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Boundary effects on the mass and coupling constant in the compactified Ginzburg–Landau model: The boundary dependent critical temperature

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We consider the Euclidean D-dimensional N-component $\lambda|\varphi|^4$ ($\lambda > 0$) model with d ($d \leq D$) compactified dimensions. Introducing temperature by means of the Ginzburg–Landau prescription in the mass term of the Hamiltonian, this model can be interpreted as describing a second-order phase transition for a system in a region of the D-dimensional space, limited by d pairs of parallel planes, orthogonal to the coordinates axis $x_1, x_2, \ldots, x_d$. The planes in each pair are separated by distances $L_1, L_2, \ldots, L_d$. Making the appropriate boundary corrections to the coupling constant, we obtain in the large-N limit the transition temperature as a function of the size of the system, $T_c(\{L_i\})$, $i=1,2,\ldots,d$. For $D=3$ we particularize this formula, taking $L_1=L_2=\cdots=L_d=L$ for the physically interesting cases $d=1$ (a film), $d=2$ (an infinitely long wire having a square cross section), and $d=3$ (a cubic grain). © *2009 American Institute of Physics.* [DOI: 10.1063/1.3204079]

I. INTRODUCTION

Studies on field theory applied to second-order phase transitions have been done in literature for a long time. A thorough account on the subject can be found in Refs. 1–10. Recent applications of similar ideas to bounded systems can also be found in Refs. 11 and 12. Under the assumption that information about general features of the behavior of systems undergoing phase transitions can be obtained in the approximation which neglects gauge field contributions in the Ginzburg–Landau model, investigations have been done with an approach different from the renormalization-group analysis. Phase transitions in bounded systems, in particular, the system confined between two parallel planes (a sample of a material in the form of a film), have been considered and the dependence of the critical temperature on the film thickness has been established,[13–16] in particular, in comparison with experimental data using ideas from Refs. 17–19.

In this paper, starting from the formalism developed in Refs. 20 and 21, the way in which the critical temperature for a second-order phase transition is affected by the presence of confining boundaries is investigated on general grounds. We focus, in particular, on the mathematical aspects of the formalism, which furnish the tools to study boundary effects on the phase transition. We consider the D-dimensional N-component Ginzburg–Landau model compactified in d ($\leq D$) of the spatial dimensions. Taking the large-N limit, which allows to take into account nonperturbatively corrections to the coupling constant, we obtain expressions for the transition temperature in the general situation. For $D=3$ and $d=1$, $d=2$, and $d=3$, we have the critical temperature ($T_c(L)$) for the system in the form of a film of thickness L, an infinitely long wire having a square cross section L^2, and for a cubic grain of volume L^3. We show that $T_c(L)$ decreases as the size L is diminished in a slightly nonlinear way. The minimal size for the suppression of the second-order

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transition is lower than the one obtained without considering coupling-constant corrections. These results generalize previous works dealing with transitions in low-dimensional compactified subspaces.[21–23]

It is worth to emphasize that this generalization can be done by using nontrivial extensions to several dimensions of the one-dimensional mode-sum regularization described in Ref. 24. These extensions require, in particular, the definition of symmetrized multidimensional Epstein–Hurwitz zeta functions with no analog in the one-dimensional case.[21,25] This allows one to get general formulas for the critical temperature as a function of the size of the system.

II. EFFECTIVE POTENTIAL WITH COMPACTIFICATION OF A *d*-DIMENSIONAL SUBSPACE

In the absence of geometrical restrictions, the N-component vector model is described by the Ginzburg–Landau Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\nabla\varphi_a\cdot\nabla\varphi_a + \frac{1}{2}m_0^2(T)\varphi_a\varphi_a + \frac{\lambda}{N}(\varphi_a\varphi_a)^2, \tag{1}$$

where λ is the coupling constant, $m_0^2(T)=\alpha(T-T_0)$ is the bare mass (with T_0 as the bulk transition temperature), and summation over repeated indices a is assumed. In the following, we will consider the model described by the Hamiltonian (1) with N components and take the large-N limit.

We here consider the system in D dimensions confined to a region of space delimited by $d \leq D$ pairs of parallel planes. Each plane of a pair j is at a distance L_j from the other member of the pair, $j=1,2,\ldots,d$, and is orthogonal to all other planes belonging to distinct pairs $\{i\}$, $i\neq j$. This may be pictured as a parallelepiped box embedded in the D-dimensional space, whose parallel faces are separated by distances $L_1,L_2,\ldots,L_d$. We use Cartesian coordinates $\mathbf{r}=(x_1,\ldots,x_d,\mathbf{z})$, where $\mathbf{z}$ is a $(D-d)$-dimensional vector, with corresponding momentum $\mathbf{k}=(k_1,\ldots,k_d,\mathbf{q})$, $\mathbf{q}$ is a $(D-d)$-dimensional vector in momentum space. The Hamiltonian thus becomes

$$\mathcal{H} = \frac{1}{2}\nabla\varphi_a\cdot\nabla\varphi_a + \frac{1}{2}\bar{m}_0^2(T;L_1,\ldots,L_d)\varphi_a\varphi_a + \frac{\lambda}{N}(\varphi_a\varphi_a)^2, \tag{2}$$

where $\bar{m}_0^2(T;L_1,\ldots,L_d)$ is a suitably defined boundary-modified mass parameter such that

$$\lim_{\{L_i\}\to\infty}\bar{m}_0^2(T;L_1,\ldots,L_d) = m_0^2(T) \equiv \alpha(T-T_0). \tag{3}$$

The $\{L_i\}$-corrections entering in the coupling constant λ are one of the main subjects of this paper and will be considered in detail later.

The generating functional of the correlation functions is written in the form

$$Z = \int \mathcal{D}\varphi \exp\left(-\int_0^{L_1}dx_1\cdots\int_0^{L_d}dx_d\int d^{D-d}\mathbf{z}\,\mathcal{H}(|\varphi|,|\nabla\varphi|)\right), \tag{4}$$

with the field $\varphi(x_1,\ldots,x_d,\mathbf{z})$ satisfying the condition of confinement inside the box, $\varphi(x_i\leq 0,\mathbf{z}) = \varphi(x_i\geq L_i,\mathbf{z})=0$. Then, following the procedure developed in Ref. 21, we are allowed to introduce a generalized Matsubara prescription, performing the following multiple replacements (compactification of a d-dimensional subspace):

$$\int\frac{dk_i}{2\pi} \to \frac{1}{L_i}\sum_{n_i=-\infty}^{+\infty}, \quad k_i \to \frac{2n_i\pi}{L_i}, \quad i=1,2,\ldots,d. \tag{5}$$

Notice that compactification can be implemented in different ways, as, for instance, by imposing specific conditions on the fields at spatial boundaries. We here choose periodic boundary conditions. We emphasize, however, that we are considering a Euclidean field theory in D *purely* spatial dimensions. Therefore, we are *not* working within the framework of finite-temperature field theory. Here, the temperature is introduced in the mass term of the Hamiltonian by means of the usual Ginzburg–Landau prescription.

