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Dissertação de Mestrado

Similarity between the Mandelbrot set and the Julia set at Misiurewicz points

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Orientador: Carlos Alberto Siqueira Lima

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Auto-similaridade entre os conjuntos de Mandelbrot e Julia em pontos Misiurewicz

Gabriel Andrés Borges Morales

Dissertação apresentada ao Colegiado do Curso de Pós-graduação em Matemática da Universidade Federal da Bahia, como requisito parcial para obtenção do Título de Mestre em Matemática.

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"Caminante, no hay camino. Se hace camino al andar".

(Antonio Machado Ruiz)

Resumo

Iremos demonstrar que o conjunto de Julia J_c é assintoticamente similar ao conjunto de Mandelbrot $\mathcal{M}$, ao redor de um parâmetro Misiurewicz $c \in \partial\mathcal{M}$, utilizando o conceito de similaridade descrito a partir da distância de Hausdorff-Chabauty. Esse resultado foi provado originalmente por Tan Lei, em 1990 [6]. Discutiremos conceitos basilares da Dinâmica Complexa e alguns resultados centrais para a demonstração desse teorema.

Palavras-chave: Pontos Misiurewicz; Autossimilaridade; Similaridade; Similaridade assintótica; Conjunto de Julia; Conjunto de Mandelbrot; Familia quadrática.

Abstract

We shall prove that the Julia set J_c is asymptotically similar to the Mandelbrot set $\mathcal{M}$ about a Misiurewicz parameter $c \in \partial\mathcal{M}$, using the concept of similarity and the Hausdorff-Chabauty distance. This result was originally proved by Tan Lei in 1990 [6]. We will discuss the basic framework from the field of Complex Dynamics and some central results needed to prove this theorem.

Keywords: Misiurewicz points; Self-similarity; Symilarity; Asymptotic similarity; Julia set; Mandelbrot set; Quadratic family.

Contents

Chapter 1

Introduction

Our main goal is to present a self-contained proof of Theorems 1.1, 1.2 and 1.3, from the original article of TAN Lei, *Similiarity Between the Mandelbrot set and Julia sets*, **Communications in Mathematical Physics** (1990).

Experimental overview

The *Julia set* $J(f)$ of a rational function $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is the closure of all *repelling periodic points* of f. If f is a polynomial, we can define the *filled-in Julia set* $K(f)$ as the set of points $z \in \mathbb{C}$ whose *orbit* is a bounded subset of $\mathbb{C}$.
The Mandelbrot set $\mathcal{M}$ is the *connectedness locus* of the quadratic family $P_c(z) = z^2 + c$. Hence $c \in \mathcal{M}$ if, and only if, $K(P_c)$ is connected. By a theorem of P. Fatou,

$$\mathcal{M} = \{c \in \mathbb{C} \mid 0 \in K(P_c)\}.$$

Definition 1.1 (Misiurewicz point). A parameter $c \in \mathcal{M}$ is a *Misiurewicz point* if the orbit of 0 under P_c is strictly preperiodic. In other words, 0 is not a periodic point of the quadratic map P_c, but some iterate P_c^k maps 0 to a periodic point of P_c.

If c is a Misiurewicz point, then $c \in J(P_c)$ and $c \in \partial M$. A preliminary experiment is presented in the following example.

Example 1.1 (Similarity at Misiurewicz points). The complex number $-0.607 + 0.605i$ is very close to a Misiurewicz point $c \in \partial\mathcal{M}$ satisfying $P_c^3(P_c^2(c)) = P_c^2(c)$. Experimentally, one can observe that by magnifying the Julia set $J(P_c)$ centered at c, with a specific factor, the images that appear are similar to each other up to a rotation.

The last three images in Figures 1.2 and 1.3 are very similar. Tan Lei [6] was the first to give a rigorous proof of this similiarity at every Misiurewicz point (see Theorems 1.1, 1.2 and 1.3).

