Universidade Federal da Bahia - UFBA
Instituto de Matemática e Estatística - IME
Programa de Pós-Graduação em Matemática
Dissertação de Mestrado

On the Most General One-Dimensional Brownian Motion with Boundary Conditions at the Origin

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Salvador-Bahia

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Dissertação de Mestrado apresentada ao Colegiado da Pós-Graduação em Matemática da Universidade Federal da Bahia como requisito parcial para obtenção do título de Mestre em Matemática.

Orientador: Prof. Dr. Tertuliano Franco Santos Franco.
Coorientador: Prof. Dr. Dirk Erhard.

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Ficha catalográfica elaborada pela Biblioteca Universitária de Ciências e Tecnologias Prof. Omar Catunda, SIBI – UFBA.

S586 Silva, Wanessa Muricy

On the Most General One-Dimensional Brownian Motion with Boundary Conditions at the Origin / Wanessa Muricy Silva. – Salvador, 2025.

182 f.

Orientador: Prof. Dr. Tertuliano Franco Santos Franco.
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Dissertação (Mestrado) – Universidade Federal da Bahia. Instituto de Matemática e Estatística, 2025.

1. Brownian Motion. 2. Snapping Out Brownian Motion 3. Markov Processes. 4. Infinitesimal Generator. I. Franco, Tertualino Franco Santos. II. Erhard, Dirk. III. Universidade Federal da Bahia. IV.Título.

CDU: 530.162

Sobre o Movimento Browniano unidimensional mais geral em uma dimensão com condições de fronteira na origem.

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Dissertação apresentada ao Colegiado do Curso de Pós-graduação em Matemática da Universidade Federal da Bahia, como requisito parcial para obtenção do Título de Mestre em Matemática.

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To my dad, who made me believe that anything is possible, because he himself made everything possible.

Agradecimentos

Em primeiro lugar, gostaria de agradecer a Deus, que fez um universo matemático e ainda me deu a felicidade de existir e ser um pouco matemática nele! Estudando todas essas belas e elegantes verdades matemáticas, não posso deixar de cantar a sua altíssima glória. Mas Deus também é muito próximo, e lembro o que ensina a Igreja: "Aquele que se esforça, com perseverança e humildade, por penetrar no segredo das coisas, é como que conduzido pela mão de Deus, que sustenta todos os seres e faz que eles sejam o que são". De fato, sem a doce e generosa mão de Deus, jamais meus insuficientes esforços me fariam terminar esse trabalho.

Gostaria de dedicar esse trabalho à memória do meu pai. Infelizmente, ele não pôde me ver concluir o mestrado, nem a graduação, mas, com certeza, se não fosse ele, não haveria nada disso. Todos os seus muitos anos de trabalhos e sua confiança na vida, e em mim também, fizeram esse sonho possível. Além disso, acho que ele seria a pessoa mais feliz em me ver aqui, e posso saber até o que ele me diria. Obrigada, pai! Agradeço igualmente a minha mãe, que me acompanha com seus carinhos maternais, não sei o que seria de mim sem a senhora. Agradeço a minha irmã, e aos meus demais familiares, sou imensamente grata pelo suporte direto e indireto que me dão, eles são essenciais.

Agradeço aos meus orientadores, Tertu e Dirk. Acho que ambos sabem olhar muito mais para o que de melhor podemos ser, do que para o que somos de fato. Comecei em Probabilidade quase do zero, mas essa confiança e paciência de vocês, me fizeram avançar, e estar agora totalmente rendida e muito feliz na área. Em meio às demandas de cada um, que já são muitas, estavam sempre prontos a me ajudar em tantos pormenores a fim de que eu também construísse a minha carreira. Mais ainda, não posso deixar de admirar os matemáticos que são, espero um dia poder ser assim e em dupla! Mas dizem que tendemos a nos identificar com o que amamos, então, por favor, me deem a licença de considerar que já tive um pouco disso em vocês. E não só como matemáticos, admiro muito as pessoas que são, e o quanto tornaram essa experiência tão valiosa. Não poderia ter sido orientada por pessoas melhores, sinto-me completamente em dívida com vocês.

Paralelamente, agradeço a todos os professores que tive o prazer de conhecer no mestrado. Tive a alegria de ter excelentes profissionais, com os quais pude aprender não só muitos conteúdos, mas também tirar muitas lições para a carreira e para a vida. Agradeço também aos membros da banca, pelo tempo e atenção que deram a essa dissertação, e as suas correções. Agradeço a CAPES pelo suporte financeiro, sem o qual seria impossível fazer o mestrado.

Agradeço aos meus amigos da UFBA, aos que começaram o mestrado comigo e aos que fui conhecendo pelo caminho. O mestrado não teria sido a mesma coisa sem vocês, não teria sido tão legal e engraçado, e seria muito difícil não poder dividir as dificuldades e contar com o apoio de todos. Cada um, e não vou ousar citar nomes porque são muitos, está guardado no meu coração e sentirei muitas saudades!

Agradeço também aos meus demais amigos, queria citar as meninas do Centro Âncora, o grupo de católicos da UFBA, as meninas que moram comigo, a comunidade da Capela Sagrada Família, pude facilmente chamar Salvador de casa por causa de vocês! Além disso, cito meus amigos que conheci em Brasília e da minha cidade natal, Senhor do Bonfim, que mesmo longe sabem se fazer presentes por pequenas atenções. Fico tão feliz por cada um na minha vida, quero-os muitíssimo! A título de exemplo, vou contar uma breve história. Antes de decidir minha área e orientador, estava bastante em dúvida, porque tendo a querer que as coisas fossem as melhores possíveis e impossíveis, logo nunca sei exatamente o que escolher. Conversando isso com a Camis, que sempre me dá bons conselhos, ela falou “ah, você tem que apostar!”. Bom, já que é para *apostar* estamos na Probabilidade! Mal sabia eu que a aposta seguia um submartingal, de fato ganhei muito mais do que eu tinha, e até do que esperava. Obrigada, Camis!

Estou certa de que não agradeci a todos, e nem o suficiente. São tantas as pessoas e situações que tornaram possível toda essa jornada até o mestrado, a qual amo apaixonadamente e que me deixa particularmente feliz. No entanto, esse panorama me põe diante daquela coisa assombrosa: as coisas são, as coisas existem! Assim, sem mais, nem menos, gratuitamente, e são muito boas! Com um autor que gosto muito, afirmo “Que maravilha que vocês existem!”. Sinto-me completamente incapaz de retribuir à altura, mas conheço alguém que pode! Que Deus vos abençoe infinitamente!

“Quem ama não conhece nada que seja difícil.”

(Santo Antônio)

“Faced with the sacredness of life and of the human person, and before the marvels of the universe, wonder is the only appropriate attitude.”

(Saint John Paul II)

Abstract

In this master's dissertation, we characterize the most general one-dimensional Brownian motion under some Markovian behavior at zero via the study of its infinitesimal generators. The class of processes here considered is defined as the class of diffusion processes that behave as the absorbed Brownian Motion up to the hitting time of zero, and at zero the process has some (Markovian) behavior, which includes jumping to an extra absorbing point Δ called cemetery. Carefully adapting techniques of Knight's book [3] we obtain two new results.

Our first main result consists on proving that the most general Brownian motion on the state space $\mathbb{R}\cup\{\Delta\}$ coincides with the *Skew Sticky Killed Brownian Motion*, whose infinitesimal generator can be found in Borodin's book [1].

Our second main result consists on the characterization of the most general Brownian motion on the state space $(-\infty, 0-] \cup [0+, \infty) \cup \{\Delta\}$. We conclude that that class of processes obtained includes, as a particular case, the *Snapping Out Brownian Motion*, a Brownian motion on $(-\infty, 0-] \cup [0+, \infty)$ recently constructed in Lejay's paper [5]. Moreover, the class of processes here obtained includes a Brownian-type process not known in the literature, which we call a *Skew Sticky Killed Snapping Out Brownian motion*.

Keywords: Brownian motion; Markov processes; Infinitesimal generator.

Contents

Chapter 1

Introduction

The Brownian motion was first observed as the erratic movement of particles suspended in a fluid. This discovery is commonly attributed to the botanist Robert Brown, who, in 1827, noticed the jittery motion of pollen grains immersed in water under a microscope. The modern explanation of this motion lies in atomic theory: the fluid is composed of molecules in constant motion, and their collisions with the suspended particles result in the observed random trajectories. Later, mathematicians such as Norbert Wiener provided a rigorous mathematical formulation of this phenomenon in terms of a stochastic process.

From a mathematical perspective, the Brownian motion is a stochastic process characterized by stationary and independent Gaussian increments and continuous sample paths. It is also a strong Markov process, and its transition semigroup is the fundamental solution of the heat equation – a key partial differential equation in mathematical physics.

In a more general context, we will examine a class of Brownian-type processes with specific boundary conditions at 0. At this point, the process may be absorbed, killed, or continue to follow a Markovian behavior. More formally, we study strong Markov processes with continuous paths up to a random lifetime, after which the process remains at an isolated point known as the cemetery state. In addition, these processes coincide with the standard Brownian motion up to the hitting time of zero.

Moreover, considering two disjoint half-lines, we construct a similar process on the state space $(-\infty, 0-] \cup [0+, \infty)$. This work provides the characterization of the infinitesimal generator of such processes, giving the exact boundary conditions of their domains, for both processes on $\mathbb{R}$ and on $(-\infty, 0-] \cup [0+, \infty)$.

In **Chapter 2**, we review some fundamental definitions and properties of the standard Brownian motion, specially the strong Markov property. Our main reference is [4] *Brownian motion, Martingales, and Stochastic Calculus* by Jean-François Le Gall, where the reader can find detailed proofs of the results presented. **Chapter** 3 introduces the analytical setup used in this work, including Feller processes and infinitesimal generators, as well as probabilistic tools from the theory of Markov processes. Alongside Le Gall's text, we refer to [6] Continuous Martingales and Brownian Motion by Revuz and Yor for results related to Markov processes.

Chapter 4 and **Chapter** 5 constitute the core of this study, where we define the class of Brownian processes under consideration and provide a complete characterization of their generator and domain. In **Chapter** 4 we prove that the most general Brownian motion on $\mathbb{R}$ coincides with the *Skew Sticky Killed Brownian Motion* described in *Handbook of Brownian motion - Facts and Formulae* by Borodin and Paavo [1, page 127, Section 13, Appendix 1]. In **Chapter** 5, we characterize the most general Brownian motion on $(-\infty, 0-] \cup [0+, \infty)$, from where we recover the *Snapping out Brownian Motion*, constructed in the recent paper of A. Lejay [5], and furthermore obtain a new Brownian type process, which we call *Skew Sticky Killed Snapping Out Brownian motion*.

Finally, it is important to mention that many techniques applied in **Chapter** 4 and **Chapter** 5 are careful adaptations of results about the most general Brownian motion on the half-line $[0, \infty)$ from Frank B. Knight's book *Essentials of Brownian motion and diffusion* [3, Chapter 6], which, in its turn, has roots on earlier works of William Feller.

Chapter 2

Brownian motion

Let $(\Omega, \mathscr{F}, P)$ be a probability space. Given $\sigma > 0, \mu \in \mathbb{R}$, we say that a random variable X is *Gaussian with $\mathscr{N}(\mu, \sigma^2)$-distribution* if X has

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big\{ -\frac{(x-\mu)^2}{2\sigma^2} \Big\}$$

as its density. Moreover, if $\mu = 0$, we say that X is a centered Gaussian variable. If $\mu = 0, \sigma = 1$, then X is called a standard Gaussian or normal variable. Finally, by extension, we set that $X = \mu$ a.s. is Gaussian with $\mathscr{N}(\mu, 0)$-distribution.

Let E be a d-dimensional space with $\langle \cdot, \cdot \rangle$ as an inner product. A random variable X with values in E is called a *Gaussian vector* if $\langle X, u \rangle$ is a real Gaussian variable, for every $u \in E$. When these Gaussian variables are centered, we say that X is a centered Gaussian vector. For instance, considering $E = \mathbb{R}^d$ with the usual inner product, and $X_1, \ldots, X_d$ independent Gaussian variables, then the random vector $(X_1, \ldots, X_d)$ is a Gaussian vector.

We define a (centered) *Gaussian space* as a linear subspace of $L^2(\Omega, \mathscr{F}, P)$ containing only centered Gaussian variables. For example, if $X = (X_1, \ldots, X_d)$ is a centered Gaussian vector in $\mathbb{R}^d$, then the subspace spanned by $\{X_1, \ldots, X_d\}$ is a Gaussian space.

Let $(E, \mathscr{E})$ be a measurable space, and let T be an arbitrary index set. The collection $(X_t)_{t \in T}$ of random variables valued in E is called a *stochastic process* with values in E. Besides the fact that the functions $\Omega \ni \omega \mapsto X_t(\omega)$ are measurable for every $t \in T$, we are mostly interested in the *sample paths*, that is, the mappings $T \in t \mapsto X_t(\omega)$, fixed $\omega \in \Omega$. Note that the sample paths form a collection of mappings from T into E, indexed by $\omega \in \Omega$.

In particular, we will consider the following type of real-valued stochastic process.

Definition 2.1. *A real-valued stochastic process $(X_t)_{t\in T}$ is a (centered) Gaussian Process if any finite linear combination of the variables $X_t, t \in T$, is a centered Gaussian.*

It follows from this definition that the linear subspace of L^2 spanned by the variables $X_t, t \in T$, is a Gaussian space, which is called the *Gaussian space generated by the process X*. Similarly, for every choice of the distinct indices $t_1, \dots, t_p$ in T, the random vector $(X_{t_1}, \dots, X_{t_p})$ is a Gaussian vector in $\mathbb{R}^p$ whose law is called a *finite-dimensional marginal distribution* of the process $(X_t)_{t\in T}$.

If $(X_t)_{t\in T}$ is a centered Gaussian process, the *covariance function of X* is the function $\Gamma : T \times T \to \mathbb{R}$ defined by $\Gamma(s,t) = \operatorname{cov}(X_s, X_t) = E[X_s X_t]$, recalling that the variables are centered Gaussian. It is possible to show that this function determines the finite-dimensional distributions of the process $(X_t)_{t\in T}$, which in turn determine the process itself.

2.1 Brownian motion

Definition 2.2. *Let $(E, \mathscr{E})$ be a measurable space, and let μ be a σ-finite measure on $(E, \mathscr{E})$. An isometry G from $L^2(E, \mathscr{E}, \mu)$ into a centered Gaussian space $\mathscr{H}$ is called a Gaussian white noise with intensity μ.*

Note that if $f, g \in L^2(E, \mathscr{E}, \mu)$, the covariance of $G(f), G(g)$ is

$$E[G(f)G(g)] = \langle G(f), G(g)\rangle_{\mathscr{H}} = \langle f, g\rangle_{L^2(E,\mathscr{E},\mu)} = \int f g \, d\mu, \tag{2.1}$$

since G is an isometry.

Definition 2.3. *Consider $\mathbb{R}_+$ endowed with its Borel σ-field, and let G be a Gaussian white noise whose intensity is the Lebesgue measure. The stochastic process $(B_t)_{t\geq 0}$ defined by*

$$B_t = G(\mathbf{1}_{[0,t]})$$

is called a pre-Brownian motion.

In particular, a pre-Brownian motion is a centered Gaussian Process. Also, by equation (2.1), it is easy to check the next proposition.

Proposition 2.4. *A pre-Brownian motion is a centered Gaussian Process with covariance*

$$K(s,t) = \min\{s,t\} := s \wedge t.$$

The preceding proposition is actually an equivalence. Moreover, there are another ways to characterize a pre-Brownian motion.

Proposition 2.5. *Let $(B_t)_{t\geq 0}$ be a stochastic process. The following are equivalent:*

- *$(B_t)_{t\geq 0}$ is a pre-Brownian motion;*
- *$(B_t)_{t\geq 0}$ is a centered Gaussian Process with covariance $K(s,t) = s \wedge t$;*
- *$B_0 = 0$ a.s., and, for every $0 \leq s < t$, the variable $B_t - B_s$ is independent of $\sigma(X_r, r \leq s)$ and follows the $\mathcal{N}(0, t-s)$-distribution.*
- *$B_0 = 0$ a.s., and, for every choice of $0 = t_0 < t_1 < \cdots < t_p$, the variables $B_{t_i} - B_{t_{i-1}}, 1 \leq i \leq p$ are independent, and $B_{t_i} - B_{t_{i-1}}$ follows the $\mathcal{N}(0, t_i - t_{i-1})$-distribution, for every $1 \leq i \leq p$.*

The last item of the above proposition allows us to compute the finite-dimensional distributions of a pre-Brownian motion.

Corollary 2.6. *Let $(B_t)_{t\geq 0}$ be a pre-Brownian motion. Then, for every choice of $0 = t_0 < t_1 < \cdots < t_p$, the distribution of the vector $(B_{t_1}, \ldots, B_{t_p})$ has density*

$$p(x_1, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}\sqrt{(t_1(t_2 - t_1)\ldots(t_n - t_{n-1})}} \exp\Big\{ -\sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})} \Big\},$$

where $x_0 = 0$.

A pre-Brownian motion has the following useful properties.

Proposition 2.7. *If $B = (B_t)_{t\geq 0}$ is a pre-Brownian motion, then:*

- *$-B$ is a pre-Brownian motion (symmetry property);*
- *For every $\lambda > 0$, the process $B_t^\lambda = \frac{1}{\lambda} B_{\lambda^2 t}$ is a pre-Brownian motion (invariance under scaling);*
- *For every $s \geq 0$, the process $B_t^s = B_{s+t} - B_s$ is a pre-Brownian motion and it is independent of $\sigma(B_r, r \leq s)$ (simple Markov property).*

We define the Brownian motion now only by asking for an essential condition on the sample paths of a pre-Brownian motion.

Definition 2.8. *A Brownian motion is a pre-Brownian motion whose sample paths are all continuous.*

Furthermore, starting from a pre-Brownian motion B, it is possible to modify B slightly to obtain a Brownian motion. To state this, we consider more generally E a metric space equipped with its Borel σ-field, and $X = (X_t)_{t\in T}$, $\tilde{X} = (\tilde{X}_t)_{t\in T}$ two stochastic processes on E.

Definition 2.9. *The process $\tilde{X}$ is said to be a modification of X if*

$$\forall\, t \in T,\ P(\tilde{X}_t = X_t) = 1.$$

In other words, for each $t \in T$, the random variables $\tilde{X}_t$ and X_t are equal a.s., in particular $\tilde{X}$ has the same finite-dimensional distributions as X. Thus, if X is a pre-Brownian motion, $\tilde{X}$ is also a pre-Brownian motion. However, the sample paths of $\tilde{X}$ may be very different from those of X. We consider then a stronger notion.

Definition 2.10. *The process $\tilde{X}$ is said to be indistinguishable from X if there exists a negligible subset N of Ω such that*

$$\forall\, \omega \in \Omega \setminus N, \forall\, t \in T,\ \tilde{X}_t(\omega) = X_t(\omega).$$

If the set $\{\tilde{X}_t = X_t, \forall\ t \in T\}$ is measurable, this definition is equivalent to saying that $P(\tilde{X}_t = X_t, \forall\, t \in T) = 1$, then the sample paths of indistinguishable process are equal almost surely. Moreover, it is easy to see that if $\tilde{X}$ is indistinguishable from X, then $\tilde{X}$ is a modification of X. In this way, we identify two indistinguishable process as the same.