In principle, the effective potential for systems with spontaneous symmetry breaking is obtained, following the analysis introduced in Ref. 26, as an expansion in the number of loops in Feynman diagrams. Accordingly, to the free propagator and to the no-loop (tree) diagrams for the coupling, radiative corrections are added with increasing number of loops. Thus, at the one-loop approximation, we get the infinite series of one-loop diagrams with all numbers of insertions of the φ^4 vertex (two external legs in each vertex).

At the one-loop approximation, the contribution from $|\varphi_0|^4$ vertices to the effective potential is obtained directly as an adaptation of the Coleman–Weinberg expression after compactification in d dimensions. In this case, we start from the well-known expression for the one-loop contribution to the zero-temperature effective potential in unbounded space,

$$U_1(\varphi_0) = \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s}[12(\lambda/N)\varphi_0^2]^s \int \frac{d^D k}{(k^2+m^2)^s}, \tag{6}$$

where m is the *physical* mass.

Then to deal with dimensionless quantities in the regularization procedures, we introduce parameters $c^2=m^2/4\pi^2\mu^2$, $b_i=(L_i\mu)^{-2}$, $g=(\lambda/N)/4\pi^2\mu^{4-D}$, and $\phi_0^2=\varphi_0^2/\mu^{D-2}$, where m is the physical mass in the absence of boundaries, φ_0 is the normalized vacuum expectation value of the field (the classical field), and μ is a mass scale (naturally, the results do not depend on μ). Performing the replacement (5), the compactified ($\{L_i\}$-dependent) one-loop contribution to the effective potential can be written as

$$U_1(\phi_0,\{b_i\}) = \mu^D \sqrt{b_1\cdots b_d}\sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s}[12g\phi_0^2]^s \sum_{n_1\cdots n_d=-\infty}^{+\infty} \int \frac{d^{D-d}q'}{(b_1 n_1^2+\cdots b_d n_d^2+\mathbf{q}'^2+c^2)^s}, \tag{7}$$

where $\{b_i\}=\{b_1,b_2,\ldots,b_d\}$ and $q'=q/2\pi\mu$ is dimensionless.

The integral over the $D-d$ noncompactified momentum variables is performed using the well-known dimensional regularization formula,[9]

$$\int \frac{d^l p}{(2\pi)^l}\frac{1}{(\mathbf{p}^2+M)^s} = \frac{\Gamma\left(s-\frac{l}{2}\right)}{(4\pi)^{l/2}\Gamma(s)M^{s-l/2}} \tag{8}$$

for $l=D-d$, we obtain

$$U_1(\phi_0,\{b_i\}) = \mu^D \sqrt{b_1\cdots b_d}\sum_{s=1}^{\infty} f(D,d,s)[12g\phi_0^2]^s Z_d^{c^2}\left(s-\frac{D-d}{2};\{b_1\}\right), \tag{9}$$

where

$$f(D,d,s) = \pi^{(D-d)/2}\frac{(-1)^{s+1}}{2s\Gamma(s)}\Gamma\left(s-\frac{D-d}{2}\right) \tag{10}$$

and $Z_d^{c^2}(\nu;\{a_i\})$ are Epstein–Hurwitz zeta functions, valid for $\mathrm{Re}(\nu)>d/2$, defined by

$$Z_d^{c^2}(\nu;\{a_i\}) = \sum_{n_1,\ldots,n_d=-\infty}^{+\infty} (a_1 n_1^2+\cdots+a_d n_d^2+c^2)^{-\nu}. \tag{11}$$

This multidimensional Epstein–Hurwitz function possesses the following analytical extension to the whole complex ν plane:[27,28]

$$Z_d^{c^2}(\nu;\{a_i\}) = \frac{\pi^{d/2}}{\sqrt{a_1\cdots a_d}\Gamma(\nu)}\Gamma\left(\nu-\frac{d}{2}\right)c^{d-2\nu} + \frac{2\pi^\nu}{\sqrt{a_1\cdots a_d}\Gamma(\nu)}\sum_{n_1,\ldots,n_d=-\infty}^{+\infty\,\prime}\left(\frac{1}{c}\sqrt{\frac{n_1^2}{a_1}+\cdots+\frac{n_d^2}{a_d}}\right)^{\nu-d/2} \times K_{d/2-\nu}\left(2\pi c\sqrt{\frac{n_1^2}{a_1}+\cdots+\frac{n_d^2}{a_d}}\right), \tag{12}$$

where the prime attached to the summation symbol means that $n_1=n_2=\cdots=n_d=0$ is excluded from the indices and $K_{d/2-\nu}(z)$ is a Bessel function of the third kind. This analytical extension is the generalization to several dimensions of the mode-sum regularization prescription described in Ref. 24, as used in Ref. 21.

Now, using Eq. (12), expanding the prime summation, and coming back to the original variables $(L_i, m, \lambda, \varphi_0)$, the one-loop contribution to the effective potential becomes

$$U_1(\varphi_0,\{L_i\}) = \sum_{s=1}^{\infty}[12(\lambda/N)\varphi_0^2]^s h(D,s)\Bigg[2^{s-D/2-2}\Gamma(s-(D/2))m^{D-2s} + \sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{m}{L_i n_i}\right)^{D/2-s}K_{D/2-s}(mL_i n_i) + 2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}\left(\frac{m}{\sqrt{L_i^2n_i^2+L_j^2n_j^2}}\right)^{D/2-s}K_{D/2-s}(m\sqrt{L_i^2n_i^2+L_j^2n_j^2}) + \cdots + 2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m}{\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}}\right)^{D/2-s}K_{D/2-s}(m\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2})\Bigg], \tag{13}$$

where

$$h(D,s) = \frac{1}{2^{D/2+s-1}\pi^{D/2}}\frac{(-1)^{s+1}}{s\Gamma(s)}. \tag{14}$$

The mass and the coupling constant are obtained from the normalization conditions

$$\left.\frac{\partial^2}{\partial\varphi_0^2}U(\varphi_0)\right|_{\varphi_0=0} = m^2 \tag{15}$$

and

$$\left.\frac{\partial^4}{\partial\varphi_0^4}U(\varphi_0)\right|_{\varphi_0=0} = \frac{\lambda}{N}, \tag{16}$$

where U is the sum of the tree-level and one-loop contributions to the effective potential.