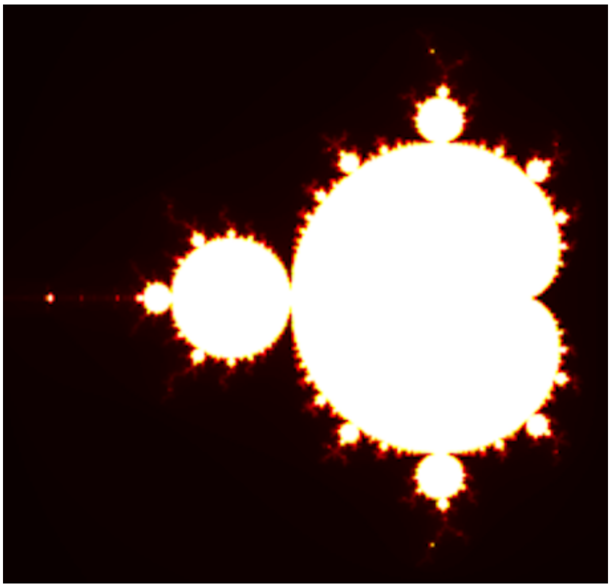

Figure 1.1: The Mandelbrot set $\mathcal{M}$ generated with a Python code. In the algorithm, a point c is coloured black if its forward orbit $c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \cdots$ diverges to infinity.

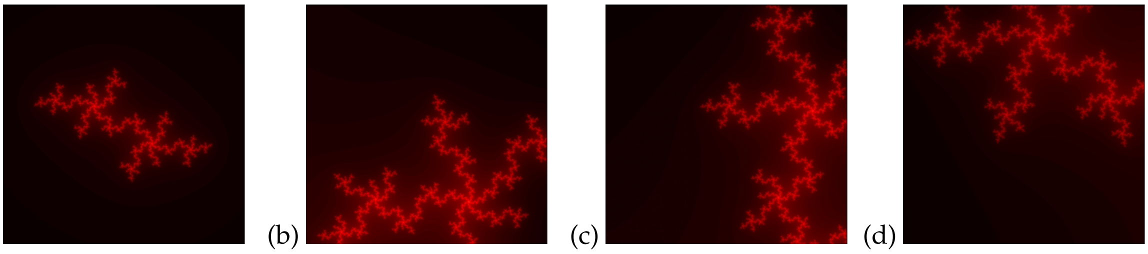

Figure 1.2: The Julia set $J(P_c)$ and successive magnifications centered at $c \approx -.607 + .605i$. The last image has width 1.6×10^{-6}.

1.1 The Hausdorff-Chabauty topology

Given a compact set $A \subset \mathbb{C}$ and $\varepsilon > 0$, the ε-neighborhood of A is the set of all $z \in \mathbb{C}$ whose distance from A is less than ε. Let d denote the Hausdorff distance function between compact subsets of $\mathbb{C}$. Recall that $d(A, B) < \varepsilon$ if, and only if, A is contained in the ε-neighborhood of B, and B is contained in the ε-neighborhood of A. Let $\mathbb{D}_r$ denote the disk $\{z \in \mathbb{C} : |z| < r\}$. If $A \subset \mathbb{C}$ is closed and $r > 0$, then we define

$$(A)_r = (A \cap \mathbb{D}_r) \cup \partial\mathbb{D}_r.$$

The *Hausdorff-Chabauty* distance

$$d_r(A, B) = d((A)_r, (B)_r)$$

is defined for every pair of closed subsets of the plane.