Suppose that T is an interval of $\mathbb{R}$. If the sample paths of both X and $\tilde{X}$ are continuous (except possibly on a negligible set of Ω), we can show also that if $\tilde{X}$ is a modification of X, then $\tilde{X}$ is indistinguishable from X. Indeed, we note that $\tilde{X}_t$ and X_t are equal almost surely at rational times t, and then we use the continuity to extend the a.s. equality for all times $t \geq 0$. Hence, for a continuous path process, it is enough to consider its modifications, which will be unique up to indistinguishability.

We are ready now to state Kolmogorov's lemma, which gives a condition to obtain a modification of a stochastic process with sample paths having better continuity properties.

Theorem 2.11 (Kolmogorov's lemma). *Let I be a bounded interval of $\mathbb{R}$, let (E, d) be a complete metric space, and let $X = (X_t)_{t \in I}$ be a stochastic process with values in E. Suppose that there exist three real numbers $q, \varepsilon, C > 0$ such that, for every $s, t \in I$,*

$$E[d(X_s, X_t)^q] \leq C|s - t|^{1+\varepsilon}.$$

Then, there is a modification $\tilde{X}$ of X whose sample paths are Hölder continuous with exponent α, for each $\alpha \in (0, \frac{\varepsilon}{q})$. In particular, $\tilde{X}$ is a modification of X with continuous sample paths, and such a modification is unique up to indistinguishability.

Remark 1. *If I is unbounded, we can apply the previous lemma for bounded subintervals of I and we get that X has a modification whose sample paths are locally Hölder continuous with exponent α, for each $\alpha \in (0, \frac{\varepsilon}{q})$.*

The next corollary applies Kolmogorov's lemma to a pre-Brownian motion $B = (B_t)_{t\geq 0}$ in order to obtain a Brownian motion. At least intuitively, we might already expect that sample paths of B would be locally Hölder continuous with exponent close to $\frac{1}{2}$. Indeed, for $s < t$,

$$E[|B_t - B_s|^2] = t - s \Rightarrow |B_t - B_s|^2 \approx t - s \Rightarrow |B_t - B_s| \approx |t - s|^{\frac{1}{2}}.$$

Corollary 2.12. *Let $B = (B_t)_{t\geq 0}$ be a pre-Brownian motion. The process B has a modification whose sample paths are locally Hölder continuous with exponent $\frac{1}{2} - \delta$, for each $\delta \in (0, \frac{1}{2})$.*

2.2 The Strong Markov Property of Brownian motion

Let $(\Omega, \mathscr{F}, P)$ be a probability space. To state the strong Markov property for Brownian motion, we need some definitions and results that will be useful throughout this text.

Definition 2.13. *A collection $(\mathscr{F}_t)_{0\leq t\leq\infty}$ of sub-σ-fields of $\mathscr{F}$ such that $\mathscr{F}_s \subseteq \mathscr{F}_t, \forall\ 0 \leq s < t \leq \infty$ is called a filtration on $(\Omega, \mathscr{F}, P)$. Also, we call $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ a filtered probability space.*

Let $(\mathscr{F}_t)$ be a filtration. We define

$$\mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s,$$

for $t \geq 0$, and $\mathscr{F}_{\infty+} = \mathscr{F}_\infty$. Note that $(\mathscr{F}_{t+})$ is also a filtration. It is easy to see that $\mathscr{F}_t \subseteq \mathscr{F}_{t+}$, for all $t \in [0, \infty]$. Moreover, when

$$\mathscr{F}_t = \mathscr{F}_{t+}, \forall\ t \geq 0,$$

we say that $(\mathscr{F}_t)$ is a right-continuous filtration. The filtration $(\mathscr{F}_{t+})$ is right-continuous.

Given any stochastic process $(X_t)_{t\geq 0}$, we construct a filtration $(\mathscr{F}_t^X)_{0\leq t\leq\infty}$ by setting $\mathscr{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$ for all $0 \leq t < \infty$, and $\mathscr{F}_\infty^X = \sigma(X_s, s \geq 0)$. This is called the *canonical filtration of X*.

Definition 2.14. *We say that a stochastic process $(X_t)_{t\geq 0}$ is adapted with respect to the filtration $(\mathscr{F}_t)$ if X_t is $\mathscr{F}_t$-measurable, for all $t \geq 0$.*

In particular, any stochastic process is adapted with respect to its canonical filtration.

Definition 2.15. *A random variable $T : \Omega \to [0, \infty]$ is a stopping time of the filtration $(\mathscr{F}_t)$, if $\{T \leq t\} \in \mathscr{F}_t$, for every $t \geq 0$.*

If T is a stopping time, we also have $\{T < t\} \in \mathscr{F}_t$ for each $t > 0$, since

$$\{T < t\} = \bigcup_{q \in [0,t) \cap \mathbb{Q}]} \{T \leq q\}.$$

Moreover, the set

$$\mathscr{F}_T = \{A \in \mathscr{F}_\infty : \forall\, t \geq 0, A \cap \{T \leq t\} \in \mathscr{F}_t\}$$

is a σ-field, called the *σ-field of the past before T*, and T is $\mathscr{F}_T$-measurable.

Given a stochastic process $(X_t)_{t\geq 0}$, we are usually interested in the hitting times

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}, \tag{2.2}$$

for some A, a measurable set. When $A = \{a\}$, we simply write τ_a. The hitting times are stopping times under some topological conditions of the sets A and continuity properties of the sample paths of X, as it is stated in the next result.

Proposition 2.16. *Let $(X_t)_{t\geq 0}$ be an adapted stochastic process with respect to $(\mathscr{F}_t)$, taking values in a metric space E.*

- *If X has right-continuous sample paths and O is an open subset of E, then the hitting time τ_O is a stopping time of $(\mathscr{F}_{t+})$;*
- *If X has continuous sample paths and F is a closed subset of E, then the hitting time τ_F is a stopping time of $(\mathscr{F}_t)$.*

For the remainder of this chapter, we fix a Brownian motion $B = (B_t)_{t\geq 0}$. Let T be a stopping time of the canonical filtration of B. We define the random variable $\mathbf{1}_{\{T<\infty\}} B_T$ by

$$\mathbf{1}_{\{T<\infty\}} B_T(\omega) = \begin{cases} B_{T(\omega)}(\omega) & \text{if } T(\omega) < \infty, \\ 0 & \text{if } T(\omega) = \infty. \end{cases}$$

which is $\mathscr{F}_T$-measurable.

Theorem 2.17 (strong Markov property). *If T is a stopping time, and $P(T < \infty) > 0$, then the process $(B_t^{(T)})_{t\geq 0}$ given by*

$$B_t^{(T)} = \mathbf{1}_{\{T<\infty\}}(B_{T+t} - B_T)$$

is a Brownian motion independent of $\mathscr{F}_T$, under the probability measure $P(\cdot \mid T < \infty)$.

One interesting application of the strong Markov property is the "reflection principle".

Theorem 2.18. *For every $t > 0$, let $S_t = \sup_{s \leq t} B_s$. Then, if $a \geq 0$ and $b \leq a$, we have*

$$P(S_t \geq a, B_t \leq b) = P(B_t \geq 2a - b).$$

In particular, S_t has the same distribution as $|B_t|$.

Now, we briefly consider some properties of the hitting times for Brownian Motion. Let $a \in \mathbb{R}$. By Proposition 2.16, τ_a is indeed a stopping time, and it is well known that $\tau_a < \infty$ almost surely. Also, one can see by the continuity of the paths that $\mathbf{1}_{\{\tau_a < \infty\}} B_{\tau_a} = a$ almost surely.

Moreover, by Theorem 2.18, the distributions of S_t and $|B_t|$ are equal, allowing us to compute easily the distribution of τ_a, for every $a \in \mathbb{R}$. Indeed,

$$P(\tau_a \leq t) = P(S_t \geq a) = P(|B_t| \geq a) = P(B_t^2 \geq a^2) = P(tB_1^2 \geq a^2) = P(\frac{a^2}{B_1^2} \leq t),$$

where, before the last equality, we have used that $B_t \sim \mathcal{N}(0, t)$, then $B_t \stackrel{d}{=} \sqrt{t} B_1$.

This proves the following corollary.

Corollary 2.19. *For every $a \in \mathbb{R}$, τ_a has the same distribution as $\frac{a^2}{B_1^2}$.*

Another useful property concerning now the hitting times $\tau_{\{a,b\}}$, for $a < 0 < b$, is the distribution of $\mathbf{1}_{\{\tau_{\{a,b\}} < \infty\}} B_{\tau_{\{a,b\}}}$.

Proposition 2.20. *For $a < 0 < b$, we have*

$$P(\tau_b < \tau_a) = \frac{-a}{b-a}, \quad P(\tau_a < \tau_b) = \frac{b}{b-a}.$$

Or equivalently,

$$P(B_{\tau_{\{a,b\}}} = b) = \frac{-a}{b-a}, \quad P(B_{\tau_{\{a,b\}}} = a) = \frac{b}{b-a}.$$

We finish by defining the Brownian motion not starting at 0.

Definition 2.21. *Let Z be a random variable. A stochastic process $(X_t)_{t \geq 0}$ is a Brownian motion started from Z, if we can write $X_t = Z + B_t$, where B is a Brownian motion started from 0 and is independent of Z.*

We will frequently consider the Brownian motion starting at $x \in \mathbb{R}$, choosing then $Z \equiv x$.

Chapter 3

Markov processes, Feller semigroups and generators

3.1 Analytic Setup

Let $(E, \mathscr{E})$ be a measurable space.

Definition 3.1. *A function $Q : E \times \mathscr{E} \to [0, 1]$ is a Markovian transition kernel, if it satisfies the following conditions:*

1. *For every $x \in E$, the mapping $\mathscr{E} \ni A \mapsto Q(x, A)$ is a probability measure on $(E, \mathscr{E})$;*
2. *For every $A \in \mathscr{E}$, the mapping $E \ni x \mapsto Q(x, A)$ is measurable with respect to $\mathscr{E}$.*

Let $B(E)$ be the vector space of all bounded measurable functions $f : E \to \mathbb{R}$, and equip it with the norm $\|f\| = \sup\{|f(x)| : x \in E\}$. Throughout this text, the notation $\|\cdot\|$ will always refer to the supremum norm. A Markovian transition kernel Q defines a linear operator on $B(E)$ by

$$Qf(x) = \int Q(x, dy)\, f(y),$$

for each $x \in E$. Indeed, it is clear that Qf is bounded, moreover it is a contraction on $B(E)$. And, for $f = \mathbf{1}_A$ with $A \in \mathscr{E}$, we have by definition that

$$x \mapsto Qf(x) = Q(x, A)$$

is measurable, thus we conclude the general case by standard approximation arguments.

Given a collection $(Q_t)_{t \geq 0}$ of transition kernels on E, we would like to obtain a semigroup on $B(E)$.

Definition 3.2. *A collection $(Q_t)_{t\geq 0}$ of transition kernels on E is a transition semigroup if the following properties hold:*

1. *For every $x \in E$, $Q_0(x, dy) = \delta_x(dy)$;*
2. *For every $t, s \geq 0$ and $A \in \mathscr{E}$,*
$$Q_{t+s}(x, A) = \int Q_t(x, dy)\, Q_s(y, A),$$
which is called the Chapman-Kolmogorov identity;
3. *For every $A \in \mathscr{E}$, the mapping $\mathbb{R}_+ \times E \ni (t, x) \mapsto Q_t(x, A)$ is measurable with respect to $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{E}$.*

By the first item of Definition 3.2, we have $Q_0 f = f$ for every $f \in B(E)$, and the second one ensures that the equation

$$Q_t(Q_s f) = Q_{t+s} f$$

is satisfied for indicator functions, so the general case follows again by approximation arguments. Hence, the family $\{Q_t, t \geq 0\}$ forms a semigroup, as desired.

In the same way, we get that the functions $(t, x) \mapsto Q_t f(x)$ are measurable with respect to $\mathscr{B}(\mathbb{R}_+) \otimes \mathscr{E}$. This allows us to define another important linear operator on $B(E)$.

Definition 3.3. *Given $\lambda > 0$, the λ-resolvent $R_\lambda : B(E) \to B(E)$ of a transition semigroup $(Q_t)_{t\geq 0}$ on E is a linear operator defined by*

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} Q_t f(x)\, dt$$

for every $f \in B(E), x \in E$.

Proposition 3.4. *The resolvent satisfies the following properties:*

- *$\lambda ||R_\lambda f|| \leq ||f||$ for each $f \in B(E), \lambda > 0$;*
- *For $\lambda, \mu > 0$, the resolvent equation holds:*
$$R_\lambda - R_\mu + (\lambda - \mu) R_\lambda R_\mu = 0.$$

From now on, we assume that E is a metrizable locally compact topological space, and also that E is countable at infinity, i.e. E is a countable union of compact sets. We consider the Borel σ-field of E.

We say that a function $f : E \to \mathbb{R}$ *tends to 0 at infinity* if, for every $\varepsilon > 0$, there exists a compact subset $K \subseteq E$ such that $|f(x)| < \varepsilon$, for all $x \in E \setminus K$. We denote $C_0(E)$ the set of all continuous functions $f : E \to \mathbb{R}$ tending to 0 at infinity. It is clear that $C_0(E) \subseteq B(E)$, and we would like to restrict the previous operators to the space $C_0(E)$.

Definition 3.5. *A transition semigroup $(Q_t)_{t\geq 0}$ is a Feller semigroup if for every $f \in C_0(E)$, we have*

- $Q_t f \in C_0(E)$;
- $||Q_t f - f|| \longrightarrow 0$ *as* $t \to 0$.

Remark 2. *It is known that the second item can be replaced by the weaker condition*

$$\forall\, f \in C_0(E), \forall\, x \in E, |Q_t f(x) - f(x)| \longrightarrow 0 \text{ as } t \to 0.$$

If $f \in C_0(E)$, one can show that $R_\lambda f \in C_0(E)$, so the resolvent R_λ is a linear operator on $C_0(E)$. Moreover, the range of R_λ in $C_0(E)$ plays a fundamental role in the theory of Feller semigroups.

Proposition 3.6. *Let $\lambda > 0$. The set $\mathscr{R} = \{R_\lambda f : f \in C_0(E)\}$ does not depend on the choice of $\lambda > 0$. Also, $\mathscr{R}$ is a dense subspace of $C_0(E)$.*

We now define the object that we will be concerned about throughout this work.

Definition 3.7. *Let $(Q_t)_{t\geq 0}$ be a Feller semigroup. A function $f \in C_0(E)$ belongs to the domain $\mathscr{D}(L)$ of the infinitesimal generator L of Q_t if the limit*

$$Lf = \lim_{t \downarrow 0} \frac{Q_t f - f}{t}$$

exists in $C_0(E)$.

In particular, the domain $\mathscr{D}(L)$ is a subspace of $C_0(E)$, and the infinitesimal generator $L : \mathscr{D}(L) \to C_0(E)$ is a linear operator.

There is a close relationship between the infinitesimal generator and the resolvent operators R_λ. In fact, we can determine the domain $\mathscr{D}(L)$ in terms of R_λ.

Proposition 3.8. *Let $\lambda > 0$.*

- *For all $g \in C_0(E)$, we have $R_\lambda g \in \mathscr{D}(L)$ and $(\lambda - L)R_\lambda g = g$;*
- *For all $f \in \mathscr{D}(L)$, we have $R_\lambda(\lambda - L)f = f$.*

Consequently, $\mathscr{D}(L) = \mathscr{R}$, and the operators $R_\lambda : C_0(E) \to \mathscr{R}$ and $\lambda - L : \mathscr{D}(L) \to C_0(E)$ are the inverse of each other.

Furthermore, Proposition 3.8 implies the next theorem, which shows the importance of the infinitesimal operator in the Feller semigroups theory.

Theorem 3.9. *A Feller semigroup $(Q_t)_{t\geq 0}$ is determined by its infinitesimal generator L.*

We emphasize that the definition of infinitesimal generator L deeply relies on its domain $\mathscr{D}(L)$. Process with the same expression for the generator (Laplacian, for instance), but under different domains can have totally different behaviors.

3.2 Probabilistic Setup

Let $(E, \mathscr{E})$ be a measurable space and $(X_t)_{t\geq 0}$ a stochastic process with values in E. We now introduce the theory of Markov processes.

Intuitively, we say that X is a Markov process when, given the present state of X at time s, the past up to time s is irrelevant to predict the future of X after time s. More precisely, the information of X until time s is given by the σ-field $\mathscr{F}_s^X = \sigma(X_r, 0 \leq r \leq s)$. Thus, if X is a Markov process, for each choice of $t > s$ and $A \in \mathscr{E}$, the conditional probability

$$P[X_t \in A \mid \sigma(X_r, 0 \leq r \leq s)]$$

is just a function of X_s. Namely, there exists $E \ni x \mapsto Q_{s,t}(x, A)$ a measurable function such that

$$P[X_t \in A \mid \sigma(X_r, 0 \leq r \leq s)] = Q_{s,t}(X_s, A). \tag{3.1}$$

It is also reasonable that the conditional probability is indeed the probability of X_t belongs to A, which implies that $\mathscr{E} \ni A \mapsto Q_{s,t}(x, A)$ is a probability measure on $(E, \mathscr{E})$. In this way, $Q_{s,t}$ will be a family of Markovian transition kernels.

We will be interested in the case when $Q_{s,t}$ does not depend specifically on the instants $s < t$, but only on the increment $t - s$. Thus, we can rewrite (3.1) by

$$P[X_t \in A \mid \sigma(X_r, 0 \leq r \leq s)] = Q_{t-s}(X_s, A), \tag{3.2}$$

for every $t > s$. This means that the process evolves homogeneously in time, so it is called a homogeneous Markov process.

As usual, by standard approximation arguments, the equation (3.2) can be generalized for any $f \in B(E)$ rather than indicator functions, so we get

$$E[f(X_{s+t}) \mid \sigma(X_r, 0 \leq r \leq s)] = Q_t f(X_s). \tag{3.3}$$

for any $t, s \geq 0$.

Recalling conditional expectation properties and applying (3.3) twice, we get, for every $t, s \geq 0$,

$$\begin{aligned} E[X_{s+t} \in A \mid \sigma(X_0)] &= E\left[E[X_{s+t} \in A \mid \sigma(X_r, 0 \leq r \leq s)] \mid \sigma(X_0)\right] \\ &= E[Q_t(X_s, A) \mid \sigma(X_0)] \\ &= Q_s(Q_t(X_0, A)) \\ &= \int Q_s(X_0, dy)\, Q_t(y, A), \end{aligned}$$

also in the last equality we used the definition of the operator Qf. On the other hand, we should have that

$$E[X_{s+t} \in A \mid \sigma(X_0)] = Q_{s+t}(X_0, A).$$

Thus,

$$Q_{s+t}(X_0, A) = \int Q_s(X_0, dy)\, Q_t(y, A),$$

so the kernels Q_t must satisfy the Chapman-Kolmogorov equation.