III. BOUNDARY EFFECTS ON THE COUPLING CONSTANT IN THE LARGE-N LIMIT

In the following, we consider the four-point function at zero external momenta, which we take as the basic object for our definition of the renormalized coupling constant. At leading order in $1/N$, it is given by the sum of all chains of one-loop diagrams, which has the formal expression[9]

$$\Gamma_D^{(4)}(\mathbf{p}=0,m,\{L_i\}) = \frac{\lambda/N}{1+\lambda\Pi(D,m,\{L_i\})}, \tag{17}$$

where, after using the prescription (5), $\Pi(D,m,\{L_i\})=\Pi(\mathbf{p}=0,D,m,\{L_i\})$ corresponds to the single bubble four-point diagram with compactification of a d-dimensional subspace.

To proceed, we use the renormalization condition (16) from which we deduce formally that the single bubble function $\Pi(D,m,\{L_i\})$ is obtained from the coefficient of the fourth power of the field ($s=2$) in Eq. (13). Then, using Eq. (16), we can write $\Pi(D,m,\{L_i\})$ in the form

$$\Pi(D,m,\{L_i\}) = H(D,m) + \Pi_R(D,m,\{L_i\}), \tag{18}$$

where the $\{L_i\}$-dependent term $\Pi_R(D,m,\{L_i\})$ comes from the second term between brackets in Eq. (13),

$$\begin{aligned}\Pi_R(D,m;\{L_i\}) = \frac{1}{(2\pi)^{D/2}}\Bigg[&\sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{m}{L_i n_i}\right)^{D/2-2} K_{D/2-2}(mL_i n_i) + 2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}\left(\frac{m}{\sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}}\right)^{D/2-2} \\ &\times K_{D/2-2}\left(m\sqrt{L_i^2 n_i^2 + L_j^2 n_j^2}\right) + \cdots + 2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m}{\sqrt{L_1^2 n_1^2 + \ldots + L_d^2 n_d^2}}\right)^{D/2-2} \\ &\times K_{D/2-2}\left(m\sqrt{L_1^2 n_1^2 + \ldots + L_d^2 n_d^2}\right)\Bigg]\end{aligned} \tag{19}$$

and $H(D,m)$ is a polar term coming from the first term between brackets in Eq. (13),

$$H(D,m) \propto \Gamma\left(2 - \frac{D}{2}\right) m^{D-4}. \tag{20}$$

We see from Eq. (20) that for even dimensions $D \geq 4$, $H(D,m)$ is divergent due to the pole of the Γ-function. Accordingly, this term must be subtracted to give the renormalized single bubble function $\Pi_R(D,m,\{L_i\})$. In order to have a coherent procedure for a generic dimension D, the subtraction of the term $H(D,m)$ should be performed even in the case of odd dimensions, where no poles of Γ-functions are present (finite renormalization). From the properties of Bessel functions, it can be seen from Eq. (19) that for any dimension D, $\Pi_R(D,m,\{L_i\})$ satisfies the conditions

$$\lim_{L_i\to\infty} \Pi_R(D,m,\{L_i\}) = 0, \quad \lim_{L_i\to 0} \Pi_R(D,m,\{L_i\}) \to \infty. \tag{21}$$

We also conclude, from the properties of Bessel functions, that $\Pi_R(D,m,\{L_i\})$ is positive for all values of D and $\{L_i\}$.

Taking inspiration from Eq. (17), let us define the $\{L_i\}$-dependent renormalized coupling constant $\lambda_R(m,D,\{L_i\})$, at the leading order in $1/N$, as

$$\Gamma_{D,R}^{(4)}(\mathbf{p}=0,m,\{L_i\}) \equiv \frac{1}{N}\lambda_R(D,m,\{L_i\}) = \frac{(\lambda/N)}{1 + \lambda\Pi_R(D,m,\{L_i\})} \tag{22}$$

and let $\lambda_R(D,m)$, the renormalized coupling constant in the absence of constraints, be defined by

$$\frac{\lambda_R(D,m)}{N} = \lim_{L_i\to\infty} \Gamma_{D,R}^{(4)}(\mathbf{p}=0,m,\{L_i\}). \tag{23}$$

From Eqs. (21)–(23), we simply get $\lambda_R(D,m)=\lambda$. In other words, we have done a choice of renormalization scheme such that the constant λ introduced in the Hamiltonian corresponds to the *renormalized* coupling constant in the absence of boundaries. From Eqs. (22) and (23) we obtain the $\{L_i\}$-dependent renormalized coupling constant

$$\lambda_R(D,m,\{L_i\}) = \frac{\lambda}{1 + \lambda\Pi_R(D,m,\{L_i\})}. \tag{24}$$

IV. BOUNDARY EFFECTS ON THE CRITICAL BEHAVIOR

Criticality is attained from the ordered phase when the inverse squared correlation length, $\xi^{-2}(\{L_i\},\phi_0)$, vanishes in the large-N gap equation

$$\xi^{-2}(\{L_i\},\varphi_0) = \bar{m}_0^2 + 12\lambda_R(D,\{L_i\})\varphi_0^2 + \frac{24\lambda_R(D,\{L_i\})}{L_1\cdots L_d}\sum_{\{n_j\}=-\infty}^{\infty}\int \frac{d^{D-d}q}{(2\pi)^{D-d}}\frac{1}{\mathbf{q}^2 + \sum_{j=1}^{d}\left(\frac{2\pi n_j}{L_j}\right)^2 + \xi^{-2}(\{L_i\},\varphi_0)}. \tag{25}$$

In the ordered-disordered border, φ_0 vanishes and the inverse correlation length equals the physical mass. The physical mass is obtained at the one-loop order from Eqs. (13) and (15) (again suppressing the polar term of U_1); after performing the change $\lambda \to \lambda_R(D,m,\{L_i\})$, where $\lambda_R(D,m,\{L_i\})$ is the renormalized $\{L_i\}$-dependent coupling constant [Eq. (24)], we get

$$\begin{aligned} m^2(D,T,\{L_i\}) = \bar{m}_0^2(T,\{L_i\}) + \frac{24\lambda_R(D,m,\{L_i\})}{(2\pi)^{D/2}}\Bigg[\sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{m}{L_in_i}\right)^{D/2-1}K_{D/2-1}(mL_in_i) \\ + 2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}\left(\frac{m}{\sqrt{L_i^2n_i^2+L_j^2n_j^2}}\right)^{D/2-1}K_{D/2-1}\left(m\sqrt{L_i^2n_i^2+L_j^2n_j^2}\right) + \cdots \\ + 2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m}{\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}}\right)^{D/2-1}K_{D/2-1}\left(m\sqrt{L_1^2n_1^2+\cdots+L_d^2n_d^2}\right)\Bigg], \end{aligned} \tag{26}$$