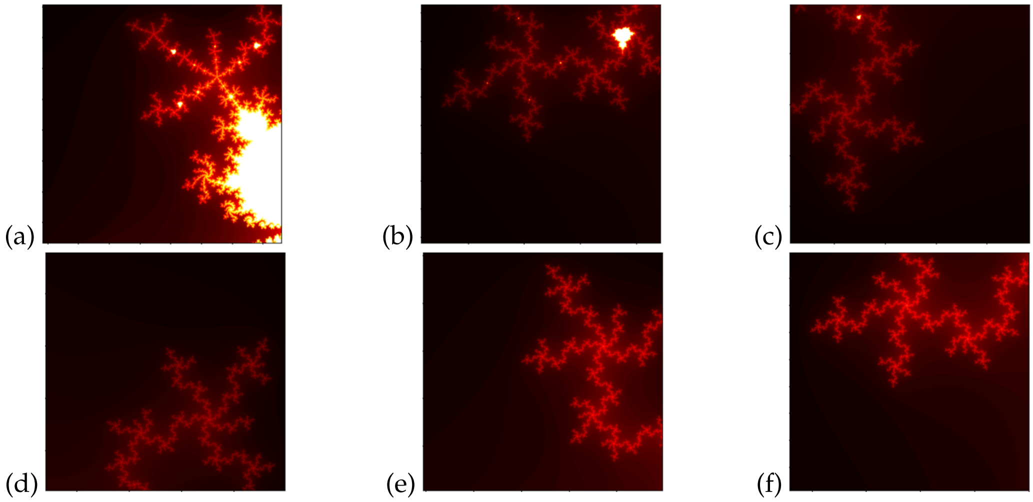

Figure 1.3: Successive magnifications of $\mathcal{M}$ centered at c. The last image has approximated width 1.1×10^{-5}.

1.2 Asymptotic similarity

Let $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. A nonempty closed subset F of the complex plane is *asymptotically self-similar about* $c \in F$ *with scale* λ if there exists $r > 0$ such that the sequence $\lambda^n(F - c)$ converges to $F - c$ with respect to the metric d_r. (For the following theorem, the concept of multiplier is detailed in section 2.1).

Theorem 1.1 (Tan Lei, 1990). *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with degree at least two, and let z_0 be a repelling periodic point of f with multiplier λ. If $z \in J(f)$ is eventually mapped to z_0 by some iterate of f, then the Julia set is asymptotically self-similar about z_0 with scale λ.*

By Proposition 2.4,

$$\bigcup_{n>0} f^{-n}(z_0)$$

is a dense subset of the Julia set for every $z_0 \in J(f)$. It follows from Theorem 1.1 that asymptotic self-similarity occurs on a dense subset of $J(f)$, for every rational function f with degree at least two.

Self-similarity at points of the Mandelbrot set

If c_0 is a Misiurewicz point for the quadratic family $P_c(z) = z^2 + c$, then by definition 0 is strictly preperiodic. There exists a first $k > 0$ such that $P^k_{c_0}(0)$ is periodic. Therefore, $P^k_{c_0}(0)$ determines a periodic orbit. We are going to show in Proposition 3.1 that this orbit is repelling. This is the *repelling cycle associated with* c_0.

Theorem 1.2 (Tan Lei, 1990). *Let c_0 be a Misiurewicz point of the quadratic family $z^2 + c$. Let λ be the multiplier of the repelling cycle associated with c_0. Then $c_0 \in \partial\mathcal{M}$. Moreover, $\mathcal{M}$ is asymptotically self-similar about c_0 with scale λ.*

Suppose A and B are nonempty closed subsets of $\mathbb{C}$ which contain a common point c. If

$$d_r(t(A-c), t(B-c)) \to 0$$

as $t \to \infty$, with $t \in \mathbb{C}$, we say that A and B are *asymptotic similar about* c.

Theorem 1.3 (Tan Lei, 1990). *For every Misiurewicz point c_0 of the quadratic family $z^2 + c$ there exists $\lambda \in \mathbb{C} - \{0\}$ such that $\mathcal{M}$ and λJ_{c_0} are asymptotically similar about c_0.*

1.3 Nonuniform hyperbolicity

Suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map with degree at least 2. We say that f is hyperbolic if the forward orbit of every critical point of f converges to an attracting cycle of f. Equivalently, f is hyperbolic precisely when there exists a conformal metric $\gamma(z)|dz|$ defined on a neighborhood of $J(f)$ such that $\|f'(z)\|_\gamma > 1$, for every z in the Julia set.