This motivates the following definition.

Definition 3.10. *Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$ be a filtered probability space and let $(Q_t)_{t\geq 0}$ be a transition semigroup on E. An adapted process $(X_t)_{t\geq 0}$ with values in E is a (homogeneous) Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$, and transition semigroup $(Q_t)_{t\geq 0}$ if for every $f \in B(E)$ and $t, s \geq 0$,*

$$E[f(X_{s+t}) \mid \mathscr{F}_s] = Q_t f(X_s).$$

It is easy to see that if $(X_t)_{t\geq 0}$ is a Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$, then it is also a Markov process with respect to the canonical filtration $(\mathscr{F}_t^X)_{t\geq 0}$. If a Markov process is mentioned without specifying the filtration, we are implicitly referring to the canonical one.

We know that a process is determined by its finite-dimensional distribution, so the following proposition says that in order to understand a Markov process it is sufficient to know its transition semigroup and its initial distribution.

Proposition 3.11. *A process $(X_t)_{t\geq 0}$ is a Markov process with respect to $(\mathscr{F}_t^X)_{t\geq 0}$ with transition semigroup $(Q_t)_{t\geq 0}$, and initial distribution μ if and only if for any $0 = t_0 < t_1 < \cdots < t_p$ and $f_0, \ldots, f_p \in B(E)$,*

$$E[f_0(X_0) f_1(X_{t_1}) \ldots f_p(X_{t_p})] = \int \mu(dx_0) f_0(x_0) \int Q_{t_1}(x_0, dx_1) f_1(x_1) \times \int Q_{t_2 - t_1}(x_1, dx_2) f_2(x_2) \cdots \int Q_{t_p - t_{p-1}}(x_{p-1}, dx_p) f_p(x_p). \tag{3.4}$$

Furthermore, the converse gives a way to construct a Markov process given a transition semigroup $(Q_t)_{t\geq 0}$ and an initial distribution μ, only asking for some topological conditions on E.

For that, let E be a Polish space, that is, a topological space that can be metrized by a complete metric and has a countable dense subset, and let $\mathscr{E}$ be its Borel σ-field. We will consider $\Omega = E^{\mathbb{R}_+}$, the set of all mappings $\omega : \mathbb{R}_+ \to E$, eqquiped with $\mathscr{F} = \mathscr{E}^{\mathbb{R}_+}$, the σ-field generated by the coordinate maps $\Omega \ni \omega \mapsto \omega(t)$, for each $t \geq 0$. Moreover, $(X_t)_{t\geq 0}$ will be the canonical process on Ω, i.e. for each $t \geq 0$, $X_t(\omega) = \omega(t)$, for all $\omega \in \Omega$. Under this setting, the following proposition can be proved by Kolmogorov's extension theorem.

Proposition 3.12. *Given a transition semigroup $(Q_t)_{t\geq 0}$ on E, for any probability measure μ on E, there exists a unique probability measure P_μ on $(\Omega, \mathscr{F})$ under which $(X_t)_{t\geq 0}$ is a Markov process with transition semigroup $(Q_t)_{t\geq 0}$, and the law of X_0 is μ.*

Given a random variable Z, we denote $E_\mu[Z]$ for the expectation of Z under P_μ. When $\mu = \delta_x$ for some $x \in E$, we simply write P_x and $E_x[Z]$.

By (3.4), for any $t \geq 0$ and $f \in B(E)$, we get

$$E_x[f(X_t)] = \int Q_t(x, dy) f(y) = Q_t f(x), \tag{3.5}$$

that is, the semigroup of a Markov process of a function f at the point x is the expectation of the process started at x composed with f. In particular, taking $f = \mathbf{1}_A$ for any $A \in \mathscr{E}$, we have

$$P_x(X_t \in A) = Q_t(x, A),$$

so the transition semigroup gives the probability that the process started at x is in A at time t, as we desired in the introduction of this section.

Moreover, $E \ni x \mapsto P_x(X_t \in A)$ is a measurable function. By a standard monotone class argument, we obtain that $E \ni x \mapsto E_x[Z]$ is measurable, for any random variable Z. Also, we have that

$$E_\mu[Z] = \int \mu(dx)\, E_x[Z].$$

This enables us to write the Markov property in a handy form. For that, we consider for each $s > 0$ an operator θ_s such that for every $t \geq 0$,

$$Y_t \circ \theta_s = Y_{t+s},$$

where Y_t is any stochastic process. By this definition, it is immediate that θ_s is measurable with respect to $\sigma(Y_t, t \geq 0)$. Note also that θ_s takes away the path of Y before time s and shifts the rest to the initial time, so we call θ_s a shift operator.

Proposition 3.13. *For every random variable Z positive or bounded and $s > 0$,*

$$E_\mu[Z \circ \theta_s \mid \mathscr{F}_s] = E_{X_s}[Z].$$

The right-hand side of this equation is the composition of $\omega \mapsto X_s(w)$ with $x \mapsto E_x[Z]$, so it is $\mathscr{F}_s$-measurable by our previous observations. Notice that for $Z = \mathbf{1}_{\{X_t \in A\}}$, the above formula gives

$$P_\mu[X_{t+s} \in A \mid \mathscr{F}_s] = P_{X_s}(X_t \in A) = Q_t(X_s, A),$$

which is exactly the definition of the Markov process, so the reader can expect that this proposition follows from applying a monotone class argument.

We briefly observe that under some topological conditions on E (metrizable, locally compact, and countable at infinity), and for a Feller semigroup $(Q_t)_{t\geq 0}$, if $(X_t)_{t\geq 0}$ is a Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$, with Feller semigroup $(Q_t)_{t\geq 0}$, we can obtain $(\tilde{X}_t)_{t\geq 0}$ a modification of $(X_t)_{t\geq 0}$ with càdlàg sample paths. This modification is still a Markov process with respect to the same transition semigroup $(Q_t)_{t\geq 0}$, but with respect to a right-continuous filtration $(\tilde{\mathscr{F}}_t)_{t\geq 0}$.

Finally, we set this entire construction as our standard definition of a Markov process.

Definition 3.14. *Let $(\mathscr{F}_t)_{t\geq 0}$ be a right-continuous filtration on $(\Omega, \mathscr{F})$ and $(X_t)_{t\geq 0}$ be an adapted process with values in E with right-continuous sample paths. For each $x \in E$, let P_x be a probability measure on $\mathscr{F}_\infty$ such that $E \ni x \mapsto P_x(S)$ is measurable, for every $S \in \mathscr{F}_\infty$. Given a probability measure μ on $(E, \mathscr{E})$, let $P_\mu = \int \mu(dx)\, P_x$.*

Then $(X_t)_{t\geq 0}$ is a normal Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$, if

- $P_x(X_0 = x) = 1$ *(in particular, μ is the initial distribution under P_μ);*
- $E_\mu[Z \circ \theta_s \mid \mathscr{F}_s] = E_{X_s}[Z]$*, for all $s > 0$, and any $\mathscr{F}^X_\infty$-measurable function Z.*

This particular version of the Markov process has a important property.

Proposition 3.15. *Let $x \in E$ and $\sigma_x = \inf\{t > 0 : X_t \neq x\}$. There exists $\lambda(x) \in [0, \infty]$ such that σ_x is exponentially distributed with parameter $\lambda(x)$ under P_x.*

With this proposition, we have a classification of points. If $\lambda(x) = \infty$, then $P_x(\sigma_x = 0) = 1$, which means that the process leaves x immediately. If $\lambda(x) = 0$, then $P_x(\sigma_x = \infty) = 1$ or equivalently $P_x(X_t = x, \forall\ t) = 1$, so the process never leaves x, and we call x a trap or an absorbing point. Finally, if $0 < \lambda(x) < \infty$, then σ_x has an exponential law with parameter $\lambda(x)$, and we say that x is a holding point or that the process stays in x for an exponential holding time. This characterization will be crucial in the following chapters.

We are interested in a stronger version of Definition 3.14, where we also can do shifts at random times, specifically using stopping times.

Definition 3.16. *Let $(X_t)_{t\geq 0}$ be a normal Markov process with respect to $(\mathscr{F}_t)_{t\geq 0}$. We say that $(X_t)_{t\geq 0}$ is a strong Markov process if for every $(\mathscr{F}_t)$-stopping time T, and any μ probability measure on $\mathscr{E}$, we have*

$$E_\mu[Z \circ \theta_T \mid \mathscr{F}_T] = E_{X_T}[Z],$$

for every $\mathscr{F}^X_\infty$-measurable function Z.

Let $x \in E$, $\sigma_x = \inf\{t > 0 : X_t \neq x\}$, and $\lambda(x)$ as in Proposition 3.15. The next result ensures that a strong Markov process only leaves holding points by a jump.

Proposition 3.17. *Let $(X_t)_{t\geq 0}$ be a strong Markov process. If $0 < \lambda(x) < \infty$, then $P_x(X_{\sigma_x} \neq x) = 1$.*

Strong Markov processes have a more probabilistic way to compute their infinitesimal generator, given by Dynkin's Formula. We will consider process $(X_t)_{t\geq 0}$ taking values on $\mathbb{R}^d$. Recall the definition of τ_A from (2.2).

Theorem 3.18 (Dynkin's Formula). *Let $(X_t)_{t\geq 0}$ be a strong Markov process taking values on $\mathbb{R}^d$ with respect to $(\mathscr{F}_t)_{t\geq 0}$ and with Feller semigroup $(Q_t)_{t\geq 0}$. For $f \in \mathscr{D}(L)$ and x not being an absorbing point, we have*

$$Lf(x) = \lim_{|A|\to 0, A\ni x} \frac{E_x[f(X_{\tau_{A^c}})] - f(x)}{E_x[\tau_{A^c}]}, \tag{3.6}$$

where $|A|$ denotes the maximum diameter of a Borel set A, restricted by $E_x[\tau_{A^c}] > 0$.

We apply now this framework to Brownian motion.

Proposition 3.19. *A Brownian motion is a strong Markov process with Feller semigroup*

$$Q_t(x, dy) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right) dy.$$

Moreover, its infinitesimal generator is given by $Lf = \frac{1}{2}f''$ and its domain is the set $\mathscr{D}(L) = \{f \in C^2(\mathbb{R}) : f, f'' \in C_0(\mathbb{R})\}$.

In the next example, we compute Dynkin's formula for the Brownian motion, since it does not have absorbing points.

Example 1. Let $(B_t)_{t\geq 0}$ be a Brownian motion starting from $x \in \mathbb{R}$. Consider $A = (x - h_1, x + h_2)$ with $h_1, h_2 \searrow 0$. Since $x \in A$ and B_t starts at x under P_x, by the continuity of the paths, B_t cannot jump out of the interval A, but it must exit precisely at one of its endpoints, then $\tau_{A^c} = \tau_{\{x-h_1, x+h_2\}}$. Thus,

$$E_x[f(B_{\tau_{A^c}})] = f(x-h_1)P_x(B_{\tau_{\{x-h_1,x+h_2\}}} = x - h_1) + f(x+h_2)P_x(B_{\tau_{\{x-h_1,x+h_2\}}} = x + h_2).$$

Using Markov property and Proposition 2.20, we get

$$\begin{aligned} E_x[f(B_{\tau_{A^c}})] &= f(x-h_1)P_0(B_{\tau_{\{-h_1,h_2\}}} = -h_1) + f(x+h_2)P_0(B_{\tau_{\{-h_1,h_2\}}} = h_2) \\ &= \frac{f(x-h_1)h_2 + f(x+h_2)h_1}{h_2 + h_1}. \end{aligned} \tag{3.7}$$

We now claim that $E_x[\tau_{\{x-h_1,x+h_2\}}] = h_1 h_2$, for every $x \in \mathbb{R}$. Indeed, by homogeneity of the Brownian motion in space, it suffices to show that $E_0[\tau_{\{-h_1,h_2\}}] = h_1 h_2$.

Now, by Proposition 3.19, if $f \in \mathscr{D}(L)$, then $f \in C^2(\mathbb{R})$, so we can use Taylor's theorem to write

$$\frac{f(h)-f(0)}{h} = f'(0) + \frac{1}{2}f''(0)h + \frac{r(h)}{h} \text{ with } \lim_{h\to 0}\frac{r(h)}{h^2},$$

for $f \in \mathscr{D}(L)$. Substituting $-h_1$ and h_2 in this equation, one can see that

$$\begin{aligned}\frac{1}{2}f''(0) &= \lim_{h_1\searrow 0, h_2\searrow 0}\left(\frac{f(-h_1)-f(0)}{h_1} + \frac{f(h_2)-f(0)}{h_2}\right)(h_1+h_2)^{-1}\\ &= \lim_{h_1\searrow 0, h_2\searrow 0}\left(\frac{f(-h_1)h_2 + f(h_2)h_1}{h_2+h_1} - f(0)\right)(h_1h_2)^{-1}\end{aligned}$$

On the other hand, if we consider equation (3.7) with $x = 0$, and the generator's value stated in Proposition 3.19, then Dynkin's Formula yields the equality:

$$\frac{1}{2}f''(0) = \lim_{h_1\searrow 0, h_2\searrow 0}\left(\frac{f(-h_1)h_2 + f(h_2)h_1}{h_2+h_1} - f(0)\right)(E_0[\tau_{\{-h_1,h_2\}}])^{-1},$$

Thus,

$$\lim_{h_1\searrow 0, h_2\searrow 0}\frac{h_1h_2}{E_0[\tau_{\{-h_1,h_2\}}]} = 1.$$

Finally, to obtain the equality without taking the limit, we use the scaling property of Brownian Motion: $\sqrt{c}B_t \overset{d}{=} B_{ct}$, for every $c > 0$ (see Proposition 2.7). Consequently, $\tau_{\{-\sqrt{c}h_1,\sqrt{c}h_2\}} \overset{d}{=} c\tau_{\{-h_1,h_2\}}$ and

$$1 = \lim_{c\searrow 0}\frac{ch_1h_2}{E_0[\tau_{\{-\sqrt{c}h_1,\sqrt{c}h_2\}}]} = \frac{h_1h_2}{E_0[\tau_{\{-h_1,h_2\}}]},$$

so the claim is proved.

Therefore, we get Dynkin's formula for the generator of a Brownian motion:

$$Lf(x) = \lim_{h_1\searrow 0, h_2\searrow 0}\left(\frac{f(x-h_1)h_2 + f(x+h_2)h_1}{h_2+h_1} - f(x)\right)(h_1h_2)^{-1}. \tag{3.8}$$

Remark 3. *It is possible to show that the limit in (3.8) is uniform in x.*

We finish this section by defining the type of processes we will examine from now on. Let $D \subseteq \mathbb{R}$ be an interval, we add to D an isolated and absorbing point Δ, called the *cemetery*.

Definition 3.20. *A diffusion process $(W_t)_{t\geq 0}$ on $D \subseteq \mathbb{R}$ is any strong Markov process with values in $D \cup \Delta$ with respect to $(\mathscr{F}^W_{t+})$ whose sample paths are continuous for $0 \leq t < \tau_\Delta$, and equal to Δ for $t \geq \tau_\Delta$, where $\tau_\Delta = \inf\{t \geq 0 : X_t = \Delta\}$.*

We will often use the continuity of the paths to guarantee that a diffusion started at x cannot reach y without hitting all the points between x and y. We recall from (2.2) the definition of the hitting times

$$\tau_x = \inf\{t \geq 0 : W_t = x\},$$

for every $x \in D \cup \Delta$. By Proposition 2.16, τ_x, $x \in D$, are stopping times, and might also be ∞, if the process is killed first. Moreover, we observe that the right-continuity of the paths on $D \cup \Delta$ is enough to ensure that $W_{\tau_x} = x$ on $\{\tau_x < \infty\}$.

Chapter 4

The most general BM on $\mathbb{R}$

From now on, we will consider our state space $E = \mathbb{R}$ with the usual topology. It is easy to see that the set $C_0(\mathbb{R})$ coincides with the subspace of continuous functions $f : \mathbb{R} \to \mathbb{R}$ having zero limit as $x \to \pm\infty$. Also, since $f \in C_0(\mathbb{R})$ is uniformly continuous on compact subsets and decays to infinity, one can see that f is uniformly continuous. Furthermore, we will add to $\mathbb{R}$ an isolated point Δ called the *cemetery*, and consider $C_0^{\Delta}(\mathbb{R})$ the set of functions in $C_0(\mathbb{R})$ extended to value 0 at Δ.

Definition 4.1. *A stochastic process $(W_t)_{t\geq 0}$ on $\mathbb{R}$ is called a general Brownian motion on $\mathbb{R}$ with boundary conditions at the origin if it satisfies the following properties:*

- *$(W_t)_{t\geq 0}$ is a strong Markov process with values in $\mathbb{R} \cup \{\Delta\}$ and it has càdlàg trajectories.*
- *The sample paths of $(W_t)_{t\geq 0}$ are continuous on the set $\big\{t \geq 0 : \lim_{s\to t^-} W_s$ or $W_t \notin \{0, \Delta\}\big\}$.*
- *The point Δ, called the cemetery, is an absorbing state.*
- *Let $\tau_0 = \inf\{t \geq 0 : W_t = 0\}$ be the hitting time of 0. For every initial point $x \in \mathbb{R}$, the law of the process $(W_{t\wedge\tau_0})_t$ coincides with the law of a standard Brownian motion on $\mathbb{R}$ absorbed at 0.*

Recalling Definition 3.20, note that a general Brownian motion on $\mathbb{R}$ is a diffusion. Also, the above definition ensures that W_t behaves like a standard Brownian motion until it hits 0, so let us investigate what happens afterwards. First, from the properties of the Brownian motion, we know that $\tau_0 < \infty$ a.s., and by the right-continuity of the paths, we have $W_{\tau_0} = 0$. Thus, since Δ is an absorbing point, it follows that W_t cannot reach Δ before 0.

Now, by Proposition 3.15, we have that $T = \inf\{t > 0 : W_t \neq 0\}$ is exponentially distributed with parameter $\lambda \in [0, \infty]$ under P_0. We distinguish three cases depending on the value of λ.

- **Case 1:** $0 < \lambda < \infty$

  Due to Proposition 3.17, the process leaves 0 by a jump, so the continuity of the paths on $\mathbb{R}$ forces W to jump to Δ. Hence, $W_t = \Delta$ for all $t \geq T$ in this case.

- **Case 2:** $\lambda = \infty$

  Here, $P_0(T = 0) = 1$, so the process leaves 0 at once. Since $P_0(W_0 = 0) = 1$ and the paths are right-continuous, the process cannot exit immediately from 0 to Δ, otherwise this would be a jump and the process would be continuous to the left, not to the right.

- **Case 3:** $\lambda = 0$

  In this situation, we have $T = \infty$ a.s., which means that 0 is a trap.

We now prove that a general Brownian motion on $\mathbb{R}$ with boundary conditions at 0 is a Feller process on $C_0^\Delta(\mathbb{R})$. In the proof, we will use the notation $W_t \notin \mathbb{R}$ meaning that $W_t = \Delta$. This will apply to more general settings where the process takes values in a partitioned state space $E = A_1 \cup A_2$ and it is allowed to exit one of the sets A_i without being killed, what will be very useful in the next chapter.