where, also in the right-hand side, $m=m(D,T,\{L_i\})$. Besides, the renormalized coupling constant $\lambda_R(D,m,\{L_i\})$ is itself a function of $m(D,T,\{L_i\})$, as given by appropriate versions of Eqs. (19) and (24), i.e.,

$$\lambda_R(D,m,\{L_i\})) = \frac{\lambda}{1+\lambda\Pi_R(D,m(D,T,\{L_i\}),\{L_i\})}, \tag{27}$$

with

$$\begin{aligned} \Pi_R(D,m(D,T,\{L_i\});\{L_i\}) = \frac{1}{(2\pi)^{D/2}}\Bigg[\sum_{i=1}^{d}\sum_{n_i=1}^{\infty}\left(\frac{m(D,T,\{L_i\})}{L_in_i}\right)^{D/2-2}K_{D/2-2}(m(D,T,\{L_i\})L_in_i) \\ + 2\sum_{i<j=1}^{d}\sum_{n_i,n_j=1}^{\infty}\left(\frac{m(D,T,\{L_i\})}{\sqrt{L_i^2n_i^2+L_j^2n_j^2}}\right)^{D/2-2} \\ \times K_{D/2-2}\left(m(D,T,\{L_i\})\sqrt{L_i^2n_i^2+L_j^2n_j^2}\right) + \cdots \\ + 2^{d-1}\sum_{n_1,\ldots,n_d=1}^{\infty}\left(\frac{m(D,T,\{L_i\})}{\sqrt{L_1^2n_1^2+\ldots+L_d^2n_d^2}}\right)^{D/2-2} \\ \times K_{D/2-2}\left(m(D,T,\{L_i\})\sqrt{L_1^2n_1^2+\ldots+L_d^2n_d^2}\right)\Bigg]. \end{aligned} \tag{28}$$

Therefore, to determine $m(D,T,\{L_i\})$, one has to solve a set of complicated, transcendental, coupled equations since $m(D,T,\{L_i\})$ appears in the right-hand sides of Eqs. (26) and (28) as part of the argument of $K_{D/2-s}$ and $\lambda_R(D,m,\{L_i\})$ depends on $m(D,T,\{L_i\})$. This set of equations has no analytical solutions in general.

Nevertheless, if we restrict ourselves to the neighborhood of criticality $(m^2(D,T,\{L_i\})\approx 0)$ for high enough values of D, we can get a solution by taking the limit $m\to 0$ in the sums of the right-hand sides of Eqs. (26) and (28). Indeed, for $D/2-s$ such that these sums are convergent and have finite limits for $m=0$, we can use the asymptotic formula for small values of the argument of the Kelvin function

$$K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \quad (z\sim 0;\quad \mathrm{Re}(\nu)>0) \tag{29}$$

to show that, in the limit $m\to 0$, the factors $m(D,T,\{L_i\})$ present in the coefficients and in the arguments of the Kelvin functions in the sums cancel out exactly giving mass-independent expressions, that is,

$$\sum_{n_1,\ldots,n_p=1}^{\infty}\left(\frac{m(D,T,\{L_i\})}{\sqrt{L_1^2n_1^2+\cdots+L_p^2n_p^2}}\right)^{D/2-s} K_{D/2-s}(m(D,T,\{L_i\})\sqrt{L_1^2n_1^2+\cdots+L_p^2n_p^2})\Bigg|_{m\to 0}$$
$$=2^{D/2-s-1}\Gamma\left(\frac{D}{2}-s\right)E_p\left(\frac{D}{2}-s;L_1,\ldots,L_p\right), \tag{30}$$

where $s=1$ and $s=2$ for, respectively, the mass in Eq. (26) and the renormalized one-loop bubble function in Eq. (28). In both cases, $p=1,2,\ldots,d$ and $E_p(D/2-s;L_1,\ldots,L_p)$ is one of the generalized Epstein zeta functions defined, in symmetric form,[21] by

$$E_p(\nu;L_1,\ldots,L_p)=\frac{1}{p!}\sum_\sigma\sum_{n_1=1}^{\infty}\cdots\sum_{n_p=1}^{\infty}[\sigma_1^2n_1^2+\cdots+\sigma_p^2n_p^2]^{-\nu}, \tag{31}$$

where $\sigma_i=\sigma(L_i)$, with σ running in the set of all permutations of the parameters $L_1,\ldots,L_p$, and the summations over $n_1,\ldots,n_p$ being taken in the given order. Notice that, for $p=1$, E_p reduces to the Riemann zeta function, i.e.,

$$E_1(\nu,L_1)=\frac{1}{L_1^{2\nu}}\zeta(2\nu). \tag{32}$$

Also, one can construct analytical continuations and recurrence relations for these multidimensional Epstein functions which permit to write them in terms of Kelvin and Riemann zeta functions;[21,25] one gets

$$E_p(\nu;L_1,\ldots,L_p)=-\frac{1}{2p}\sum_{i=1}^{p}E_{p-1}(\nu;\ldots,\widehat{L_i},\ldots)+\frac{\sqrt{\pi}}{2p\Gamma(\nu)}\Gamma\left(\nu-\frac{1}{2}\right)\sum_{i=1}^{p}\frac{1}{L_i}E_{p-1}\left(\nu-\frac{1}{2};\ldots,\widehat{L_i},\ldots\right)$$
$$+\frac{2\sqrt{\pi}}{p\Gamma(\nu)}W_p\left(\nu-\frac{1}{2},L_1,\ldots,L_p\right), \tag{33}$$

where the hat over the parameter L_i in the functions E_{p-1} means that it is excluded from the set $\{L_1,\ldots,L_p\}$ (the others being the $p-1$ parameters of E_{p-1}), and

$$W_p(\eta;L_1,\ldots,L_p)=\sum_{i=1}^{p}\frac{1}{L_i}\sum_{n_1,\ldots,n_p=1}^{\infty}\left(\frac{\pi n_i}{L_i\sqrt{(\cdots+\widehat{L_i^2n_i^2}+\cdots)}}\right)^{\eta}K_\eta\left(\frac{2\pi n_i}{L_i}\sqrt{(\cdots+\widehat{L_i^2n_i^2}+\cdots)}\right), \tag{34}$$

with $(\cdots+\widehat{L_i^2n_i^2}+\cdots)$ representing the sum $\Sigma_{j=1}^{p}L_j^2n_j^2-L_i^2n_i^2$.