If c_0 is a Misiurewicz point for the quadratic family $f_c(z) = z^2 + c$, then c_0 belongs to the boundary of the Mandelbrot set and f_{c_0} is not hyperbolic, since the critical point 0 belongs to the Julia set in this case (obviously, the Julia set does not contain attracting cycles). Nevertheless, we shall see in section 3.2 that f_{c_0} expands a conformal metric defined on a neighborhood of $J(f_{c_0})$, except for finitely many singularities in the Julia set. Therefore, Misiurewicz points are associated with very simple examples of *nonuniform hyperbolicity.*

1.4 Misiurewicz points are dense in $\partial\mathcal{M}$

A component U of the interior of the Mandelbrot set $\mathcal{M}$ is *hyperbolic* if every point $c \in U$ yields a hyperbolic map $z^2 + c$. The following is a version of the Fatou conjecture (proposed by Pierre Fatou in 1920) specialized to the quadratic family.

Fatou conjecture (1920). *Every component of the interior of the Mandelbrot set is hyperbolic.*

Concerning the dynamics of the quadratic family, one of the most difficult questions is: what kind of behavior has the function f_c when c is in the boundary of the Mandelbrot set? A partial and incomplete classification includes parameters in $\partial\mathcal{M}$ which are *semi-hyperbolic, parabolic, Collet-Eckmann, Siegel, Cremer, Misiurewicz, infinitely renormalizable* and *Feigenbaum*. In a certain sense, the majority of points on $\partial\mathcal{M}$ does not fit in any of these classes. Nevertheless, according to J. Milnor [10]:

Theorem 1.4 (Milnor, 1989). *The set of Misiurewicz points of the quadratic family $z^2 + c$ is dense in $\partial\mathcal{M}$.*

1.5 Hausdorff dimension of the boundary of the Mandelbrot set

A rational map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with degree at least 2 is *semi-hyperbolic* if $J(f)$ does not contain the set of recurrent critical points and parabolic points of f. We denote the set of semi-hyperbolic parameters for the quadratic family $z^2 + c$ by $\mathcal{S}$.

Every Misiurwicz parameter belongs to $\mathcal{S}$. Since Misiurewicz points are dense in the boundary of the Mandelbrot set, it follows that $\mathcal{S}$ is a dense subset of $\partial\mathcal{M}$ as well. Surprisingly, if we remove the Misiurewicz points from $\mathcal{S}$ the remaining set is a subset of $\partial\mathcal{M}$ with Hausdorff dimension 2. (See Theorem 1.7).

Multibrot sets

The *multibrot set* $\mathcal{M}_d$ is the connected locus of the family $z^d + c$, where $d > 1$ is an integer. Hence a point c belongs to $\mathcal{M}_d$ precisely when the orbit of the critical point 0 under $z^d + c$ is bounded. Rivera-Letelier has generalized Tan Lei's result of asymptotic similarity to semi-hyperbolic points [11].

Theorem 1.5 (Rivera-Letelier, 2001). *Let $d > 1$ be an integer. Suppose c is a parameter for which $z^d + c$ is semi-hyperbolic. Let J_c denote the Julia set of $z^d + c$. There exists a complex number $\lambda \neq 0$ such that λJ and $\mathcal{M}_d$ are asymptotic similar about c.*

The area of $\partial\mathcal{M}$

Theorem 1.5 has some very important implications for the study of the Hausdorff dimension of $\partial\mathcal{M}$. Indeed, studying the Hausdorff dimension of Julia sets is easier than in the parameter space. Roughly speaking, Theorem 1.5 reveals that at every semi-hyperbolic parameter we can "see" some pieces of Julia sets in $\partial\mathcal{M}$ with Hausdorff dimension approximately 2.

Theorem 1.6 (Shishikura, 1998). *For every nonempty open set $U \subset \mathbb{C}$ intersecting $\partial\mathcal{M}$, the Hausdorff dimention of $U \cap \partial\mathcal{M}$ is* 2.