In the present case, the proof could be simplified noting that $f \in C_0^\Delta(\mathbb{R})$ vanishes on the sets $\{W_t \notin \mathbb{R}\}$, but we avoid this in view of the results in Chapter 5.

Proposition 4.2. *Every general Brownian motion on* $\mathbb{R}$ *with boundary conditions at the origin has a Feller semigroup on* $C_0^\Delta(\mathbb{R})$.

Proof. To prove the result, we need to verify whether the semigroup satisfies Definition 3.5. Recall from equation (3.5) that the semigroup is given by $Q_t f(x) = E_x[f(W_t)]$, for each $t \geq 0$ and $x \in \mathbb{R} \cup \Delta$. Thus, we have to show the uniform convergence,

$$||Q_t f - f|| \longrightarrow 0 \text{ as } t \to 0,$$

and the stability of $C_0^\Delta(\mathbb{R})$ for the process semigroup, that is, if $f \in C_0^\Delta(\mathbb{R})$, then $Q_t f \in C_0^\Delta(\mathbb{R})$. Hence, for $f \in C_0^\Delta(\mathbb{R})$, we shall prove that $E_x[f(W_t)]$ is continuous in $x \in \{\Delta\} \cup \mathbb{R}$ and tends to 0 at infinity,

$$\lim_{x \to \infty} E_x[f(W_t)] = 0.$$

Also, $Q_t f(x)$ must vanish at Δ, but this is trivial, since Δ is an absorbing point and $f(\Delta) = 0$, hence $E_\Delta[f(W_t)] = 0$, for all $t \geq 0$.

We begin by proving the uniform convergence. By Remark 2, this comes from the pointwise convergence. Then, it suffices to show that

$$|E_x[f(W_t)] - f(x)| \longrightarrow 0 \text{ as } t \to 0, \tag{4.1}$$

for all $f \in C_0^\Delta(\mathbb{R})$ and $x \in \mathbb{R} \cup \Delta$. For $x = \Delta$, this is immediate since both f and $E_x[f(W_t)]$ vanish at Δ. So, fix $x \in \mathbb{R}$ and $f \in C_0^\Delta(\mathbb{R})$. For any $t \geq 0$, we have

$$|E_x[f(W_t)] - f(x)| \leq E_x[|f(W_t) - f(x)|\mathbf{1}_{\{W_t \in \mathbb{R}\}}] + E_x[|f(W_t) - f(x)|\mathbf{1}_{\{W_t \notin \mathbb{R}\}}]. \tag{4.2}$$

We now prove separately that these two terms from the right-hand side of (4.2) go to 0 as $t \to 0$, hence we get (4.1). Since f is bounded, for the second term on the right hand side of (4.2) it is sufficient to show that

$$\lim_{t \to 0} P_x(W_t \notin \mathbb{R}) = 0, \tag{4.3}$$

Since $P_x(W_0 = x) = 1$, then $P_x(\tau_\Delta > 0) = 1$. Consequently,

$$\lim_{t \to 0} \mathbf{1}_{\{W_t \notin \mathbb{R}\}} = 0$$

P_x-almost surely and we get (4.3).

Now, for the first term on the right hand side of (4.2) , recall that f is uniformly continuous, so for every $\varepsilon > 0$, we can find a $\delta = \delta(\varepsilon)$ such that

$$\begin{aligned}
&E_x[|f(W_t) - f(x)|\mathbf{1}_{\{W_t \in \mathbb{R}\}}] \\
&= E_x[|f(W_t) - f(x)|\mathbf{1}_{\{|W_t - x| < \delta\} \cap \{W_t \in \mathbb{R}\}}] + E_x[|f(W_t) - f(x)|\mathbf{1}_{\{|W_t - x| \geq \delta\} \cap \{W_t \in \mathbb{R}\}}] \\
&< \varepsilon + 2\|f\| P_x(|W_t - x| \geq \delta, W_t \in \mathbb{R}).
\end{aligned}$$

Again, it is enough to prove that for all $\delta > 0$,

$$\lim_{t \to 0} P_x(|W_t - x| \geq \delta, W_t \in \mathbb{R}) = \lim_{t \to 0} P_x(|W_t - W_0| \geq \delta, W_t \in \mathbb{R}) = 0,$$

which follows from the continuity of the paths, since for every $\delta > 0$, it holds that $\mathbf{1}_{\{|W_t - W_0| \geq \delta\} \cap \{W_t \in \mathbb{R}\}} = 0$ whenever t is sufficiently close to 0.

We proceed by verifying that $C_0^\Delta(\mathbb{R})$ is stable for the process semigroup. First, fixed some $f \in C_0^\Delta(\mathbb{R})$ and $t \geq 0$, we will show that

$$\lim_{x \to \infty} E_x[f(W_t)] = 0.$$

Indeed, given $\varepsilon > 0$ there exists $M > 0$ such that $|f(x)| < \varepsilon$ for $x > M$. Thus,

$$\begin{aligned}
&|E_x[f(W_t)]| \\
&\leq E_x[|f(W_t)|\mathbf{1}_{\{W_t > M\} \cap \{W_t \in \mathbb{R}\}}] + E_x[|f(W_t)|\mathbf{1}_{\{W_t \leq M\} \cap \{W_t \in \mathbb{R}\}}] + E_x[|f(W_t)|\mathbf{1}_{\{W_t \notin \mathbb{R}\}}]| \\
&\leq \varepsilon + \|f\| P_x(W_t \leq M, W_t \in \mathbb{R}) + \|f\| P_x(W_t \notin \mathbb{R}).
\end{aligned}$$

So, we want to show that for every $t \geq 0$ and $M > 0$,

$$\lim_{x\to\infty} P_x(W_t \leq M, W_t \in \mathbb{R}) = 0 \quad \text{and} \quad \lim_{x\to\infty} P_x(W_t \notin \mathbb{R}) = 0. \tag{4.4}$$

To do so, let $y \in \mathbb{R}$ and $(B_t)_{t\geq 0}$ be a standard Brownian motion, we denote $\tau_y = \inf\{t \geq 0 : W_t = y\}$ and $\tau_y^B = \inf\{t \geq 0 : B_t = y\}$. Let $x > M$, starting from x, W_t cannot reach zero without passing through M, thus by the continuity of the paths, we have that $P_x(\tau_0 > \tau_M) = 1$. In particular, the definition of W_t implies that $\tau_M \stackrel{d}{=} \tau_M^B$. Also, if $W_t \leq M$ then W_t has already hit M by time t, so $\tau_M^B \leq t$ under P_x, whose probability we can estimate using the proof of Corollary 2.19. Thus,

$$\begin{aligned} P_x(W_t \leq M, W_t \in \mathbb{R}) &\leq P_x(\tau_M^B \leq t) \\ &= P_0(\tau_{x-M}^B \leq t) \\ &= 2P_0(B_t \geq x - M) \\ &= 2\int_{x-M}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{y^2}{2t}\Big)\,dy \longrightarrow 0 \ \text{ as } x \to \infty. \end{aligned}$$

We will also use this to achieve the second limit in (4.4). Since the process does not exit $\mathbb{R}$ without passing through 0 and it is equal in distribution to a Brownian motion in time $[0, \tau_0]$, we have

$$P_x(W_t \notin \mathbb{R}) \leq P_x(\tau_0^B \leq t) \longrightarrow 0 \ \text{ as } x \to \infty,$$

as before. By symmetry of the Brownian motion, the same argument holds to prove that $E_x[f(W_t)]$ goes to 0 as $x \to -\infty$.

Finally, we will show that $E_x[f(W_t)]$ is continuous in $x \in \{\Delta\} \cup \mathbb{R}$, fixed some $f \in C_0^{\Delta}(\mathbb{R})$ and $t \geq 0$. Since the cemetery is an isolated point, we can assume $x \in \mathbb{R}$. Let us construct a coupling of all processes W_t^y starting from any $y \in \mathbb{R}$. Let $(B_t)_{t\geq 0}$ be a Brownian motion starting from 0 independent of W_t^0, a general Brownian motion with boundary conditions at the origin, also starting from zero. We define

$$W_t^y = \begin{cases} y + B_t\,, & \text{if } 0 \leq t < \tau_{-y}^B; \\ W_{t-\tau_{-y}^B}^0\,, & \text{if } t \geq \tau_{-y}^B. \end{cases}$$

In this way, before the time τ_{-y}^B (when B_t first hits $-y$ so that W_t^y first hits 0), the processes W_t^y follow the same Brownian motion, but each one starting from y. After τ_{-y}^B, the processes W_t^y are coupled with W_t^0.

Without loss of generality, we can suppose that $x \in [0, \infty)$, since the other case is symmetric. First, fixed $x \in (0, \infty)$ and $t \geq 0$, we have for any $y \in \mathbb{R}$ that

$$\begin{aligned} &|E[f(W_t^y)] - E[f(W_t^x)]| \\ &\leq E[|f(W_t^y) - f(W_t^x)|\mathbf{1}_{\{t<\tau_{-y}^B\wedge\tau_{-x}^B\}}] + E[|f(W_t^y) - f(W_t^x)|\mathbf{1}_{\{t\geq\tau_{-y}^B\wedge\tau_{-x}^B\}}] \\ &\leq E[|f(y + B_t) - f(x + B_t)|] + E[|f(W_t^y) - f(W_t^x)|\mathbf{1}_{\{t\geq\tau_{-y}^B\wedge\tau_{-x}^B\}}]. \end{aligned}$$

By the uniform continuity of f, we have $E[|f(y+B_t)-f(x+B_t)|] \longrightarrow 0$ as $y \to x$. For the remaining part, without loss of generality, we can suppose that $0 < x < y$ (otherwise, we can choose $0 < y < x$, and the proof is totally symmetric). Thus, $\tau^B_{-x} < \tau^B_{-y}$, and

$$\begin{aligned}
&E[|f(W^y_t) - f(W^x_t)|\mathbf{1}_{\{t \geq \tau^B_{-y} \wedge \tau^B_{-x}\}}] \\
&= E[|f(W^y_t) - f(W^x_t)|\mathbf{1}_{\{\tau^B_{-x} \leq t \leq \tau^B_{-y}\}}] + E[|f(W^y_t) - f(W^x_t)|\mathbf{1}_{\{\tau^B_{-y} < t\}}] \\
&\leq 2\|f\|P(\tau^B_{-x} \leq t \leq \tau^B_{-y}) + E[|f(W^0_{t-\tau^B_{-y}}) - f(W^0_{t-\tau^B_{-x}})|\mathbf{1}_{\{\tau^B_{-y} < t\}}].
\end{aligned}$$

Due to the limit $\lim_{y\to x} \tau^B_{-y} = \tau^B_{-x}$ a.s., the sequence of sets $\{\tau^B_{-x} \leq t \leq \tau^B_{-y}\}$ decreases to $\{\tau^B_{-x} = t\}$ as y decreases to x. Thus,

$$\lim_{y \uparrow x} P(\tau^B_{-y} \leq t \leq \tau^B_{-x}) = P(\tau^B_{-x} = t) = 0,$$

where the last equality follows from the fact that τ^B_{-x} is a continuous random variable (see Corollary 2.19).

Now, denoting $\tau^0_\Delta = \inf\{t \geq 0 : W^0_t = \Delta\} = \inf\{t \geq 0 : W^0_t \notin \mathbb{R}\}$, note that

$$\begin{aligned}
&E[|f(W^0_{t-\tau^B_{-y}}) - f(W^0_{t-\tau^B_{-x}})|\mathbf{1}_{\{\tau^B_{-y} < t\}}] \\
&= E[|f(W^0_{t-\tau^B_{-y}}) - f(W^0_{t-\tau^B_{-x}})|\mathbf{1}_{\{\tau^B_{-x}+\tau^0_\Delta = t\}}\mathbf{1}_{\{\tau^B_{-y} < t\}}] \\
&\quad + E[|f(W^0_{t-\tau^B_{-y}}) - f(W^0_{t-\tau^B_{-x}})|\mathbf{1}_{\{\tau^B_{-x}+\tau^0_\Delta \neq t\}}\mathbf{1}_{\{\tau^B_{-y} < t\}}].
\end{aligned}$$

Since $P(\tau^B_{-x} + \tau^0_\Delta = t) = 0$ (see Remark 4 below), the first term vanishes. It remains to analyze the second term. Note that the only discontinuity allowed on the paths of the process is when it jumps from 0 to the cemetery, which occurs for W^y_t when $W^0_{t-\tau^B_{-y}} = W^0_{\tau^0_\Delta}$, that is $\{\tau^B_{-y} + \tau^0_\Delta = t\}$. Observe that on the event $\{\tau^B_{-x} + \tau^0_\Delta \neq t\}$, we can choose y sufficiently close to x such that $\tau^B_{-y} + \tau^0_\Delta \neq t$. On this set, the paths W^0_t are continuous, then

$$W^0_{t-\tau^B_{-y}} \longrightarrow W^0_{t-\tau^B_{-x}} \qquad \text{as } y \to x$$

almost surely. Since f is continuous and bounded, the Dominated Convergence Theorem yields

$$E[|f(W^0_{t-\tau^B_{-y}}) - f(W^0_{t-\tau^B_{-x}})|\mathbf{1}_{\{\tau^B_{-x}+\tau^0_\Delta \neq t\}}\mathbf{1}_{\{\tau^B_{-y} < t\}}] \longrightarrow 0 \qquad \text{as } y \to x.$$

Therefore, we conclude that

$$|E[f(W^y_t)] - E[f(W^x_t)]| \longrightarrow 0 \qquad \text{as } y \to x,$$

for $x > 0$.

For $x = 0$, we have that $\tau_0^B = 0$ a.s. under P_0 and W_t^x coincides with W_t^0. For any $y \in \mathbb{R}$, note that

$$\begin{aligned}
&|E[f(W_t^y)] - E[f(W_t^0)]| \\
&\leq E[|f(y + B_t) - f(W_t^0)|\mathbf{1}_{\{\tau_{-y}^B > t\}}] + E[|f(W_{t-\tau_{-y}^B}^0) - f(W_t^0)|\mathbf{1}_{\{\tau_{-y}^B \leq t\}}] \\
&\leq 2\|f\|P(\tau_{-y}^B > t) + E[|f(W_{t-\tau_{-y}^B}^0) - f(W_{t-\tau_{-x}^B}^0)|\mathbf{1}_{\{\tau_{-y}^B \leq t\}}].
\end{aligned}$$

For the first term in the right-hand side of the equation, we use the limit $\lim_{y\to 0} \tau_{-y}^B = \tau_0^B = 0$ a.s. to obtain

$$\lim_{y\to 0} P(\tau_{-y}^B > t) = P(\varnothing) = 0.$$

For the second term, we argue as before, using the continuity of the sample paths and the function f, and the Dominated Convergence Theorem. This completes the proof of the continuity of $Q_t f(x)$ in x, and the lemma follows. □

Remark 4. The claim $P(\tau_{-x}^B + \tau_\Delta^0 = t) = 0$ follows from the fact that if X, Y are independent random variables and X is continuous (i.e. its distribution function F_X is continuous), then $X + Y$ is continuous.

Indeed, for $t \in \mathbb{R}$, $\delta > 0$, note that

$$\begin{aligned}
P(X + Y = t) &\leq E[\mathbf{1}_{\{t-\delta < X+Y \leq t\}}] \\
&= E[E[\mathbf{1}_{\{t-\delta-Y < X \leq t-Y\}}|Y]] \\
&= E[g(Y)],
\end{aligned}$$

where $g(y) = E[\mathbf{1}_{\{t-\delta-y < X \leq t-y\}}]$, using the substitution rule (see [2, Example 5.1.5]), since X, Y are independent. On the other hand, for all $y \in \mathbb{R}$ and $\varepsilon > 0$,

$$E[\mathbf{1}_{\{t-\delta-y < X \leq t-y\}}] = F_X(t - y) - F_X(t - \delta - y) < \varepsilon,$$

choosing $\delta = \delta(\varepsilon) > 0$ small enough. In fact, F_X is uniformly continuous, because it is continuous, $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$. Since $\varepsilon > 0$ was arbitrary, we can conclude.

Since each Brownian motion on $\mathbb{R}$ with boundary conditions at the origin is a Feller process, we are naturally led to the problem of computing its infinitesimal generator.

Proposition 4.3. *For f in the domain $\mathscr{D}_W(L)$ of a general Brownian motion W_t on $\mathbb{R}$ with boundary conditions at the origin, the infinitesimal generator values*

$$Lf(x) = \frac{1}{2}f''(x),$$

at $x \neq 0$ and $Lf(\Delta) = 0$. Moreover, $\mathscr{D}_W(L)$ is contained in the space of functions whose second derivative on $\mathbb{R} \setminus \{0\}$ extends to an element of $C_0^\Delta(\mathbb{R})$.

Proof. Let $x \neq 0$, then x is not a trap and we can apply the Dynkin's formula given in Theorem 3.18. Suppose that $x > 0$ and consider $A = (x - h_1, x + h_2)$ with $h_1, h_2 \searrow 0$. For h_1 sufficiently small and starting from point x, for W_t to reach 0 it must exit the set A. Therefore, W_t behaves like a Brownian motion in A, and their Dynkin's formulas are the same. Thus, as in Example 1, we have

$$\begin{aligned} Lf(x) &= \lim_{h_1 \searrow 0, h_2 \searrow 0} \left(\frac{f(x-h_1)h_2 + f(x+h_2)h_1}{h_2 + h_1} - f(x) \right) (h_1 h_2)^{-1} \\ &= \lim_{h_1 \searrow 0, h_2 \searrow 0} \left(\frac{f(x-h_1) - f(x)}{h_1} + \frac{f(x+h_2) - f(x)}{h_2} \right) (h_1 + h_2)^{-1}. \end{aligned} \tag{4.5}$$

Since h_1 and h_2 go to 0 independently, we can take first the limit on h_2 and we get

$$Lf(x) = \lim_{h_1 \searrow 0} \left(\frac{f(x-h_1) - f(x)}{h_1} + \lim_{h_2 \searrow 0} \frac{f(x+h_2) - f(x)}{h_2} \right) (h_1)^{-1}, \tag{4.6}$$

so that the limit $\lim_{h_2 \searrow 0} \frac{f(x+h_2)-f(x)}{h_2}$ exists, which is precisely $f'_+(x)$ the right derivative of f at x. Similarly, the limit $\lim_{h_1 \searrow 0} \frac{f(x-h_1)-f(x)}{h_1}$ exists and it is equal to $-f'_-(x)$, where $f'_-(x)$ denotes the left derivative of f. Moreover, for the limit in (4.6) to be finite, the numerator must necessarily approach zero, implying that $f'_-(x) = f'_+(x)$. This ensures the existence of the derivative $f'(x)$ for $x > 0$. A completely analogous argument shows that $f'(x)$ also exists for $x < 0$.