Inserting the appropriate versions of Eq. (30) into Eqs. (26) and (28), we obtain expressions for the physical mass and the renormalized coupling constant at criticality $(m^2\approx 0)$,

$$m^2(D,T,\{L_i\}) \approx \bar{m}_0^2(T,\{L_i\}) + \frac{6\lambda_R(D,\{L_i\})}{\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right)\left[\sum_{i=1}^{d}\frac{1}{L_i^{D-2}}\zeta(D-2) + 2\sum_{i<j=1}^{d}E_2\left(\frac{D}{2}-1;L_i,L_j\right) + \cdots + 2^{d-1}E_d\left(\frac{D}{2}-1;L_1,\ldots,L_d\right)\right] \tag{35}$$

and

$$\lambda_R(D,\{L_i\}) \approx \frac{\lambda}{1+\lambda C(D)\left[\Sigma_{i=1}^{d}L_i^{4-D}\zeta(D-4) + 2\Sigma_{i<j=1}^{d}E_2\left(\frac{D-4}{2};L_i,L_j\right) + \cdots + 2^{d-1}E_d\left(\frac{D-4}{2};L_1,\ldots,L_d\right)\right]}, \tag{36}$$

where

$$C(D) = \frac{1}{8\pi^{D/2}}\Gamma\left(\frac{D}{2}-2\right). \tag{37}$$

Substituting Eq. (36) into Eq. (35) and imposing the criticality condition, $m^2(D,T,\{L_i\})=0$, we obtain the critical temperature as a function of the distances between the parallel plane boundaries $\{L_i\}$. This allows us to analyze, taking $d=1,2,\ldots,D$, finite size effects on the critical temperature for the cases where Eqs. (35) and (36) are both well defined.

The cases of physical interest, $D=3$ with $d=1,2,3$ corresponding respectively to films, rectangular wires, and parallelepiped grains undergoing second-order phase transitions, however, cannot be considered by taking $D=3$ directly in Eq. (35); doing so, one faces the divergence of $\zeta(D-2)$ as $D\to 3$. Notice that the same sort of problem would appear if one tries to use Eq. (30) with $D=3$ and $d=1$, for example.

To get meaningful results for $D=3$, we have to perform an additional mass *zeta*-function renormalization. First, we analytically extend the expressions of m^2 and λ_R close to criticality [Eqs. (35) and (36)] to generic dimension D by considering the analytic extensions of the gamma function and the generalized Epstein and Riemann zeta functions; in this way, $m^2(D,T,\{L_i\})$ and $\lambda_R(D,\{L_i\})$ become meromorphic functions of D, as usually happens in dimensional and zeta-function regularization techniques. Then, we choose the bare mass parameter, $\bar{m}_0^2(T,\{L_i\})$, in such way that it eliminates the divergence coming from the pole that might exist at $D=3$. This procedure is applied in Secs. V–VII to obtain the dependence of the critical temperature on $\{L_i\}$ for films, wires, and grains, respectively.

V. BOUNDARY EFFECTS ON THE TRANSITION TEMPERATURE FOR FILMS

Let us now consider the case of films, where we have one spatial dimension compactified $(d=1)$ and make $L_1\equiv L$. In this case, Eqs. (35) and (36), using Eq. (37), reduce to

$$m^2(D,T,L) \approx \bar{m}_0^2(T,L) + \frac{6\lambda_R(D,L)}{\pi^{D/2}L^{D-2}}\Gamma\left(\frac{D}{2}-1\right)\zeta(D-2) \tag{38}$$

and

$$\lambda_R(D,L) \approx \frac{8\pi^{D/2}\lambda}{8\pi^{D/2}+\lambda\Gamma(D/2-2)\zeta(D-4)L^{4-D}}. \tag{39}$$

In these equations, the Γ- and ζ-functions are to be taken as their usual analytic extensions.

As $D\to 3$, Eq. (39) is finite, and using $\Gamma(-1/2)=-2\sqrt{\pi}$ and $\zeta(-1)=-1/12$, we get

$$\lambda_R(L) = \frac{48\pi}{48\pi + \lambda L}. \tag{40}$$

On the other hand, $\zeta(D-2)$ has a single pole at $D=3$ and Eq. (38), as it stands, is meaningless for $D=3$. However, it can be made physically meaningful if we perform a mass renormalization procedure to suppress this divergence, as follows.

In a vicinity of $D=3$, $\zeta(D-2)$ possesses the following Laurent expansion:

$$\zeta(D-2) = \frac{1}{D-3} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(D-3)^n, \tag{41}$$

where γ_n are the Stieltjes constants, $\gamma_0 \equiv \gamma \approx 0.5772$ being the Euler–Mascheroni constant. Thus, using that $\Gamma(1/2)=\sqrt{\pi}$, for $D\approx 3$, Eq. (38) can be written as

$$m^2(D,T,L) \approx \bar{m}_0^2(T,L) + \frac{6\lambda_R(L)}{\pi L}\left\{\frac{1}{D-3} + \gamma - \gamma_1(D-3) + \cdots\right\}. \tag{42}$$

We then define the L-dependent bare mass $\bar{m}_0^2(T,L)$, for $D\approx 3$, in such a way that the pole at $D=3$ in Eq. (42) is suppressed, that is, we take

$$\bar{m}_0^2(T,L) \approx M(T) - \frac{1}{(D-3)}\frac{6\lambda_R(L)}{\pi L}, \tag{43}$$

where $M(T)$ is the finite part. Inserting Eq. (43) in Eq. (42) and taking the limit $D\to 3$, we obtain

$$m^2(T,L) \approx M(T) + \frac{6\gamma\lambda_R(L)}{\pi L}. \tag{44}$$

To get the renormalized mass close to criticality, for $D=3$, we need to fix the finite term $M(T)$. This has to be done ensuring that condition (3) is satisfied. From Eq. (43), it follows that $\bar{m}_0^2(T,L)\to M(T)$ as $L\to\infty$ so that the simplest choice that guarantees (3) is

$$M(T) = m_0^2(T) = \alpha(T - T_0), \tag{45}$$

where T_0 is the bulk critical temperature. This leads to the renormalized mass m^2, close to criticality, in the Ginzburg–Landau form

$$m^2(T,L) \approx \alpha(T - T_c(L)), \tag{46}$$

where

$$T_c(L) = T_0 - \frac{48\pi C_1 \lambda}{48\pi\alpha L + \alpha\lambda L^2} \tag{47}$$

is the modified, L-dependent, transition temperature of the film and

$$C_1 = \frac{6\gamma}{\pi} \approx 1.1024. \tag{48}$$

The dependence of the critical temperature on the thickness of the film [Eq. (47)] has two important features. First, $T_c(L)\to T_0$ as $L\to\infty$ recovering the bulk sample critical temperature. Second, $T_c(L)$ decreases as L diminishes and there is a minimal allowed film thickness for the existence of the transition, obtained by solving $T_c(L)=0$,

$$L'_{\min} = \frac{24\pi}{\lambda}\left[\sqrt{1+\frac{\lambda L_{\min}}{12\pi}}-1\right], \tag{49}$$

where

$$L_{\min} = \frac{\mathcal{C}_1\lambda}{\alpha T_0} \tag{50}$$

is the corresponding minimal allowed thickness when no boundary corrections to the coupling constant are taken into account.[22]

