw = wS'_w$, there exist $s, s' \in S'_w$ satisfying (1) and (2). This says that the projection of $\mathrm{Diag}(E)$ along the ith coordinate when restricted to S is D. Thus if $\{E_{ii}\}_{i=1}^n$ are those elements of $\mathrm{Diag}(E)$ with 1 in the ith position on the diagonal and 0's elsewhere then $\{E_{ii}\}$ is a left and right basis for $\mathrm{Diag}(D)$ over S. Since E is both a left and right free module over $\mathrm{Diag}(D)$ of dimension n, we find that the left and right dimensions of E over S are finite and equal. Thus

$$[E : S] = \dim_T E \cdot \dim_S T = \dim(E)_T \cdot \dim(T)_S.$$

Since T is finitely generated over its center and $\dim_S T$ is finite, the left and right dimensions of T over S are equal. This yields

$$\dim_T E = \dim\,(E)_T.$$

Let $\{f_i\}$ be a basic set of primitive orthogonal idempotents with $f_1 + \cdots + f_s = 1$. Since the set is basic, $f_i R f_j \subseteq J f_j$ for $(i \neq j)$. Thus $f_i R f_j J = 0$ $(i \neq j)$. This implies that each $(f_1 + \cdots + f_r) J$ is a two-sided ideal, where $r \leqslant s$. Therefore since $J^2 = 0$,

$$0 \leqslant f_1 J \leqslant (f_1 + f_2) J \leqslant \cdots \leqslant J$$

is a two-sided composition series in the sense of [9, 2.6]. This observation and our previous remarks show that the hypotheses of [9, Theorem 2.7] are satisfied. Therefore, the composition lengths of fR and the injective hull $E(Rf/Jf)$ of Rf/Jf are the same for each primitive idempotent f. This shows that every indecomposable injective left R-module is uniserial.

COROLLARY 3.11. *Let R be an Artinian ring such that R/J is finitely generated over its center. Then R is serial if and only if every indecomposable injective R-module is uniserial.*

Proof. Apply Theorem 3.10.

Recall that a left uniserial ring is defined to be *cleft* in the case where it is an abelian group direct sum of its radical and a skew field. Given a skew field D and a ring monomorphism $\varphi : D \to D$ we can always construct a left uniserial ring as follows: Let $V = Dx_1 \oplus \cdots \oplus Dx_m$ be a D-vector space with $\dim_D(V) = m$, and define $A(D, V, \varphi)$ to be the ring $D \oplus V$ with multiplication given by $d_1 x_s d_2 x_t = d_1 \varphi^s(d_2)\, x_{s+t}$ with x_0 the identity on D, φ^0 the identity map on D, and $x_r = 0$, $r > m$. Ivanov [5, Theorem 12] has shown that any cleft left uniserial ring R is isomorphic to some $A(D, V, \varphi)$. We observe that $R/J^2 = D \oplus Dx_1$ and $Dx_1 = J/J^2$ is the radical of

$$S = A(D, V, \varphi)/J^2(A(D, V, \varphi)).$$

It is easily seen that if $ax_1 = x_1 b = \varphi(b)\, x_1$ then $a = \varphi(b)$ so that $S_{x_1}/J = \varphi(D)$. Thus we have the following proposition.

PROPOSITION 3.12. *Every $A(D, V, \varphi)$ is a cleft left uniserial ring and every cleft left uniserial ring is isomorphic to some $A(D, V, \varphi)$. Moreover $S_{x_1}/J \cong \varphi(D)$.*

Remarks. (1) In light of this proposition, we can construct classes of left uniserial rings with $S_w/J = \varphi(D)$ for a given monomorphism $\varphi : D \to D$ and a fixed $w \in J/J^2$. Thus by letting D be as in the examples constructed in Section 2 we can easily construct classes of left uniserial rings with S_w/J inner in $D = R/J$ and such that D is not finitely generated over its center. These rings satisfy the right analogue of Theorem 3.8 and so have their right indecomposable injective module uniserial.

(2) Xue [10] has constructed an example of a left uniserial ring whose right indecomposable injective module is not uniserial, and in [11] he has constructed a non-serial ring whose indecomposable injective modules are all uniserial. Thus Theorem 3.10 and Corollary 3.11 do not extend to rings R such that R/J is not finitely generated over its center.

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