Now, we will compute the second derivative. Fix $0 < \varepsilon < x$, we apply (4.5) to the points $\varepsilon < y < x$ with $h_1 = h_2 = h_n = \frac{x-\varepsilon}{n}$, for each $n \in \mathbb{N}$, so

$$Lf(y) = \lim_{n \to \infty} \frac{1}{2h_n} \left(\frac{f(y-h_n) - f(y)}{h_n} + \frac{f(y+h_n) - f(y)}{h_n} \right).$$

By Remark 3, the above limit is uniform in $\varepsilon < y < x$, so we may exchange the limit and the integral below:

$$\int_\varepsilon^x Lf(y)\, dy = \lim_{n \to \infty} \int_\varepsilon^x \frac{1}{2h_n} \left(\frac{f(y-h_n) - f(y)}{h_n} + \frac{f(y+h_n) - f(y)}{h_n} \right) dy.$$

Observe that

$$\begin{aligned} &\int_\varepsilon^x \frac{1}{2h_n} \left(\frac{f(y-h_n) - f(y)}{h_n} + \frac{f(y+h_n) - f(y)}{h_n} \right) dy \\ &= \frac{1}{2h_n^2} \left(\int_\varepsilon^x f(y-h_n)\, dy - \int_\varepsilon^x f(y)\, dy + \int_\varepsilon^x f(y+h_n)\, dy - \int_\varepsilon^x f(y)\, dy \right) \\ &= \frac{1}{2h_n^2} \left(\int_{\varepsilon-h_n}^{x-h_n} f(y)\, dy - \int_\varepsilon^x f(y)\, dy \right) + \frac{1}{2h_n^2} \left(\int_{\varepsilon+h_n}^{x+h_n} f(y)\, dy - \int_\varepsilon^x f(y)\, dy \right) \\ &= \frac{1}{2h_n^2} \left(\int_{\varepsilon-h_n}^{\varepsilon} f(y)\, dy - \int_{x-h_n}^x f(y)\, dy \right) + \frac{1}{2h_n^2} \left(\int_x^{x+h_n} f(y)\, dy - \int_\varepsilon^{\varepsilon+h_n} f(y)\, dy \right). \end{aligned}$$

Define $g(x) = \int_0^x f(y)\,dy$ for $x > 0$. Since f is differentiable on $(0, \infty)$, we have that g is twice differentiable and $g'' = f'$ for $x > 0$. We can rewrite the above expression in terms of g as follows:

$$\frac{1}{2h_n^2}\left(\int_x^{x+h_n} f(y)\,dy - \int_{x-h_n}^{x} f(y)\,dy\right) - \frac{1}{2h_n^2}\left(\int_\varepsilon^{\varepsilon+h_n} f(y)\,dy - \int_{\varepsilon-h_n}^{\varepsilon} f(y)\,dy\right)$$
$$= \frac{1}{2h_n}\left(\frac{g(x+h_n)-g(x)}{h_n} + \frac{g(x-h_n)-g(x)}{h_n}\right)$$
$$- \frac{1}{2h_n}\left(\frac{g(\varepsilon+h_n)-g(\varepsilon)}{h_n} + \frac{g(\varepsilon-h_n)-g(\varepsilon)}{h_n}\right),$$

and applying Taylor's theorem (similarly as we did in Example 1), we obtain

$$\lim_{n\to\infty} \frac{1}{2h_n}\left(\frac{g(x+h_n)-g(x)}{h_n} + \frac{g(x-h_n)-g(x)}{h_n}\right) = \frac{1}{2}g''(x) = \frac{1}{2}f'(x),$$

and

$$\lim_{n\to\infty} \frac{1}{2h_n}\left(\frac{g(\varepsilon+h_n)-g(\varepsilon)}{h_n} + \frac{g(\varepsilon-h_n)-g(\varepsilon)}{h_n}\right) = \frac{1}{2}f'(\varepsilon).$$

Thus, combing all these expressions, we have

$$\int_\varepsilon^x Lf(y)\,dy = \frac{1}{2}(f'(x) - f'(\varepsilon)),$$

and the Fundamental Theorem of Calculus implies that f' is differentiable and

$$Lf(x) = \frac{1}{2}f''(x),$$

for $x > \varepsilon > 0$. Since $\varepsilon \in (0, x)$ was arbitrary, this identity extends to all $x > 0$. The proof for $x < 0$ is analogous.

Now, to compute the generator at Δ, using that it is an absorbing state, we obtain trivially

$$Lf(\Delta) = \lim_{t\to 0^+} \frac{E_\Delta[f(W_t)] - f(\Delta)}{t} = 0.$$

Therefore, for $f \in \mathscr{D}_W(L)$, we have by definition that $Lf \in C_0^\Delta(\mathbb{R})$, and we have compute that

$$Lf(x) = f''(x) \text{ for } x \neq 0 \text{ and } Lf(\Delta) = 0.$$

Since Lf is continuous and Δ is isolated, there exists a natural extension of f'' to 0 and Δ, given by

$$\lim_{x\to 0^+} f''(x) = \lim_{x\to 0^-} f''(x) = 2Lf(0),$$
$$f''(\Delta) = 2Lf(\Delta) = 0.$$

In particular, this extension will belong to $C_0^\Delta(\mathbb{R})$. This completes the proof. □

Now, in order to fully characterize the general Brownian motion on $\mathbb{R}$ (with boundary conditions at the origin), we must determine the domain of its infinitesimal generator. Thus, by Theorem 3.9, we will determine all such processes. Naturally, we expect that the domain of our general Brownian motion will have certain restrictions at 0. The next lemma confirms our expectations by showing that the domain $\mathscr{D}_W(L)$ is determined by a condition in the neighborhood of $x = 0$. Moreover, we will see later that this boundary condition depends on the behavior of the process when it hits 0.

Lemma 4.4. *If $f_1 \in \mathscr{D}_W(L)$ and $f_2 \in C_0^{\Delta}(\mathbb{R})$ is such that f_2'' exists in $\mathbb{R} \setminus \{0\}$, admits an extension to an element of $C_0^{\Delta}(\mathbb{R})$, and $f_1 = f_2$ in $(-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$, then $f_2 \in \mathscr{D}_W(L)$.*

Proof. We need to show that the limit

$$\lim_{t\to 0^+} \frac{Q_t f_2(x) - f_2(x)}{t} = \lim_{t\to 0^+} \frac{E_x[f_2(W_t)] - f_2(x)}{t}$$

exists and it is uniform in x. We will consider two cases: $|x| > \varepsilon/2$ and $|x| < \varepsilon/2$.

Let $|x| > \varepsilon/2$. Since the process W_t is equal in distribution to B_t until the hitting time of zero, we can compare the quotient in the generator limit for both process. Let $(B_t^0)_{t\geq 0}$ be an absorbed Brownian motion at 0 starting from x, then

$$\begin{aligned}
&\left| \frac{E_x[f_2(W_t)] - f_2(x)}{t} - \frac{E_x[f_2(B_t^0)] - f_2(x)}{t} \right| \\
&= \left| \frac{E_x[f_2(B_t^0)\mathbf{1}_{\{\tau_0 > t\}}] + E_x[f_2(W_t)\mathbf{1}_{\{\tau_0 < t\}}] - f_2(x)}{t} - \frac{E_x[f_2(B_t^0)] - f_2(x)}{t} \right| \\
&= \left| \frac{E_x[f_2(W_t)\mathbf{1}_{\{\tau_0 < t\}}] - E_x[f_2(B_t^0)\mathbf{1}_{\{\tau_0 < t\}}]}{t} \right| \\
&\leq 2\|f_2\| \frac{P_x(\tau_0 < t)}{t},
\end{aligned}$$

Now, by the proof of Corollary 2.19, the hitting time τ_0 for a Brownian motion is such that

$$\begin{aligned}
P_x(\tau_0 < t) = P_0(\tau_x < t) &= 2P_0(B_t \geq x) \\
&= 2\int_x^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\Big(-\frac{y^2}{2t}\Big)\, dy \\
&= 2\int_0^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\Big\{-\frac{(y-x)^2}{2t}\Big\}\, dy = o(t^n) \ \text{ as } t \to 0^+, \qquad (4.7)
\end{aligned}$$

for any $n \in \mathbb{N}$ and $|x| > \varepsilon$, for all $\varepsilon > 0$. Therefore, choosing $n = 1$, we conclude that the uniform limits

$$\lim_{t\to 0^+} \frac{E_x[f_2(W_t)] - f_2(x)}{t}, \quad \lim_{t\to 0^+} \frac{E_x[f_2(B_t^0)] - f_2(x)}{t}.$$

must to be equal for $|x| > \varepsilon/2$ whenever they exist.

Now, we have that $f_2 \in C_0(\mathbb{R})$ and is twice differentiable in $\mathbb{R} \setminus \{0\}$ with second derivative extending to an element of $C_0^{\Delta}(\mathbb{R})$. Since we are considering only x away from the origin, we get

$$\lim_{t \to 0^+} \frac{E_x[f_2(B_t^0)] - f_2(x)}{t} = \frac{1}{2} f_2''(x),$$

finishing the argument.

It remains to prove that the same limit extends uniformly near to $x = 0$. Since $f_1 = f_2$ on $(-\varepsilon, \varepsilon)$, we have for $-\varepsilon < x < \varepsilon$

$$\left| \frac{E_x[f_2(W_t)] - f_2(x)}{t} - \frac{E_x[f_1(W_t)] - f_1(x)}{t} \right| = \left| \frac{E_x[f_2(W_t)\mathbf{1}_{\{|W_t| > \varepsilon\}}] - E_x[f_1(W_t)\mathbf{1}_{\{|W_t| > \varepsilon\}}}{t} \right|.$$

Since the process started at $-\varepsilon < x < \varepsilon$, if $|W_t| > \varepsilon$, then W has already hit ε or $-\varepsilon$ by time t. Thus, setting $T_\varepsilon = \tau_{\{-\varepsilon, \varepsilon\}}$, we have $T_\varepsilon < t$. Observing also that $|f_2(x) - f_1(x)| \leq \max_{|x| \geq \varepsilon} |f_2(x) - f_1(x)|$ which is finite, we get

$$\left| \frac{E_x[f_2(W_t)] - f_2(x)}{t} - \frac{E_x[f_1(W_t)] - f_1(x)}{t} \right| \leq \max_{|x| \geq \varepsilon} |f_2(x) - f_1(x)| \frac{P_x(T_\varepsilon < t)}{t}.$$

We now estimate this hitting probability and aim to find a uniform upper bound. Considering $-\varepsilon/2 \leq x \leq \varepsilon/2$, we have $P_x(T_{\varepsilon/2} \leq T_\varepsilon) = 1$ by continuity of the paths (note that this also includes the case when $T_\varepsilon = \infty$ when the process is killed before hitting $-\varepsilon$ or ε). Now, recall that the shift operators θ_s removes the portion of a path before time s and shift the remaining path at time zero. Thus, if W reaches $-\varepsilon$ or ε before time t, then the shifted process $W \circ \theta_{T_{\varepsilon/2}}$ also reaches $-\varepsilon$ or ε before time t. That is,

$$\mathbf{1}_{\{T_\varepsilon < t\}} \leq \mathbf{1}_{\{T_\varepsilon < t\}} \circ \theta_{T_{\varepsilon/2}}.$$

So an application of the strong Markov property yields

$$\begin{aligned} P_x(T_\varepsilon < t) &\leq E_x[\mathbf{1}_{\{T_\varepsilon < t\}} \circ \theta_{T_{\varepsilon/2}}] \\ &= E_x[E_{W_{T_{\varepsilon/2}}}[\mathbf{1}_{\{T_\varepsilon < t\}}]] \\ &= E_x[\mathbf{1}_{\{W_{T_{\varepsilon/2}} = -\varepsilon/2\}} E_{-\varepsilon/2}[\mathbf{1}_{\{T_\varepsilon < t\}}]] + E_x[\mathbf{1}_{\{W_{T_{\varepsilon/2}} = \varepsilon/2\}} E_{\varepsilon/2}[\mathbf{1}_{\{T_\varepsilon < t\}}]] \\ &= P_x(W_{T_{\varepsilon/2}} = -\varepsilon/2) P_{-\varepsilon/2}(T_\varepsilon < t) + P_x(W_{T_{\varepsilon/2}} = \varepsilon/2) P_{\varepsilon/2}(T_\varepsilon < t) \\ &\leq P_{-\varepsilon/2}(T_\varepsilon < t) + P_{\varepsilon/2}(T_\varepsilon < t). \end{aligned}$$

Now, we compare this to the Brownian motion hitting time. Let $T_{\{0,\varepsilon\}} = \inf\{t \geq 0 : W_t \in \{-\varepsilon, 0, \varepsilon\}\}$ and $T^B_{\{0,\varepsilon\}} = \inf\{t \geq 0 : B_t \in \{-\varepsilon, 0, \varepsilon\}\}$, where $(B_t)_{t \geq 0}$ is a standard Brownian motion. Observe that

$$\begin{aligned} P_{\varepsilon/2}(T_\varepsilon < t) &\leq P_{\varepsilon/2}(T_\varepsilon < t, T_{\{0,\varepsilon\}} < t) \\ &\leq P_{\varepsilon/2}(T_{\{0,\varepsilon\}} < t) \\ &= P_{\varepsilon/2}(T^B_{\{0,\varepsilon\}} < t), \end{aligned}$$

using that W_t is equal in distribution to a Brownian motion in $[0, T_{\{0,\varepsilon\}}]$. Similarly, $P_{-\varepsilon/2}(T_\varepsilon < t) \le P_{-\varepsilon/2}(T^B_{\{0,\varepsilon\}} < t)$. Also, as we did in (4.7), we have

$$P_{-\varepsilon/2}(T^B_{\{0,\varepsilon\}} < t) = P_{\varepsilon/2}(T^B_{\{0,\varepsilon\}} < t) = o(t^n) \quad \text{as } t \to 0^+$$

for all $n > 0$.

So putting all this together, we get

$$\left|\frac{E_x[f_2(W_t)] - f_2(x)}{t} - \frac{E_x[f_1(W_t)] - f_1(x)}{t}\right|$$
$$\le \max_{x \ge \varepsilon} |f_2(x) - f_1(x)| \frac{P_{-\varepsilon/2}(T^B_{\{0,\varepsilon\}} < t) + P_{\varepsilon/2}(T^B_{\{0,\varepsilon\}} < t)}{t}$$

which goes to 0 independently of x, as $t \to 0^+$. Therefore, both expressions have the same limit. Recalling that $f_1 \in \mathscr{D}_W(L)$, we obtain that

$$\lim_{t\to 0^+} \frac{E_x[f_2(W_t)] - f_2(x)}{t} = \lim_{t\to 0^+} \frac{E_x[f_1(W_t)] - f_1(x)}{t} = \frac{1}{2} f_1''(x)$$

uniformly in $|x| \le \varepsilon/2$. This completes the proof. □

From now on, we denote $g(0+) = \lim_{x\to 0^+} g(x)$ and $g(0-) = \lim_{x\to 0^-} g(x)$, when these limits exist. For $f \in \mathscr{D}_W(L)$, we claim that

$$f'(0+) = f'_+(0) \quad \text{and} \quad f'(0-) = f'_-(0).$$

Indeed, let $f \in \mathscr{D}_W(L)$. We have seen that f is twice differentiable on $\mathbb{R} \setminus \{0\}$, and the second derivative f'' extends to an element of $C_0^\Delta(\mathbb{R})$, in particular f'' is bounded. Thus, for any sequence $c_n \downarrow 0$, we have

$$|f'(c_n) - f'(c_m)| = \left|\int_{c_m}^{c_n} f''(x)\,dx\right| \le \|f''\| |c_n - c_m| \longrightarrow 0,$$

for n, m large enough. This proves that the limit $\lim_{c\to 0^+} f'(c)$ exists.

Now, for any $h > 0$, we have that f is continuous on $[0, h]$ and differentiable on $(0, h)$, so using the Mean Value Theorem, we can found a $c = c(h) \in (0, h)$ such that

$$f'(c) = \frac{f(h) - f(0)}{h}.$$

In particular, since $c(h) \to 0$ as $h \to 0$, it follows that

$$f'_+(0) = \lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{c\to 0^+} f'(c) = f'(0+), \tag{4.8}$$

as desired. The other limit can be computed similarly.

We will deduce the expression for the domain $\mathscr{D}_W(L)$ of a general Brownian motion on $\mathbb{R}$ from a expression in terms of some measures on $(-\infty, 0)$ and $(0, \infty)$. We start with the following lemma.

Lemma 4.5. *There exist constants* $c_1, c_2^-, c_2^+, c_3 \geq 0$ *and measures* $\nu^-(dx)$ *and* $\nu^+(dx)$ *on* $(-\infty, 0)$ *and* $(0,\infty)$*, respectively, satisfying*

$$c_1 + c_2^- + c_2^+ + c_3 + \int_{-\infty}^0 (1 \wedge -x)\, \nu^-(dx) + \int_0^\infty (1 \wedge x)\, \nu^+(dx) = 1, \tag{4.9}$$

so that f *belongs to* $\mathscr{D}_W(L)$ *if and only if:* f *belongs to* $C_0^\Delta(\mathbb{R})$*, it is twice differentiable in* $\mathbb{R} \setminus \{0\}$ *with second derivative extending to an element of* $C_0^\Delta(\mathbb{R})$*, and the equation*

$$\begin{aligned} &c_1 f(0) + c_2^- f'(0-) - c_2^+ f'(0+) + \frac{c_3}{2} f''(0+) \\ &= \int_{(-\infty,0)} (f(x) - f(0))\, \nu^-(dx) + \int_{(0,\infty)} (f(x) - f(0))\, \nu^+(dx) \end{aligned} \tag{4.10}$$

holds.

To prove the previous lemma we need the next result concerning weak convergence of measures.

Proposition 4.6. *Let* $(\mu_\varepsilon)_{\varepsilon>0}$ *be a family of measures on* $(0,\infty)$ *such that* $\mu_\varepsilon((0,\infty)) \leq 1$ *for all* $\varepsilon > 0$*, and let* $C[0,\infty]$ *be the space of continuous functions on* $[0,\infty]$*. Then, there exist a subsequence* $\varepsilon_n \searrow 0$ *and a measure* μ *on* $[0,\infty]$ *such that*

$$\lim_{n\to\infty} \int_{(0,\infty)} f(x)\, \mu_{\varepsilon_n}(dx) = \int_{[0,\infty]} f(x)\, \mu(dx), \tag{4.11}$$

for every $f \in C[0,\infty]$.