VI. BOUNDARY EFFECTS ON THE TRANSITION TEMPERATURE FOR WIRES

We now focus on the situation where two spatial dimensions are compactified, $d=2$. Close to criticality ($m^2 \approx 0$), for dimension D high enough to ensure that both sides of Eq. (30) are finite, Eqs. (35) and (36), using Eq. (37), become

$$m^2(D,T,L_1,L_2) \approx \bar{m}_0^2(T,L_1,L_2) + \frac{6\lambda_R(D,L_1,L_2)}{\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right) \times\left[\left(\frac{1}{L_1^{D-2}}+\frac{1}{L_2^{D-2}}\right)\zeta(D-2)+2E_2\left(\frac{D}{2}-1;L_1,L_2\right)\right] \tag{51}$$

and

$$\lambda_R(D,L_1,L_2) \approx \frac{8\pi^{D/2}\lambda}{8\pi^{D/2}+\lambda\Gamma\left(\frac{D-4}{2}\right)\left[\sum_{i=1}^{2}\frac{\zeta(D-4)}{L_i^{D-4}}+2E_2\left(\frac{D}{2}-2;L_1,L_2\right)\right]}, \tag{52}$$

where the two-dimensional Epstein zeta function E_2 is defined by Eq. (31) with $p=2$.

Using Eq. (33), the analytical extension of $E_2(D/2-s;L_1,L_2)$, for $s=1,2$ is

$$E_2\left(\frac{D}{2}-s;L_1,L_2\right) = -\frac{1}{4}\left(\frac{1}{L_1^{D-2s}}+\frac{1}{L_2^{D-2s}}\right)\zeta(D-2s) + \frac{\sqrt{\pi}}{4\Gamma\left(\frac{D}{2}-s\right)}\left(\frac{1}{L_1L_2^{D-2s-1}}+\frac{1}{L_1^{D-2s-1}L_2}\right)\Gamma\left(\frac{D-2s-1}{2}\right)\zeta(D-2s-1) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{D}{2}-s\right)}W_2\left(\frac{D}{2}-s-\frac{1}{2};L_1,L_2\right), \tag{53}$$

which is a meromorphic function of D, symmetric in the parameters L_1 and L_2. The function $W_2(D/2-s-\frac{1}{2};L_1,L_2)$, appearing in the finite part in Eq. (53), is the particular case of Eq. (34) for $p=2$ and $\eta=D/2-s-\frac{1}{2}$. The poles of $E_2(D/2-s;L_1,L_2)$ are the poles of the Γ- and ζ-functions in (53). However, in handling the second term in Eq. (53), one has to be aware of the fact that the analytical extensions of the Γ- and ζ-functions are constructed in such a way that the reflection formula,

$$\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{z-1/2}\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z) \tag{54}$$

is satisfied; it should be used to calculate the limiting behavior as $D\to 3$.

For $s=2$, taking the limit $D\to 3$ in Eq. (53) and using that $\Gamma(3/2)=\sqrt{\pi}/2$, we obtain

$$E_2\left(-\frac{1}{2};L_1,L_2\right) = \frac{1}{48}(L_1+L_2) - \frac{\zeta(3)}{16\pi^2}\left(\frac{L_1^2}{L_2}+\frac{L_2^2}{L_1}\right) - \frac{1}{2}W_2(-1;L_1,L_2), \tag{55}$$

where

$$W_2(-1;L_1,L_2) = \frac{1}{\pi}\sum_{n_1,n_2=1}^{\infty}\left\{L_1\frac{n_1}{n_2}K_1\left(2\pi\frac{L_1}{L_2}n_1n_2\right) + L_2\frac{n_2}{n_1}K_1\left(2\pi\frac{L_2}{L_1}n_1n_2\right)\right\}. \tag{56}$$

Since $\zeta(D-4)$ is finite for $D\to 3$, this shows that expression (52) for $\lambda_R(D,L_1,L_2)$ is finite in this limit. On the other hand, using Eq. (41) and the analytical extension of $\Gamma(D-3/2)$,

$$\Gamma\left(\frac{D-3}{2}\right) = \frac{2}{D-3} - \gamma + \frac{1}{4}\left(\gamma^2+\frac{\pi^2}{6}\right)(D-3) + \cdots, \tag{57}$$

we obtain, for $D\approx 3$,

$$E_2\left(\frac{D}{2}-1;L_1,L_2\right) = -\frac{1}{2}\left(\frac{1}{L_1}+\frac{1}{L_2}\right)\left(\frac{1}{D-3}+\frac{\gamma}{4}\right) + W_2(0;L_1,L_2) + \mathcal{O}(D-3), \tag{58}$$

with the function $W_2(0;L_1,L_2)$ given by

$$W_2(0;L_1,L_2) = \sum_{n_1,n_2=1}^{\infty}\left\{\frac{1}{L_1}K_0\left(2\pi\frac{L_2}{L_1}n_1n_2\right) + \frac{1}{L_2}K_0\left(2\pi\frac{L_1}{L_2}n_1n_2\right)\right\}. \tag{59}$$

One sees that $E_2(D/2-1;L_1,L_2)$ has a simple pole at $D=3$, which, in principle, would require a mass renormalization to give meaning to Eq. (51). However, this divergence is exactly canceled out by the divergence of the ζ-function of the first term in the square bracket of Eq. (51) and no mass renormalization is really needed; in this case, we can choose any finite bare mass that satisfies condition (3), but the simplest choice is

$$\bar{m}_0^2(T,L_1,L_2) = m_0^2(T). \tag{60}$$

We could proceed substituting Eq. (55) into Eq. (52) to get $\lambda_R(L_1,L_2)$ and inserting Eqs. (41) and (58) into Eq. (51) to obtain a finite expression for $m^2(T,L_1,L_2)$ when $D=3$; the condition $m^2=0$ would then produce an expression for the (L_1,L_2)-dependent transition temperature for a rectangular wire, $T_c(L_1,L_2)$. However, the resulting expression has a very complicated dependence on the lengths L_1 and L_2 because their ratios appear in the arguments of the Kelvin functions in the infinite double sums of Eqs. (56) and (59), which define the functions $W_2(-1;L_1,L_2)$ and $W_2(0;L_1,L_2)$. Instead, we consider a wire with square transversal section for which these double sums can be performed, leading to much simpler expressions for the W_2-functions.