Proof of Proposition 4.6. Considering Alexandroff compactification, $[0,\infty]$ is compact, so the space $C[0,\infty]$ has a countable dense subset in the supremum norm, which we denote by $\{f_k\}_{k=1}^\infty$. We will show first that there exists a subsequence $\varepsilon_n \searrow 0$ such that the limit

$$\lim_{n\to\infty} \int_{(0,\infty)} f_k(x)\, \mu_{\varepsilon_n}(dx)$$

exists, for all $k \in \mathbb{N}$. We will proceed via a diagonal argument. Letting $k = 1$, we have

$$\int_{(0,\infty)} f_1(x)\, \mu_\varepsilon(dx) \leq \|f_1\| \mu_\varepsilon((0,\infty)) \leq \|f_1\|,$$

for all $\varepsilon > 0$. Thus, we can find a sequence $\varepsilon_n^1 \searrow 0$ such that $\int_{(0,\infty)} f_1(x)\, \mu_{\varepsilon_n^1}(dx)$ converges as $n \to \infty$. Also, for $k = 2$, we have

$$\int_{(0,\infty)} f_2(x)\, \mu_{\varepsilon_n^1}(dx) \leq \|f_2\|,$$

for all $\varepsilon_n^1 > 0$, then we can extract a subsequence (ε_n^2) of (ε_n^1) such that $\int_0^\infty f_2(x)\, \mu_{\varepsilon_n^2}(dx)$ converges as $n \to \infty$ with $\varepsilon_n^2 \searrow 0$. Applying this successively, we will have found for each k a sequence $(\varepsilon_n^k) \subseteq (\varepsilon_n^{k-1}) \searrow 0$ such that $\int_0^\infty f_k(x)\, \mu_{\varepsilon_n^k}(dx)$ converges as $n \to \infty$.

Thus, fixing k and taking the diagonal sequence $\varepsilon_n = \varepsilon_n^n$, we obtain

$$\left| \int_{(0,\infty)} f_k(x)\, \mu_{\varepsilon_n^n}(dx) - \int_{(0,\infty)} f_k(x)\, \mu_{\varepsilon_n^k}(dx) \right| \longrightarrow 0,$$

as $n \to \infty$, since for n sufficiently large, we have $(\varepsilon_n^n) \subseteq (\varepsilon_n^k)$ and the second integral converges. This completes the diagonal argument.

Next, we can extend this limit for all $f \in C[0,\infty]$ using the density of $\{f_k\}_{k=1}^{\infty}$. In fact, for every function $f \in C[0,\infty]$, there exists a subsequence $(f_{k_n})_n$ of $\{f_k\}_{k=1}^{\infty}$ such that $f_{k_n} \longrightarrow f$ as $n \to \infty$ in the supremum norm. Then, for all $\varepsilon > 0$, we can choose n large enough such that $\|f_{k_n} - f\|_\infty < \varepsilon$. Hence,

$$\left| \int_{(0,\infty)} f(x)\, \mu_{\varepsilon_n}(dx) - \int_{(0,\infty)} f_{k_n}(x)\, \mu_{\varepsilon_n}(dx) \right| \le \|f_{k_n} - f\|_\infty < \varepsilon,$$

and the limit $\lim_{n\to\infty} \int_0^\infty f(x)\, \mu_{\varepsilon_n}(dx)$ exists for every $f \in C[0,\infty]$, for the same sequence $\varepsilon_n \searrow 0$.

Now, we may define the operator $C[0,\infty] \ni f \mapsto \lim_{n\to\infty} \int_0^\infty f(x)\, \mu_{\varepsilon_n}(dx)$, which is positive linear. Also, since $C[0,\infty]$ consists of bounded continuous functions on a compact space, Riesz Representation Theorem ensures that there exists a measure $\mu(dx)$ on $[0,\infty]$ such that

$$\lim_{n\to\infty} \int_{(0,\infty)} f(x)\, \mu_{\varepsilon_n}(dx) = \int_{[0,\infty]} f(x)\, \mu(dx),$$

for all $f \in C[0,\infty]$, which completes the proof. □

Remark 5. *Note that the same result holds when considering a family $(\mu_\varepsilon)_{\varepsilon>0}$ of measures on $(-\infty,0)$ with $\mu_\varepsilon((-\infty,0)) \le 1$, for all $\varepsilon > 0$, and the space $C[-\infty,0]$.*

Now, we come back to the proof of the Lemma 4.5.

Proof of Lemma 4.5. First, recall that $T = \inf\{t > 0 : W_t \neq 0\}$ is exponentially distributed with parameter $\lambda \in [0,\infty]$ under P_0. We will split the proof into three cases: $\lambda = 0, 0 < \lambda < \infty$, and $\lambda = \infty$.

Case $\lambda = 0$:

Here, $T = \infty$ a.s. and 0 is a trap, so the process W_t must coincide with the absorbed Brownian motion at 0. Also, since 0 is absorbing, we have

$$Lf(0) = \lim_{t\to 0^+} \frac{E_0[f(W_t)] - f(0)}{t} = 0,$$

for any $f \in \mathscr{D}_W(L)$. Thus, $f''(0-) = f''(0+) = 0$, and choosing $c_1 = c_2^- = c_2^+ = 0$, $c_3 = 1$, and $\nu^- = \nu^+ = 0$, the lemma holds trivially.

Case $0 < \lambda < \infty$:

We will show that $\frac{1}{2}f''(0+) = -\lambda f(0)$ is the boundary condition for $f \in \mathscr{D}_W(L)$. Recall from Proposition 3.17 that for exponential holding points, the process leaves 0 jumping to the cemetery, so for $f \in C_0^\Delta(\mathbb{R})$, we get

$$\begin{aligned} E_0[f(W_t)] &= E_0[f(W_t)\mathbf{1}_{\{T<t\}}] + E_0[f(W_t)\mathbf{1}_{\{T>t\}}] \\ &= E_0[f(\Delta)\mathbf{1}_{\{T<t\}}] + E_0[f(0)\mathbf{1}_{\{T>t\}}] \\ &= f(0)P_0(T > t) \\ &= f(0)\exp(-\lambda t). \end{aligned}$$

Thus, for $f \in \mathscr{D}_W(L)$,

$$Lf(0) = \lim_{t\to 0+} \frac{E_0[f(W_t)] - f(0)}{t} = f(0) \lim_{t\to 0+} \frac{\exp(-\lambda t) - 1}{t} = -\lambda f(0),$$

which implies that $\frac{1}{2}f''(0+) = -\lambda f(0)$, as desired. With this boundary condition, it is enough to choose $c_1 = \frac{\lambda}{1+\lambda}, c_3 = \frac{1}{1+\lambda}, c_2^- = c_2^+ = 0$, and $\nu^- = \nu^+ = 0$ to obtain the result.

Case $\lambda = \infty$.

In this situation, $T = 0$ a.s., so the process leaves 0 at once, then 0 is not a trap, so we can compute the generator at $x = 0$ via the Dynkin formula (see Theorem 3.18). Thus, let $A = (-\varepsilon, \varepsilon)$ with $\varepsilon \searrow 0$. Starting from 0, W exits A at $-\varepsilon, \varepsilon$ or Δ. Setting $T_\varepsilon = \tau_{-\varepsilon} \wedge \tau_\varepsilon \wedge \tau_\Delta$, the Dynkin formula becomes

$$Lf(0) = \lim_{\varepsilon\searrow 0} \frac{E_0[f(W_{T_\varepsilon})] - f(0)}{E_0[T_\varepsilon]},$$

for $f \in \mathscr{D}_W(L)$. Note that

$$\begin{aligned} \frac{E_0[f(W_{T_\varepsilon})]}{E_0[T_\varepsilon]} &= \int \frac{1}{E_0[T_\varepsilon]} f(W_{T_\varepsilon})\, dP_0 \\ &= \int_{\mathbb{R}\cup\Delta} f(x)\, \frac{1}{E_0[T_\varepsilon]} P_0(W_{T_\varepsilon} \in dx) = \int_{\mathbb{R}} f(x)\, \frac{1}{E_0[T_\varepsilon]} P_0(W_{T_\varepsilon} \in dx), \end{aligned}$$

where in the last equality we used that f vanishes at Δ.

Consider the finite measure $\nu_\varepsilon(dx) = \frac{1}{E_0[T_\varepsilon]} P_0(W_{T_\varepsilon} \in dx)$ on $\mathbb{R} \cup \{\Delta\}$. Noticing that $\nu_\varepsilon(R \cup \{\Delta\}) = \frac{1}{E_0[T_\varepsilon]}$, we can express the generator in terms of ν_ε via

$$Lf(0) = \lim_{\varepsilon\searrow 0} \left\{ \left[\int_{\mathbb{R}} (f(x) - f(0))\, \nu_\varepsilon(dx) \right] - f(0)\nu_\varepsilon(\Delta) \right\}. \tag{4.12}$$

In order to obtain our normalized constants, let

$$K_\varepsilon = 1 + \nu_\varepsilon(\Delta) + \int_{-\infty}^{0} (1 \wedge -x)\, \nu_\varepsilon(dx) + \int_{0}^{\infty} (1 \wedge x)\, \nu_\varepsilon(dx).$$

Note that $1 \le K_\varepsilon < \infty$, so (4.12) implies

$$\lim_{\varepsilon \searrow 0} \frac{\nu_\varepsilon(\Delta)}{K_\varepsilon} f(0) + \frac{Lf(0)}{K_\varepsilon} - \int_{\mathbb{R}} (f(x) - f(0)) \frac{\nu_\varepsilon(dx)}{K_\varepsilon} = 0. \tag{4.13}$$

To construct our desired measures, we now investigate some weak measure limits (in the sense of (4.11)). Consider the measures

$$\mu_\varepsilon^-(A) = \int_A (1 \wedge -x) \frac{\nu_\varepsilon(dx)}{K_\varepsilon} \text{ on } (-\infty, 0) \qquad \text{and}$$
$$\mu_\varepsilon^+(A) = \int_A (1 \wedge x) \frac{\nu_\varepsilon(dx)}{K_\varepsilon} \text{ on } (0, \infty).$$

Observe that $\mu_\varepsilon^+(dx) < 1$ for all $\varepsilon > 0$, so Proposition 4.6 assures that there is a measure $\mu^+(dx)$ on $[0, \infty]$ and a subsequence $\varepsilon_n \searrow 0$ such that

$$\lim_{n\to\infty} \int_{(0,\infty)} f(x)\, \mu_{\varepsilon_n}^+(dx) = \int_{[0,\infty]} f(x)\, \mu^+(dx), \tag{4.14}$$

for all $f \in C[0, \infty]$. In view of Remark 5, applying the same argument to $\mu_{\varepsilon_n}^-$ and passing to a further subsequence if necessary, we get a measure $\mu^-(dx)$ on $[-\infty, 0]$ such that

$$\lim_{n\to\infty} \int_{(-\infty,0)} f(x)\, \mu_{\varepsilon_n}^-(dx) = \int_{[-\infty,0]} f(x)\, \mu^-(dx),$$

for every $f \in C[-\infty, 0]$.

We are ready now to return to the equation (4.13). Note that $0 \le \frac{\nu_{\varepsilon_n}(\Delta)}{K_{\varepsilon_n}} \le 1$ and $0 < \frac{1}{K_{\varepsilon_n}} \le 1$, for all $n \in \mathbb{N}$, so passing to a subsequence once more if required, we may assume

$$\lim_{n\to\infty} \frac{\nu_{\varepsilon_n}(\Delta)}{K_{\varepsilon_n}} = p_1 \text{ and } \lim_{n\to\infty} \frac{1}{K_{\varepsilon_n}} = p_2,$$

with $0 \le p_1, p_2 \le 1$.

Let $f \in \mathscr{D}_W(L)$. Since $f \in C_0^\Delta(\mathbb{R})$, we have that the function $\frac{f(x)-f(0)}{1\wedge x}$ is continuous on $(0, \infty)$ and $\lim_{x\to\infty} \frac{f(x)-f(0)}{1\wedge x} = -f(0) < \infty$. Also, we know that f is differentiable, so $\lim_{x\to 0^+} \frac{f(x)-f(0)}{1\wedge x} = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = f'(0+) < \infty$. In that way, we can extend $\frac{f(x)-f(0)}{1\wedge x}$ continuously to $[0, \infty]$.

Thus, using equation (4.14) with the function $\frac{f(x)-f(0)}{1\wedge x} \in C[0, \infty]$ and observing that $1 \wedge x$ is the Radon-Nikodym derivative of $\mu_{\varepsilon_n}^+(dx)$ with respect to $\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx)$, we get

$$\begin{aligned}
\int_{[0,\infty]} \frac{f(x) - f(0)}{1 \wedge x} \mu^+(dx) &= \lim_{n\to\infty} \int_{(0,\infty)} \frac{f(x) - f(0)}{1 \wedge x} \mu_{\varepsilon_n}^+(dx) \\
&= \lim_{n\to\infty} \int_{(0,\infty)} \frac{f(x) - f(0)}{1 \wedge x} (1 \wedge x) \frac{1}{K_{\varepsilon_n}} \nu_{\varepsilon_n}(dx) \\
&= \lim_{n\to\infty} \int_{(0,\infty)} (f(x) - f(0)) \frac{1}{K_{\varepsilon_n}} \nu_{\varepsilon_n}(dx) \\
&= \lim_{n\to\infty} \int_{[0,\infty)} (f(x) - f(0)) \frac{1}{K_{\varepsilon_n}} \nu_{\varepsilon_n}(dx),
\end{aligned}$$

where in the last equation we simply used that $f(x) - f(0)$ vanishes at 0, recalling that $\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx)$ is a measure on $\mathbb{R} \cup \Delta$.

Similarly, $\frac{f(x)-f(0)}{1\wedge -x}$ is continuous on $(-\infty, 0)$, $\lim_{x\to\infty} \frac{f(x)-f(0)}{1\wedge -x} = -f(0) < \infty$, and $\lim_{x\to 0^-} \frac{f(x)-f(0)}{1\wedge -x} = \lim_{x\to 0^-} \frac{f(x)-f(0)}{-x} = -f'(0-) < \infty$. Thus,

$$\int_{[-\infty,0]} \frac{f(x)-f(0)}{1\wedge -x}\,\mu^-(dx) = \lim_{n\to\infty} \int_{(-\infty,0)} (f(x)-f(0))\,\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx).$$

Substituting these limits into equation (4.13), we obtain

$$p_1 f(0) + p_2 Lf(0) - \int_{[-\infty,0]} \frac{f(x)-f(0)}{1\wedge -x}\,\mu^-(dx) - \int_{[0,\infty]} \frac{f(x)-f(0)}{1\wedge x}\,\mu^+(dx) = 0.$$

Also, considering the limits of $\frac{f(x)-f(0)}{1\wedge -x}$ and $\frac{f(x)-f(0)}{1\wedge x}$, we get

$$\begin{aligned}&(p_1 + \mu^-(-\infty) + \mu^+(\infty))f(0) + \mu^-(0)f'(0-) - \mu^+(0)f'(0+) + p_2 Lf(0)\\ &= \int_{(-\infty,0)} \frac{f(x)-f(0)}{1\wedge -x}\,\mu^-(dx) + \int_{(0,\infty)} \frac{f(x)-f(0)}{1\wedge x}\,\mu^+(dx).\end{aligned}$$

Defining the measures $\nu^-(A) = \int_A \frac{1}{1\wedge -x}\mu^-(dx)$ on $(-\infty, 0)$, $\nu^+(A) = \int_A \frac{1}{1\wedge x}\mu^+(dx)$ on $(0,\infty)$, and setting $c_1 = p_1 + \mu^-(-\infty) + \mu^+(\infty)$, $c_2^- = \mu^-(0)$, $c_2^+ = \mu^+(0)$, $c_3 = p_2$ yields

$$\begin{aligned}&c_1 f(0) + c_2^- f'(0-) - c_2^+ f'(0+) + \frac{c_3}{2} f''(0+)\\ &= \int_{(-\infty,0)} (f(x)-f(0))\,\nu^-(dx) + \int_{(0,\infty)} (f(x)-f(0))\,\nu^+(dx),\end{aligned}$$

which is our result missing only to check (4.9). To see this, note that integrating against the constant function equals to 1, we have $\mu^-([-\infty,0]) = \lim_{n\to\infty}\mu^-_{\varepsilon_n}((-\infty,0))$ and $\mu^+([0,\infty]) = \lim_{n\to\infty}\mu^+_{\varepsilon_n}((0,\infty))$. Thus,

$$\begin{aligned}&p_1 + p_2 + \mu^-([-\infty,0]) + \mu^+([0,\infty])\\ &= \lim_{n\to\infty} \frac{\nu_{\varepsilon_n}(\Delta)}{K_{\varepsilon_n}} + \frac{1}{K_{\varepsilon_n}} + \mu^-_{\varepsilon_n}((-\infty,0)) + \mu^+_{\varepsilon_n}((0,\infty))\\ &= \lim_{n\to\infty} \frac{\nu_{\varepsilon_n}(\Delta) + 1 + \int_{-\infty}^0 (1\wedge -x)\,\nu_{\varepsilon_n}(dx) + \int_0^\infty (1\wedge x)\,\nu_{\varepsilon_n}(dx)}{K_{\varepsilon_n}} = 1,\end{aligned}$$

and we finally get

$$\begin{aligned}&c_1 + c_2^- + c_2^+ + c_3 + \int_{-\infty}^0 (1\wedge -x)\,\nu^-(dx) \int_0^\infty (1\wedge x)\,\nu^+(dx)\\ &= c_1 + c_2^- + c_2^+ + c_3 + \mu^-((-\infty,0)) + \mu^+((0,\infty))\\ &= p_1 + p_2 + \mu^-([-\infty,0]) + \mu^+([0,\infty]) = 1.\end{aligned}$$

Now, we will show the converse. Let $\mathscr{D}'$ be the set of functions satisfying equation (4.10) for f belonging to $C_0^\Delta(\mathbb{R})$ and having second derivatives in $\mathbb{R}\setminus\{0\}$ which admit an extension to an element of $C_0^\Delta(\mathbb{R})$. We will show that $\mathscr{D}' \subseteq \mathscr{D}_W(L)$. Indeed, consider the following equation on $\mathbb{R}\setminus\{0\}$

$$\beta f - \frac{1}{2}f'' = g, \tag{4.15}$$

for $\beta > 0$ and $g \in C_0(\mathbb{R})$. This equation has a solution in $\mathscr{D}_W(L)$. In fact, by Proposition 3.8 we have that $R_\beta g \in \mathscr{D}_W(L)$ for $g \in C_0(\mathbb{R})$. Also, the operator R_β is the inverse of $\beta - L$. Thus, taking $f = R_\beta g$, we have

$$\beta f - \frac{1}{2}f'' = (\beta - L)f = g$$

on $\mathbb{R}\setminus\{0\}$, as desired.

Now, let $f \in \mathscr{D}'$. Observing that $f \in C_0(\mathbb{R})$, $f'' \in C_0(-\infty,0), C_0(0,\infty)$ and $f''(0+) = f''(0-)$, there will exist a function $g \in C_0(\mathbb{R})$ such that (4.15) holds for that f. Assuming that f does not belong to $\mathscr{D}_W(L)$, since we proved that $\mathscr{D}_W(L) \subseteq \mathscr{D}'$, there will exist two functions $f_1, f_2 \in \mathscr{D}'$ satisfying the same equation. Thus,

$$\beta(f_1 - f_2) - \frac{1}{2}(f_1'' - f_2'') = 0, \tag{4.16}$$

for $x \neq 0$. But we know that the solution for (4.16) is given by

$$f_1(x) - f_2(x) = d_1 \exp(-\sqrt{2\beta}x) + d_2 \exp(\sqrt{2\beta}x),$$

for some constants $d_1, d_2 \in \mathbb{R}$. However, $f_1 - f_2$ also goes to 0 at infinity, then $d_1 = 0$ for $x < 0$ and $d_2 = 0$ for $x > 0$. Thus,

$$f_1(x) - f_2(x) = \begin{cases} d_2 \exp(-\sqrt{2\beta}x), & \text{if } x > 0; \\ d_1 \exp(\sqrt{2\beta}x), & \text{if } x < 0. \end{cases}$$

Moreover, the continuity of f_1, f_2 forces $d_1 = d_2 = d$ and $f_1(0) - f_2(0) = d$. Also, $d \neq 0$ because f_1, f_2 are supposed to be different.

Since $f_1, f_2 \in \mathscr{D}'$, we get

$$\begin{aligned} &c_1(f_1(0) - f_2(0)) + c_2^-(f_1'(0-) - f_2'(0-)) - c_2^+(f_1'(0+) - f_2'(0+)) + \frac{1}{2}c_3(f_1''(0+) - f_2''(0+)) \\ &= \int_{-\infty}^0 (f_1(x) - f_2(x)) - (f_1(0) - f_2(0))\, \nu^-(dx) \\ &\qquad + \int_0^\infty (f_1(x) - f_2(x)) - (f_1(0) - f_2(0))\, \nu^+(dx). \end{aligned}$$

Computing the limits at 0 for the side derivatives of $f_1 - f_2$ and dividing both sides by d, we obtain

$$\begin{aligned} &c_1 + \sqrt{2\beta}(c_2^- + c_2^+) + c_3\beta \\ &= \int_{-\infty}^0 (\exp(\sqrt{2\beta}x) - 1)\, \nu^-(dx) + \int_0^\infty (\exp(-\sqrt{2\beta}x) - 1)\, \nu^+(dx), \end{aligned}$$

which is a contradiction, since both integrands are non positive whereas $c_1, c_2^-, c_2^+, c_3 \geq 0$. Moreover, these constants and measures can not be 0 at the same time, otherwise the constants would have to sum 1. Therefore, $f \in \mathscr{D}_W(L)$ and we can conclude. $\square$

We are now ready to determine the domain of the infinitesimal generator for any general Brownian motion on $\mathbb{R}$.

Theorem 4.7. *For every general Brownian motion W on $\mathbb{R}$ with boundary conditions at the origin, its infinitesimal generator is given by*

$$Lf(x) = \frac{1}{2}f''(x) \text{ for } x \neq 0 \text{ and } Lf(0) = \frac{1}{2}f''(0-) = \frac{1}{2}f''(0+),$$

with domain

$$\mathscr{D}_W(L) = \{f \in C_0^{\Delta}(\mathbb{R}) : Lf \in C_0^{\Delta}(\mathbb{R}),\ c_1 f(0) + c_2^- f'(0-) - c_2^+ f'(0+) + \frac{1}{2}c_3 f''(0+) = 0\},$$

for some constants $c_1, c_2^-, c_2^+, c_3 \geq 0$ with $c_1 + c_2^- + c_2^+ + c_3 = 1$, and $c_1 \neq 1$.

Remark 6. *We observe that $c_1 \neq 1$, otherwise the domain's equation simplifies to $f(0) = 0$, but this set of functions does not define a dense subset, so it cannot be a domain of a generator.*

Proof. By Lemma 4.5, it suffices to prove that $\nu^- = \nu^+ = 0$ what we really obtained for $0 \leq \lambda < \infty$. Arguing by contradiction, suppose that $\nu^-(-\infty, -\varepsilon), \nu^+(\varepsilon, \infty) > 0$ for some $\varepsilon > 0$. Then, choosing any $f_1 \in \mathscr{D}_W(L)$, we can slightly modify it to obtain $f_2 = f_1$ on $[-\varepsilon, \varepsilon]$, but $f_2(x) < f_1(x)$ for $|x| > \varepsilon$ so that f_2 still belongs to $C_0^{\Delta}(\mathbb{R})$, f_2'' exists in $\mathbb{R} \setminus \{0\}$ and can be extended to an element of $C_0^{\Delta}(\mathbb{R})$. Thus, according to Lemma 4.4, we have $f_2 \in \mathscr{D}_W(L)$. However, since f_1, f_2 and its derivatives coincide at 0, Lemma 4.5 yields

$$\int_{(-\infty,0)} (f_1(x) - f_2(x))\, \nu^-(dx) + \int_{(0,\infty)} (f_1(x) - f_2(x))\, \nu^+(dx) = 0$$
$$\Rightarrow \int_{(-\infty,-\varepsilon)} (f_1(x) - f_2(x))\, \nu^-(dx) = -\int_{(\varepsilon,\infty)} (f_1(x) - f_2(x))\, \nu^+(dx).$$

Since both integrand are positive, this implies that

$$\int_{(-\infty,0)} (f_1(x) - f_2(x))\, \nu^-(dx) = \int_{(0,\infty)} (f_1(x) - f_2(x))\, \nu^+(dx) = 0$$
$$\Rightarrow (f_1 - f_2)\mathbf{1}_{\{(-\infty,\varepsilon)\}} = 0 \ \nu^-\text{-a.e. and } (f_1 - f_2)\mathbf{1}_{\{(\varepsilon,\infty)\}} = 0 \ \nu^+\text{-a.e.},$$

which is not possible since both $f_1 - f_2$ and ν^- are positive on $(-\infty, \varepsilon)$, and both $f_1 - f_2$ and ν^+ are positive on (ε, ∞).

Therefore, $\nu^-(-\infty, -\varepsilon) = \nu^+(\varepsilon, \infty) = 0$, for all $\varepsilon > 0$. Letting $\varepsilon \searrow 0$ we get $\nu^-(-\infty, 0) = \nu^+(0, \infty) = 0$, and the theorem is proved. $\square$

Consider a general Brownian motion on $\mathbb{R}$ such that its constants c_1, c_2^-, c_2^+, c_3 are positive. Dividing the domain's equation by $c_2^- + c_2^+$, we get

$$\frac{c_1}{c_2^- + c_2^+} f(0) + \frac{c_2^-}{c_2^- + c_2^+} f'(0-) - \frac{c_2^+}{c_2^- + c_2^+} f'(0+) + \frac{c_3}{2(c_2^- + c_2^+)} f''(0+) = 0\,.$$

Setting $\gamma = \frac{c_1}{c_2^- + c_2^+}$, $\beta = \frac{c_2^+}{c_2^- + c_2^+}$, $c = \frac{c_3}{2(c_2^- + c_2^+)}$, we have

$$cf''(0+) = \beta f'(0+) - (1-\beta) f'(0-) - \gamma f(0),$$

which corresponds exactly to the domain's boundary condition of the Brownian Motion skew at 0, sticky at 0 and killed elastically at 0, given without proof in Borodin's book [1, page 127, Section 13, Appendix 1]. Hence, we have proved that the general Brownian motion on $\mathbb{R}$ with boundary conditions at the origin coincides with a Skew Sticky Killed Brownian Motion at 0.

Chapter 5

The most general BM on $(-\infty, 0-] \cup [0+, \infty)$

We will now consider our state space $E = (-\infty, 0-] \cup [0+, \infty)$. That is, we split the line $\mathbb{R}$ into two disjoint half-lines, the positive and the negative one, with 0 corresponding to both $0-$ and $0+$, but seen as distinct points. Also, each half-line has the topology of the usual one. As in the previous chapter, the set $C_0(E)$ coincides with the subspace of continuous functions $f : E \to \mathbb{R}$ having zero limit as $x \to \pm\infty$. Also, we will add to E the *cemetery* Δ, and consider $C_0^{\Delta}(E)$ the set of functions in $C_0(E)$ with $f(\Delta) = 0$. Note that Δ is an isolated point.

Definition 5.1. *A stochastic process $(W_t)_{t\geq 0}$ on $E = (-\infty, 0-] \cup [0+, \infty)$ is called a general Brownian motion on E with boundary conditions at the origin if it satisfies the following properties:*

- *$(W_t)_{t\geq 0}$ is a strong Markov process with values in $E \cup \{\Delta\}$ and it has càdlàg trajectories.*
- *The sample paths of $(W_t)_{t\geq 0}$ are continuous on the set $\{t \geq 0 : \lim_{s\to t^-} W_s$ or $W_t \notin \{0+, 0-, \Delta\}\}$.*
- *The point Δ, called the cemetery, is an absorbing state.*
- *Let $\tau_{0+} = \inf\{t \geq 0 : W_t = 0+\}$ be the hitting time of $0+$. For every initial point $x \in [0+, \infty)$, the law of the process $(W_{t\wedge\tau_{0+}})_t$ coincides with the law of a standard Brownian motion on $[0, \infty)$ absorbed at 0, where we are identifying $0+$ with 0. Similarly, for every starting point $x \in (-\infty, 0-]$, the law of $(W_{t\wedge\tau_{0-}})_t$ coincides with that of a Brownian motion on $(-\infty, 0]$ absorbed at 0, where we are identifying $0-$ with 0 and $\tau_{0-} = \inf\{t \geq 0 : W_t = 0-\}$ is the hitting time of $0-$.*

Since each half-line $(-\infty, 0-]$ and $[0+, \infty)$ can be identified with the standard negative and positive half-lines, respectively, some results from Chapter 4 apply directly to this setting. We will state them now.

First, the hitting time of 0 for Brownian motion is finite almost surely, yet Δ is an absorbing point, then W_t cannot reach Δ before $0-$ or $0+$. Also, the Proposition 3.15 says us that $T_+ = \inf\{t > 0 : W_t \neq 0+\}$ and $T_- = \inf\{t > 0 : W_t \neq 0+\}$ are exponentially distributed under the probabilities P_{0+}, P_{0-}, with parameters λ_+ and λ_-, respectively. We distinguish the following cases depending on the value of λ_+. The situation for λ_- is analogous.

- **Case 1:** $0 < \lambda_+ < \infty$

  In this case, by Proposition 3.17, the process leaves $0+$ via a jump, and the second condition in Definition 5.1 enables W only to jump to $0-$ or to Δ. Hence, at time T_+, the process is killed or arises on the opposite half-line.

- **Case 2:** $\lambda_+ = \infty$

  Here, $P_{0+}(T_+ = 0) = 1$ and the process leaves $0+$ at once. Since the process starts at $0+$ under P_{0+} and the process is càdlàg, thus right continuous, the process cannot go immediately from $0+$ to Δ or $0-$, since this would be a jump and the process would be continuous to the left.

- **Case 3:** $\lambda_+ = 0$

  Here, we have $T_+ = \infty$ a.s. and $0+$ is an absorbing point.

Moreover, by the observations before Proposition 4.2, we can give the same proof to show that the general Brownian motion on E is a Feller process (considering for example $x \in [0, \infty)$ and checking if the process has already left $[0, \infty)$ or not). The reader could also notice that Proposition 4.3 was proved considering $x \in [0, \infty)$ or $x \in (-\infty, 0]$ separately and was based mostly in analytic arguments, so it adapts easily to Brownian motion on the state space $E = (-\infty, 0-] \cup [0+, \infty)$. Indeed, in this case, we can say even more.

Proposition 5.2. *Let $f \in \mathscr{D}_W(L)$, the domain of a general Brownian motion on E with boundary conditions at the origin. Then, $f'' \in C_0^{\Delta}(E)$ and infinitesimal generator values*

$$Lf(x) = \frac{1}{2} f''(x), \tag{5.1}$$

for $x \in E$. Also, $Lf(\Delta) = 0$.

Proof. In view of our previous observations, it remains only to prove that

$$Lf(0+) = \frac{1}{2}f''(0+), \quad Lf(0-) = \frac{1}{2}f''(0-)$$

for $f \in \mathscr{D}_W(L)$, where the second and first derivatives at $0+$ and $0-$ are interpreted as the right-hand derivative and left-hand derivatives, respectively.

To prove that, we give a completely analytic argument. First, we identify $[0+, \infty]$ with the interval $[0, \infty]$, and apply the mean value theorem to obtain

$$f'_+(0) = \lim_{h \searrow 0} \frac{f(h) - f(0)}{h} = \lim_{c \searrow 0} f'(c),$$

as was done in the context of (4.8) in Chapter 4.

Returning to the notation $[0+, \infty]$, we have

$$f'(0+) = \lim_{h \searrow 0+} \frac{f(h) - f(0+)}{h} = \lim_{c \searrow 0+} f'(c),$$

which shows that f' is continuous at $0+$.

We now apply the same reasoning again. We have that f' is continuous on $[0, h]$ and differentiable on $(0, h)$, so we can find $c = c(h) \in (0, h)$ such that

$$f''(c) = \frac{f'(h) - f'(0)}{h}.$$

We already know that $\lim_{c \searrow 0} f''(c) = 2Lf(0)$, so we deduce that

$$f''_+(0) = \lim_{h \searrow 0} \frac{f'(h) - f'(0)}{h} = 2Lf(0),$$

that is,

$$f''(0+) = \lim_{h \searrow 0+} \frac{f'(h) - f'(0+)}{h} = 2Lf(0+).$$

An identical argument applied to the interval $(-\infty, 0-]$ shows similarly that

$$f''(0-) = 2Lf(0-).$$

□

We now set up that for a function to belong to the domain $\mathscr{D}_W(L)$, it is sufficient to know its behavior in a neighborhood of $0+$ and $0-$. The proof is almost identical to that of Lemma 4.4, and is therefore omitted.

Lemma 5.3. *If $f_1 \in \mathscr{D}_W(L)$, $f_2, f''_2 \in C_0^\Delta(E)$, and $f_1 = f_2$ in $(-\varepsilon, 0-] \cup [0+, \varepsilon)$, for some $\varepsilon > 0$, then $f_2 \in \mathscr{D}_W(L)$.*

To obtain the domain's boundary conditions at $0+$ and $0-$, we proceed similarly as we did for the general Brownian Motion on $\mathbb{R}$, expressing them first in terms of measures on $(-\infty, 0-]$ and $[0+, \infty)$. The proof is close to the one for general Brownian motion on $\mathbb{R}$, but we do some suitable adaptations.

Lemma 5.4. *There exists nonnegative constants* a^+, a^-, c_i^+, c_i^-, $i = 1, 2, 3$ *and measures* ν_j^+ *on* $(0+, \infty)$, ν_j^- *on* $(-\infty, 0-)$, $j = 1, 2$ *for which*

$$c_1^+ + a^+ + c_2^+ + c_3^+ + \int_{(0+,\infty)} (1 \wedge x)\, \nu_1^+(dx) + \nu_1^-((-\infty, 0-)) = 1, \tag{5.2}$$

$$c_1^- + a^- + c_2^- + c_3^- + \int_{(-\infty,0-)} (1 \wedge -x)\, \nu_2^-(dx) + \nu_2^+((0+, \infty)) = 1, \tag{5.3}$$

such that f *belongs to* $\mathscr{D}_W(L)$ *if and only if* $f, f'' \in C_0^\Delta(E)$, *and it satisfies*

$$\begin{aligned} &c_1^+ f(0+) + a^+(f(0+) - f(0-)) - c_2^+ f'(0+) + \frac{c_3^+}{2} f''(0+) \\ &= \int_{(0+,\infty)} (f(x) - f(0+))\, \nu_1^+(dx) + \int_{(-\infty,0-)} (f(x) - f(0+))\, \nu_1^-(dx), \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &c_1^- f(0-) + a^-(f(0-) - f(0+)) + c_2^- f'(0-) + \frac{c_3^-}{2} f''(0-) \\ &= \int_{(0+,\infty)} (f(x) - f(0-))\, \nu_2^+(dx) + \int_{(-\infty,0-)} (f(x) - f(0-))\, \nu_2^-(dx). \end{aligned} \tag{5.5}$$

Proof. Recall from Proposition 3.15 that $T_+ = \inf\{t > 0 : W_t \neq 0+\}$ and $T_- = \inf\{t > 0 : W_t \neq 0-\}$ are exponentially distributed with parameters λ_+ and λ_-, under the probabilities P_{0+}, P_{0-}, respectively. We examine the following cases, supposing first that $f \in \mathscr{D}_W(L)$.

Case $\lambda_+ = 0$ or $\lambda_- = 0$.

If $\lambda_+ = 0$, then $0+$ is an absorbing point, so $f''(0+) = Lf(0+) = 0$, and we may choose $c_1^+ = a^+ = c_2^+ = 0$, $c_3^+ = 1$, and $\nu_1^- = \nu_1^+ = 0$. The case $\lambda_- = 0$ is analogous.

Case $0 < \lambda_+ < \infty$ or $0 < \lambda_- < \infty$:

Again we will consider only $0 < \lambda_+ < \infty$. In this case, the process waits an exponential time at $0+$, then it jumps to $0-$ or Δ. In particular, $0+$ is not a trap under P_{0+}, so we may apply Theorem 3.18.

Let $A = [0+, \varepsilon)$ with $\varepsilon \searrow 0$. Starting from $0+$, the process exits A precisely when it jumps from $0+$ to the points $0-$ or Δ. Thus,

$$\begin{aligned} Lf(0+) &= \lim_{\varepsilon \searrow 0} \frac{E_{0+}[f(W_{T_+})] - f(0+)}{E_{0+}[T_+]} \\ &= \lambda_+[f(0-)P_{0+}(W_{T_+} = 0-) + f(\Delta)P_{0+}(W_{T_+} = \Delta) - f(0+)]. \end{aligned}$$

Recalling that f vanishes at Δ, we rewrite the equation as follows

$$\begin{aligned} Lf(0+) &= \lambda_+ P_{0+}(W_{T_+} = 0-)(f(0-) - f(0+)) - \lambda_+(1 - P_{0+}(W_{T_+} = 0-))f(0+) \\ &= \lambda_+ P_{0+}(W_{T_+} = 0-)(f(0-) - f(0+)) - \lambda_+ P_{0+}(W_{T_+} = \Delta)f(0+). \end{aligned}$$

Since $Lf(0+) = \frac{1}{2}f''(0+)$, we get

$$\frac{1}{2}f''(0+) + \lambda_+ P_{0+}(W_{T_+} = \Delta)f(0+) + \lambda_+ P_{0+}(W_{T_+} = 0-)(f(0+) - f(0-)) = 0. \quad (5.6)$$

Setting $c_1^+ = \frac{\lambda_+ P_{0+}(W_{T_+}=\Delta)}{1+\lambda_+}$, $a^+ = \frac{\lambda_+ P_{0+}(W_{T_+}=0-)}{1+\lambda_+}$, $c_2^+ = 0$, $c_3^+ = \frac{1}{1+\lambda_+}$ and $\nu_1^- = \nu_1^+ = 0$, the lemma follows.

Note also that we can see the equation (5.6) as

$$\frac{1}{2}f''(0+)$$
$$= -\frac{1}{E_{0+}[T_+]}[P_{0+}(W_{T_+} = \Delta)(f(0+) - f(\Delta)) + P_{0+}(W_{T_+} = 0-)(f(0+) - f(0-))].$$

Case $\lambda_+ = \infty$ **or** $\lambda_- = \infty$.