For a square wire, taking $L_1=L_2=L=\sqrt{A}$, Eqs. (51) and (52) reduce, in the limit $D\to 3$, to

$$m^2(T,\sqrt{A}) = m_0^2(T) + \mathcal{C}_2\frac{\lambda_R(\sqrt{A})}{\sqrt{A}}, \tag{61}$$

$$\lambda_R(\sqrt{A}) = \frac{48\pi\lambda}{48\pi+\mathcal{E}_2\lambda\sqrt{A}}, \tag{62}$$

where

$$\mathcal{C}_2 = \frac{9\gamma}{\pi} + \frac{24}{\pi}\sum_{n_1,n_2=1}^{\infty}K_0(2\pi n_1n_2) \approx 1.660\,62, \tag{63}$$

$$\mathcal{E}_2 = 1 + \frac{3\zeta(3)}{\pi^2} + \frac{24}{\pi} \sum_{n_1,n_2=1}^{\infty} \frac{n_1}{n_2} K_1(2\pi n_1 n_2) \approx 1.372\,41. \tag{64}$$

With $m_0^2(T)=\alpha(T-T_0)$ and using Eq. (62), we can rewrite Eq. (61) as

$$m^2(T,\sqrt{A}) = \alpha(T - T_c(\sqrt{A})), \tag{65}$$

where

$$T_c(\sqrt{A}) = T_0 - \frac{48\pi\mathcal{C}_2\lambda}{48\pi\alpha\sqrt{A} + \mathcal{E}_2\alpha\lambda(\sqrt{A})^2}. \tag{66}$$

Similar to the case of films, we infer from Eq. (66) that there is a minimal area for transversal section of a square wire below which the ordered phase is not sustained

$$A'_{\min} = \left[\frac{24\pi}{\mathcal{E}_2\lambda}\left(\sqrt{1 + \frac{\mathcal{E}_2\lambda\sqrt{A_{\min}}}{12\pi}} - 1\right)\right]^2, \tag{67}$$

where

$$A_{\min} = \left(\frac{\mathcal{C}_2\lambda}{\alpha T_0}\right)^2 \tag{68}$$

is the corresponding quantity with no boundary corrections to the coupling constant.[22]

VII. BOUNDARY EFFECTS ON THE TRANSITION TEMPERATURE FOR GRAINS

We now turn our attention to the case where three spatial dimensions are compactified, corresponding to the system confined in a box of sides L_1, L_2, L_3. For high enough D, close to criticality ($m^2 \approx 0$), taking $d=3$ in Eqs. (35) and (36), we get

$$m^2(D,T,L_1,L_2,L_3) \approx \bar{m}_0^2(T,L_1,L_2,L_3) + \frac{6\lambda_R(D,L_1,L_2,L_3)}{\pi^{D/2}}\Gamma\left(\frac{D}{2}-1\right) \times \left[\sum_{i=1}^{3}\frac{\zeta(D-2)}{L_i^{D-2}} + 2\sum_{i<j=1}^{3} E_2\left(\frac{D}{2}-1;L_i,L_j\right) + 4E_3\left(\frac{D}{2}-1;L_1,L_2,L_3\right)\right] \tag{69}$$

and using Eq. (37),

$$\lambda_R(D,L_1,L_2,L_3) \approx \frac{8\pi^{D/2}\lambda}{8\pi^{D/2} + \lambda\Gamma\left(\frac{D-4}{2}\right)\left[\sum_{i=1}^{3}\frac{\zeta(D-4)}{L_i^{D-4}} + 2\sum_{i<j=1}^{3} E_2\left(\frac{D-4}{2};L_i,L_j\right) + 4E_3\left(\frac{D-4}{2};L_1,L_2,L_3\right)\right]}. \tag{70}$$

The analytical extensions of the functions E_2 are given by Eq. (53), for $s=1,2$, while the analytical structure of the functions E_3 can be obtained from the general symmetrized recurrence relation given by Eqs. (33) and (34); explicitly, one has

$$E_3\left(\frac{D}{2}-s;L_1,L_2,L_3\right) = -\frac{1}{6}\sum_{i<j=1}^{3} E_2\left(\frac{D}{2}-s;L_i,L_j\right) + \frac{\sqrt{\pi}\Gamma\left(\frac{D}{2}-s-\frac{1}{2}\right)}{6\Gamma\left(\frac{D}{2}-s\right)} \times \sum_{i,j,k=1}^{3} \frac{|\varepsilon_{ijk}|}{2L_i} E_2\left(\frac{D-1}{2}-s;L_j,L_k\right) + \frac{2\sqrt{\pi}}{3\Gamma\left(\frac{D}{2}-s\right)} W_3\left(\frac{D-1}{2}-s;L_1,L_2,L_3\right), \tag{71}$$

where ε_{ijk} is the totally antisymmetric symbol and the functions W_3, particular cases of Eq. (34), are given by

$$W_3\left(\frac{D-1}{2}-s;L_1,L_2,L_3\right) = \sum_{i,j,k=1}^{3} \frac{|\varepsilon_{ijk}|}{2L_i} \sum_{n_1,n_2,n_3=1}^{\infty} \left(\frac{\pi n_i}{L_i\sqrt{L_j^2 n_j^2 + L_k^2 n_k^2}}\right)^{(D-1)/2-s} \times K_{(D-1)/2-s}\left(\frac{2\pi n_i}{L_i}\sqrt{L_j^2 n_j^2 + L_k^2 n_k^2}\right). \tag{72}$$

Using Eqs. (55), (56), (58), and (59), together with Eqs. (72) and (77), in Eqs. (69) and (70) leads to big expressions for the mass and the renormalized coupling constant which have very complicated dependence on the compactification lengths L_1, L_2, L_3. For this general case, in the limit $D \to 3$, one finds results similar to those for wires: it follows directly that $\lambda_R(L_1, L_2, L_3)$ is finite, while the divergences (pole parts) of the terms in the square bracket of Eq. (69) cancel out exactly (as $D \to 3$) leaving $m^2(T, L_1, L_2, L_3)$ finite. Then, fixing the bare mass as

$$\bar{m}_0^2(T,L_1,L_2,L_3) = m_0^2(T) = \alpha(T-T_0), \tag{73}$$

one would obtain the critical temperature for a parallelepiped grain. The expression of $T_c(L_1, L_2, L_3)$ is a very complicated formula, involving infinite multiple sums, which makes almost impossible a general analytical study for arbitrary parameters L_1, L_2, L_3. Instead of treating the general case, we restrict ourselves to the simplest situation where all the compactification lengths are equal, the system corresponding to a cubic grain.