If $\lambda_+ = \infty$, then the process leaves $0+$ immediately, so we can apply Theorem 3.18 again. Let $A = [0+, \varepsilon)$ with $\varepsilon \searrow 0$. Starting from $0+$, the process can leave A only at $\varepsilon, \Delta, 0-$. Then considering $T_\varepsilon = \tau_\varepsilon \wedge \tau_\Delta \wedge \tau_{0-}$, by Dynkin Formula, we have

$$Lf(0+) = \lim_{\varepsilon \searrow 0} \frac{E_{0+}[f(W_{T_\varepsilon})] - f(0+)}{E_{0+}[T_\varepsilon]}.$$

We rewrite this equation in terms of the pull-back measure $\nu_\varepsilon(dx) = \frac{1}{E_{0+}[T_\varepsilon]} P_{0+}(W_{T_\varepsilon} \in dx)$ on $E \cup \Delta$. Thus,

$$Lf(0+) = \lim_{\varepsilon \searrow 0} \int_E (f(x) - f(0+))\, \nu_\varepsilon(dx) - f(0+)\nu_\varepsilon(\Delta).$$

We consider $K_\varepsilon = 1 + \nu_\varepsilon(\Delta \cup (-\infty, 0-]) + \int_{(0+,\infty)} (1 \wedge x)\, \nu_\varepsilon(dx)$, so the above equation implies

$$\lim_{\varepsilon \searrow 0} \frac{\nu_\varepsilon(\Delta)}{K_\varepsilon} f(0+) + \frac{Lf(0+)}{K_\varepsilon} - \int_E (f(x) - f(0+))\, \frac{\nu_\varepsilon(dx)}{K_\varepsilon} = 0, \quad (5.7)$$

since $1 \leq K_\varepsilon < \infty$. We now look at the measures

$$\mu_\varepsilon^+(A) = \int_A (1 \wedge x)\, \frac{\nu_\varepsilon(dx)}{K_\varepsilon} \text{ on } (0+, \infty) \quad \text{and}$$
$$\mu_\varepsilon^-(A) = \frac{\nu_\varepsilon(A)}{K_\varepsilon} \text{ on } (-\infty, 0-].$$

Identifying $[0+, \infty)$ with $[0, \infty)$ and noticing that $\mu_\varepsilon^+((0+, \infty)) < 1$, we use Proposition 4.6 to obtain a sequence $(\varepsilon_n)_n$ and a measure $\mu^+(dx)$ on $[0+, \infty]$ such that

$$\lim_{n\to\infty} \int_{(0+,\infty)} f(x)\, \mu_{\varepsilon_n}^+(dx) = \int_{[0+,\infty]} f(x)\, \mu^+(dx),$$

for all $f \in C[0+, \infty]$..

Applying the same reasoning and Remark 5, we also obtain a measure $\mu^-(dx)$ on $[-\infty,0-]$ such that, possibly after refining the sequence, we have

$$\lim_{n\to\infty}\int_{(-\infty,0-]} f(x)\,\mu^-_{\varepsilon_n}(dx)=\int_{[-\infty,0-]} f(x)\,\mu^-(dx),$$

for all $f\in C[-\infty,0-]$.

Now, we use these limits with appropriated functions. For $f\in\mathscr{D}_W(L)$, the function $\frac{f(x)-f(0+)}{1\wedge x}$ is continuous on $(0+,\infty)$ and $\lim_{x\to\infty}\frac{f(x)-f(0+)}{1\wedge x}=-f(0+)<\infty$. Also, $\lim_{x\downarrow 0+}\frac{f(x)-f(0+)}{1\wedge x}=\lim_{x\downarrow 0+}\frac{f(x)-f(0+)}{x}=f'(0+)<\infty$. Thus, taking its extension to $[0+,\infty]$, we get

$$\begin{aligned}\int_{[0+,\infty]}\frac{f(x)-f(0+)}{1\wedge x}\,\mu^+(dx)&=\lim_{n\to\infty}\int_{(0+,\infty)}\frac{f(x)-f(0+)}{1\wedge x}\,\mu^+_{\varepsilon_n}(dx)\\&=\lim_{n\to\infty}\int_{(0+,\infty)}\frac{f(x)-f(0+)}{1\wedge x}(1\wedge x)\,\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx)\\&=\lim_{n\to\infty}\int_{(0+,\infty)}(f(x)-f(0+))\,\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx)\\&=\lim_{n\to\infty}\int_{[0+,\infty)}(f(x)-f(0+))\,\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx),\end{aligned}$$

where we have used that $1\wedge x$ is the Radon-Nikodym derivative of $\mu^+_{\varepsilon_n}(dx)$ with respect to $\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx)$.

Similarly, one can see that

$$\int_{[-\infty,0-]}(f(x)-f(0+))\,\mu^-(dx)=\lim_{n\to\infty}\int_{(-\infty,0-]}(f(x)-f(0+))\,\frac{1}{K_{\varepsilon_n}}\nu_{\varepsilon_n}(dx),$$

for all $f\in\mathscr{D}_W(L)$.

Moreover, the sequences $\frac{\nu_{\varepsilon_n}(\Delta)}{K_{\varepsilon_n}}$ and $\frac{1}{K_{\varepsilon_n}}$ are bounded, so we may assume

$$\lim_{n\to\infty}\frac{\nu_{\varepsilon_n}(\Delta)}{K_{\varepsilon_n}}=p_1 \text{ and } \lim_{n\to\infty}\frac{1}{K_{\varepsilon_n}}=p_2,$$

passing to a further subsequence, if necessary. Applying all these limits, equation (5.7) simplifies to

$$\begin{aligned}&p_1f(0+)+p_2Lf(0+)\\&=\int_{[0+,\infty]}\frac{f(x)-f(0+)}{1\wedge x}\,\mu^+(dx)+\int_{[-\infty,0-]}(f(x)-f(0+))\,\mu^-(dx).\end{aligned}$$

Hence,

$$\begin{aligned}&(p_1+\mu^+(\infty)+\mu^-(-\infty))f(0+)\\&+\mu^-(0-)(f(0+)-f(0-))-\mu^+(0+)f'(0+)+\frac{p_2}{2}f''(0+)\\&=\int_{(0+,\infty)}\frac{f(x)-f(0+)}{1\wedge x}\,\mu^+(dx)+\int_{(-\infty,0-)}(f(x)-f(0+))\,\mu^-(dx).\end{aligned}$$

Setting the constants

$$\begin{aligned}
c_1^+ &= p_1 + \mu^+(\infty) + \mu^-(-\infty), \\
a^+ &= \mu^-(0-), \\
c_2^+ &= \mu^+(0+), \\
c_3^+ &= p_2,
\end{aligned}$$

and considering the measures

$$\begin{aligned}
\nu_1^+(A) &= \int_A \frac{1}{1 \wedge x}\, \mu^+(dx) \text{ on } (0+, \infty), \\
\nu_1^-(A) &= \mu^-(A) \text{ on } (-\infty, 0-)
\end{aligned}$$

yields the desired equation (5.4). It is only missing to check (5.2). This follows from

$$\begin{aligned}
&p_1 + p_2 + \mu^-([-\infty, 0-]) + \mu^+([0+, \infty]) \\
&= \lim_{n\to\infty} \frac{\nu_{\varepsilon_n}(\Delta)}{K_{\varepsilon_n}} + \frac{1}{K_{\varepsilon_n}} + \mu^-_{\varepsilon_n}((-\infty, 0-]) + \mu^+_{\varepsilon_n}((0+, \infty)) \\
&= \lim_{n\to\infty} \frac{\nu_{\varepsilon_n}(\Delta) + 1 + \nu^-_{\varepsilon_n}((-\infty, 0-]) + \int_{(0+,\infty)} (1 \wedge x)\, \nu_{\varepsilon_n}(dx)}{K_{\varepsilon_n}} = 1.
\end{aligned}$$

For the case $\lambda_- = \infty$, the argument is analogous. We just emphasizes that, following this procedure, we will obtain an equation taking the form

$$\begin{aligned}
&p_1 f(0-) + p_2 Lf(0-) \\
&= \int_{[-\infty,0-]} \frac{f(x) - f(0-)}{1 \wedge -x}\, \mu^-(dx) + \int_{[0+,\infty]} (f(x) - f(0-))\, \mu^+(dx).
\end{aligned}$$

Now, since $\lim_{x\uparrow 0-} \frac{f(x)-f(0-)}{1\wedge -x} = \lim_{x\uparrow 0-} \frac{f(x)-f(0-)}{-x} = -f'(0-)$, this leads to the sign change in the coefficient of c_2^-, as compared to c_2^+ in the statement of the lemma.

For the converse, let $\mathscr{D}'$ be the set of functions $f, f'' \in C_0^\Delta(E)$ satisfying the equations (5.4) and (5.5). We want to show that $\mathscr{D}' \subseteq \mathscr{D}_W(L)$. In fact, as in the proof of Lemma 4.5, we have that the equation

$$\beta f - \frac{1}{2} f'' = g, \tag{5.8}$$

for $\beta > 0$ and $g \in C_0(E)$, has a solution in $\mathscr{D}_W(L)$.

Let $f \in \mathscr{D}'$. Since f satisfy (5.8) for some $\beta > 0$ and $g \in C_0(E)$, and $\mathscr{D}_W(L) \subseteq \mathscr{D}'$, if f does not belong to $\mathscr{D}_W(L)$, then there exist two different functions $f_1, f_2 \in \mathscr{D}'$ satisfying the same equation. Consequently,

$$\beta(f_1 - f_2) - \frac{1}{2}(f_1'' - f_2'') = 0 \text{ on } E. \tag{5.9}$$

Now, considering the known solution of (5.9) and the fact that $f_1 - f_2$ decays to 0 at infinity, we obtain

$$f_1(x) - f_2(x) = \begin{cases} d^+ \exp(-\sqrt{2\beta}x), & \text{if } x \in [0+, \infty); \\ d^- \exp(\sqrt{2\beta}x), & \text{if } x \in (-\infty, 0-], \end{cases}$$

for some $d^+, d^- \in \mathbb{R}$, defining naturally $\exp(0+) = \exp(0-) = 1$. Also, since f_1 and f_2 are distinct, then d^+, d^- cannot be both equal to 0.

Since $f_1, f_2 \in \mathscr{D}'$, subtracting the equations (5.4) derived from f_1 and f_2, and computing $f_1 - f_2$ and its derivatives at $0+$ or $0-$ results in

$$\begin{aligned} &c_1^+ d^+ + a^+ d^+ - a^+ d^- + c_2^+ d^+ \sqrt{2\beta} + c_3^+ d^+ \beta \\ &= \int_{(0+,\infty)} d^+ [\exp(-\sqrt{2\beta}x) - 1]\, \nu_1^+(dx) + \int_{(-\infty,0-)} [d^- \exp(\sqrt{2\beta}x) - d^+]\, \nu_1^-(dx). \end{aligned} \tag{5.10}$$

Now, doing analogous computations using equation (5.5), we obtain

$$\begin{aligned} &c_1^- d^- + a^- d^- - a^- d^+ + c_2^- d^- \sqrt{2\beta} + c_3^- d^- \beta \\ &= \int_{(0+,\infty)} [d^+ \exp(-\sqrt{2\beta}x) - d^-]\, \nu_2^+(dx) + \int_{(-\infty,0-)} d^- [\exp(\sqrt{2\beta}x) - 1]\, \nu_2^-(dx). \end{aligned} \tag{5.11}$$

We will show that these equations cannot be both true. Indeed, suppose that $d^+ = 0$, then $d^- \neq 0$, so (5.11) simplifies to

$$\begin{aligned} &c_1^- + a^- + c_2^- \sqrt{2\beta} + c_3^- \beta \\ &= \int_{(0+,\infty)} (-1)\, \nu_2^+(dx) + \int_{(-\infty,0-)} (\exp(\sqrt{2\beta}x) - 1)\, \nu_2^-(dx), \end{aligned}$$

since both integrands are negative, whereas the constants c_1^-, a^-, c_2^-, c_3^- are all nonnegative, the only possibility for the equation to hold is that all these constants and measures are simultaneously zero. However, (5.3) must hold, leading to a contradiction.

For $d^- = 0$ the argument is similar, so we can assume $d^+, d^- \neq 0$. Without loss of generality suppose $d^- \leq d^+$, then (5.10) implies

$$\begin{aligned} &c_1^+ + a^+ (1 - \frac{d^-}{d^+}) + c_2^+ \sqrt{2\beta} + c_3^+ \beta \\ &= \int_{(0+,\infty)} [\exp(-\sqrt{2\beta}x) - 1]\, \nu_1^+(dx) + \int_{(-\infty,0-)} \left[\frac{d^-}{d^+} \exp(\sqrt{2\beta}x) - 1\right] \nu_1^-(dx), \end{aligned}$$

and we have the problem that the left-hand side of the equation is nonnegative, whereas the right side is non positive, giving the same contradiction as before.

Therefore, $f \in \mathscr{D}_W(L)$ and the proof is complete. □

Combining Lemmas 5.3 and 5.4, we found the domain of any general Brownian motion on $(-\infty, 0-] \cup [0+, \infty)$ with boundary conditions at the origin.

Theorem 5.5. *For each general Brownian motion on* $(-\infty, 0-] \cup [0+, \infty)$ *with boundary conditions at the origin, there correspond nonnegative constants* a^+, a^-, c_i^+, c_i^-, $i = 1, 2, 3$ *such that the domain* $\mathscr{D}_W(L)$ *consists of functions with* $f'' \in C_0^{\Delta}(E)$ *that satisfy*

$$c_1^+ f(0+) + a^+(f(0+) - f(0-)) - c_2^+ f'(0+) + \frac{c_3^+}{2} f''(0+) = 0 \quad \textit{and}$$
$$c_1^- f(0-) + a^-(f(0-) - f(0+)) + c_2^- f'(0-) + \frac{c_3^-}{2} f''(0-) = 0.$$

Proof. We want to show that the measures ν_j^+, ν_j^-, for $j = 1, 2$, in Lemma 5.4 are identically zero, what was true for $\lambda_+, \lambda_- \neq \infty$. To illustrate the argument, let $j = 1$. We assume that $\nu_1^+(\varepsilon, \infty), \nu_1^-(-\infty, -\varepsilon) > 0$, for some $\varepsilon > 0$. Take any $f_1 \in \mathscr{D}_W(L)$ and adjust it to obtain $f_2 = f_1$ on $[-\varepsilon, 0-] \cup [0+, \varepsilon]$, but $f_2(x) < f_1(x)$ for $|x| > \varepsilon$ in such a way that f_2 also belongs to $C_0^{\Delta}(E)$ and $f'' \in C_0^{\Delta}(E)$. Thus, by Lemma 5.3, we also have $f_2 \in \mathscr{D}_W(L)$ and Lemma 5.4 applies to both f_1, f_2. Since they agree in a neighborhood of $0+$ and $0-$, equation (5.4) reduces to

$$\int_{(0+,\infty)} (f_1(x) - f_2(x))\, \nu_1^+(dx) + \int_{(-\infty,0-)} (f_1(x) - f_2(x))\, \nu_1^-(dx) = 0$$

which implies

$$\int_{(\varepsilon,\infty)} (f_1(x) - f_2(x))\, \nu_1^+(dx) = -\int_{(-\infty,-\varepsilon)} (f_1(x) - f_2(x))\, \nu_1^-(dx).$$

Since both integrands are positive, this leads to

$$\int_{(\varepsilon,\infty)} (f_1(x) - f_2(x))\, \nu_1^+(dx) = \int_{(-\infty,-\varepsilon)} (f_1(x) - f_2(x))\, \nu_1^-(dx) = 0$$

implying that

$$(f_1 - f_2)\mathbf{1}_{\{(\varepsilon,\infty)\}} = 0 \ \nu_1^+\text{-a.e. and } (f_1 - f_2)\mathbf{1}_{\{(-\infty,-\varepsilon)\}} = 0 \ \nu_1^-\text{-a.e.},$$

which is not possible since both the measures and the functions are positive on each of their respective intervals. Hence, $\nu_1^+(\varepsilon, \infty) = \nu_1^-(-\infty, -\varepsilon) = 0$. Since $\varepsilon > 0$ was arbitrary, it follows that $\nu_1^+ \equiv 0$, $\nu_1^- \equiv 0$, as desired. □

Remark 7. Let us do some final observations. As consequence of [5, Proposition 1], which characterizes the resolvent family of the *Snapping Brownian Motion* , it is possible to check the infinitesimal generator of the *Snapping Brownian Motion* on $E = (-\infty, 0-] \cup [0+, \infty)$ is given by $Lf = \frac{1}{2} f''$ whose the domain $\mathscr{D}(L)$ consists of functions such that $f'' \in C_0^{\Delta}(E)$ and

$$f'(0+) \ = \ f'(0-) \ = \ \frac{\kappa}{2}(f(0+) - f(0-))$$

where κ is a positive constant. This is a particular case of the class of processes obtained in our Theorem 5.5, by taking $a^+ = a^- = \kappa/2$, $c_2^+ = c_2^- = 1$ and $c_1^+ = c_1^- = c_3^+ = c_3^- = 0$.

The coefficients appearing in the definition of the $\mathscr{D}_W(L)$ in the statement of Theorem 5.5 can be interpreted as follows.

The coefficient c_1^+ (respectively c_1^-) is related to the rate at which the process jumps from $0+$ (respectively $0-$) to the cemetery; that is, it is a rate of "killing".

The coefficient a^+ (respectively a^-) is related to the rate at which the process jumps from $0+$ to $0-$ (respectively from $0-$ to $0+$); that is, it is the rate at each the process switch of half-line. The original Snapping Out Brownian Motion of Lejay is symmetric, that is, $a^+ = a^-$. Here we allow the case $a^+ \neq a^-$, which can be understood as a *Skew Snapping Out BM*.

The coefficient c_2^+ (respectively c_2^-) is related to the reflection strength at $0+$ (respectively $0-$); and the coefficient c_3^+ (respectively c_3^-) is related to stickiness of the process at $0+$(respectively $0-$). Note that the equations defining the domain of the generator in the Theorem 5.5 are homogeneous, so they can be normalized. This is important when thinking about the fact that stickiness at $0\pm$ and reflection at $0\pm$ are not independent: they compete.

The discussion above, thus inspires us to call the class of processes obtained in Theorem 5.5 of *Skew Sticky Killed Snapping Out Brownian Motion*, which is a new Brownian type process and, as proved here, is the most general Brownian motion on the state space $(-\infty, 0-] \cup [0+, \infty)$.

Bibliography

[1] Andrei N Borodin and Paavo Salminen. Handbook of brownian motion-facts and formulae. *Probability and Its Applications*, 2002.

[2] Richard Durrett. *Probability: Theory and Examples*. Cambridge University Press, 4th edition edition, 2010.

[3] Frank B Knight. *Essentials of Brownian motion and diffusion*. Number 18. American Mathematical Soc., 1981.

[4] Jean-François Le Gall. *Brownian motion, martingales, and stochastic calculus*. Springer, 2016.

[5] Antoine Lejay. The snapping out brownian motion. *Annals of applied probability: an official journal of the Institute of Mathematical Statistics*, 26(3):1727–1742, 2016.

[6] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.

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