For a cubic grain, we take $L_1 = L_2 = L_3 = L = V^{1/3}$ and Eqs. (69) and (70) reduce, in the limit $D \to 3$, to

$$m^2(T, V^{1/3}) = m_0^2(T) + \mathcal{C}_3 \frac{\lambda_R(V^{1/3})}{V^{1/3}}, \tag{74}$$

$$\lambda_R(V^{1/3}) = \frac{48\pi\lambda}{48\pi + \mathcal{E}_3 \lambda V^{1/3}}, \tag{75}$$

where

$$\mathcal{C}_3 = 1 + \frac{9\gamma}{\pi} + \frac{12}{\pi}\sum_{n_1,n_2=1}^{\infty} \frac{e^{-2\pi n_1 n_2}}{n_1} + \frac{48}{\pi}\sum_{n_1,n_2=1}^{\infty} K_0(2\pi n_1 n_2) + \frac{48}{\pi}\sum_{n_1,n_2,n_3=1}^{\infty} K_0\left(2\pi n_1\sqrt{n_2^2+n_3^2}\right) \approx 2.6757 \tag{76}$$

and

$$\mathcal{E}_3 = 1 + \frac{\pi}{15} + \frac{3\zeta(3)}{\pi^2} + \frac{48}{\pi}\sum_{n_1,n_2=1}^{\infty}\frac{n_1}{n_2}K_1(2\pi n_1 n_2) + \frac{24}{\pi}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_1}{n_2}\right)^{3/2}K_{3/2}(2\pi n_1 n_2)$$
$$+ \frac{48}{\pi}\sum_{n_1,n_2,n_3=1}^{\infty}\frac{\sqrt{n_1^2+n_2^2}}{n_3}K_1(2\pi n_3\sqrt{n_1^2+n_2^2}) \approx 1.5996. \tag{77}$$

Using condition (73), Eq. (74) can be rewritten as

$$m^2(T,V^{1/3}) = \alpha(T - T_c(V^{1/3})), \tag{78}$$

where the boundary dependent critical temperature is given by

$$T_c(V^{1/3}) = T_0 - \frac{48\pi\mathcal{C}_3\lambda}{48\pi\alpha V^{1/3} + \mathcal{E}_3\alpha\lambda(V^{1/3})^2} \tag{79}$$

We find that the minimal volume of the grain allowing the transition is

$$V'_{\text{min}} = \left[\frac{24\pi}{\mathcal{E}_3\lambda}\left(\sqrt{1 + \frac{\mathcal{E}_3\lambda(V_{\text{min}})^{1/3}}{12\pi}} - 1\right)\right]^3, \tag{80}$$

where

$$V_{\text{min}} = \left(\frac{\mathcal{C}_3\lambda}{\alpha T_0}\right)^3$$

corresponds to the minimal volume for the situation where boundary corrections to the coupling constant are ignored.[22]

VIII. GENERAL REMARKS AND CONCLUSIONS

In all cases studied, there exist lowest lengths, L_{min}, $\sqrt{A_{\text{min}}}$ and $(V_{\text{min}})^{1/3}$, below which the broken phase cannot be sustained and the dependence of T_c on these lengths follows the same pattern. Let us take the minimal film thickness as the length scale, $L_0 = L_{\text{min}} = \mathcal{C}_1\lambda/\alpha T_0$ and define the *reduced critical temperature* t_c and the *reduced length* l, respectively, by

$$t_c = \frac{T_c}{T_0}, \quad l = \frac{L}{L_0}. \tag{81}$$

We know from Ref. 22 that the reduced transition temperature as a function of the reduced length, in the case where no correction to the coupling constant is included, for films, square wires, and cubic grains ($d=1,2,3$, respectively), can be written in the form

$$t_c^{(d)}(l) = 1 - \frac{\mathcal{C}^{(d)}}{l} \tag{82}$$

with $\mathcal{C}^{(1)}=1$, $\mathcal{C}^{(2)}=\mathcal{C}_2/\mathcal{C}_1 \approx 1.5064$, and $\mathcal{C}^{(3)}=\mathcal{C}_3/\mathcal{C}_1 \approx 2.4272$; that is, for all values of d, the reduced temperature t_c scales with the inverse of the reduced length l^{-1}. In other words, the overall behavior of the reduced temperature does not depend on the number of compactified dimensions but only on the dimension of the Euclidian space, here $D=3$.

Considering the coupling-constant correction, the reduced transition temperatures are written as,

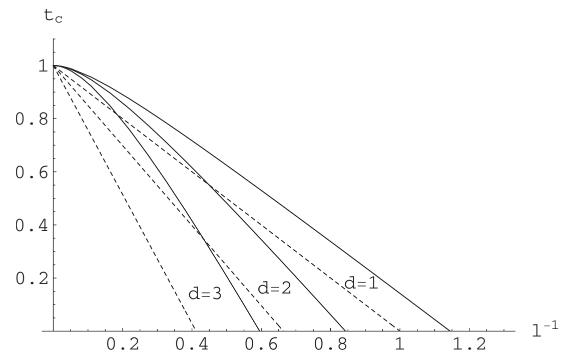

FIG. 1. Reduced transition temperature (t_c) as a function of the inverse of the reduced compactification length (l) for films (d=1), square wires (d=2), and cubic grains (d=3). The full and dashed lines correspond to the results with and without correction of the coupling constant, respectively.

$$t_c^{(d)}(l) = 1 - \frac{48\pi C^{(d)}}{48\pi l + \mathcal{E}_d \xi l^2}, \tag{83}$$

where $\mathcal{E}_1=1$, $\mathcal{E}_2 \approx 1.3724$, $\mathcal{E}_3 \approx 1.5996$, and $\xi = \lambda L_{\text{min}}$. In Fig. 1, we plot the reduced transition temperature as a function of the reduced length for all cases (films, square wires and cubic grains), fixing $\xi=25$. We also plot for comparison the corresponding curves for the cases where no corrections to the coupling constant are considered.

We have presented in this paper a general formalism which, in the framework of the Ginzburg–Landau model, is able to describe phase transitions for systems defined in spaces of arbitrary dimension, some of them being compactified. Such a generalization is not trivial since it involves the extension to several dimensions of the one-dimensional mode-sum regularization procedure of Ref. 24. This extension requires, in particular, the definition of symmetrized multidimensional Epstein-Hurwitz functions with no analog in the one-dimensional case. When combined with the boundary dependent coupling constant, this generates sets of coupled equations for the renormalized mass, which can be solved only at criticality. This leads to the critical temperature as a function of the size of the system. It is this kind of mathematical framework that allows us to obtain the general formulas (26)–(28), which are particularized to films, wires and grains, thereby implying the peculiar forms of the critical temperature as a function of the size of the system for these three physically interesting cases.

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