The original proof of Shishikura [12] is based on bifurcation of parabolic points. The same result can be obtained with different methods (Kawahira and Kisaka [5]) using the concept of semi-hyperbolicity:

Theorem 1.7 (Kawahira and Kisaka, 2021). *Let $\tilde{\mathcal{S}}$ denote the set of semi-hyperbolic parameters in the boundary of the Mandelbrot set which are not Misiurewicz. Then the Hausdorff dimension of $\tilde{\mathcal{S}}$ is* 2.

As far as we know, determining the value of the area of $\partial\mathcal{M}$ is still an open problem.[1]

1.6 Universality

From the experimental point of view, generating codes for visualizing the Mandelbrot set is something quite trivial. This allows us to perform magnifications of the Mandelbrot set at very small scales, such as 10^{-20} (not even subatomic particles are in such scales). Using such algorithms, it is possible to observe infinitely many copies of the Mandelbrot set inside itself. Explaining this phenomenon in pure mathematical terms is a hard job that is way beyond the scope of this exposition. We defer to the original papers of Eckmann and Epstein [4] as well as Douady and Hubbard [3] for a full presentation of this subject. We can at least (informally) say that *infinitely many copies of the Mandelbrot set appear in a neighborhood of every Misiurewicz point.* Since Misiurewicz points are dense in the boundary of the Mandelbrot set, it follows that $\partial\mathcal{M}$ is almost entirely made of copies of itself.

Bifurcation locus and general families of rational maps

We shall now explain what we mean by *universality* in our context. (For one-dimensional real dynamics the same term is used with another meaning).

Let $(f_\lambda)_{\lambda\in\Lambda}$ be a one-parameter family of rational maps parameterized on a connected and complex manifold Λ. It is usual to identify this family with the map $f : \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by $(\lambda, z) \mapsto f_\lambda(z)$. The family is said to be *holomorphic* if f is holomorphic.

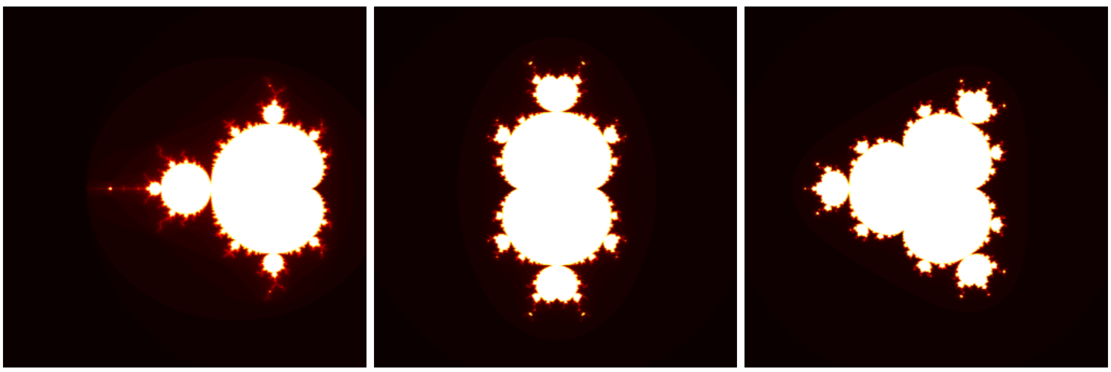

Figure 1.4: The Multibrot sets $\mathcal{M}_d$ with $d \in \{2, 3, 4\}$.

As described by McMullen in [8], for general holomorphic families $f : \Lambda \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ it is possible to define a *bifurcation locus* $B(f)$ that generalizes the notion of the Multibrot

[1]If you ask this question on Google, Artificial Intelligence claims, pretty confidently, that the area of the Mandelbrot set (not the boundary) is 2.089 and gives a formula for calculating it. This is not true.

sets; that is, if f is represented by the family $z^d + c$, then $\partial\mathcal{M}_d = B(f)$.

The Multibrot sets $\mathcal{M}_d$ are universal in the sense that every bifurcation locus $B(f)$ contains a copy of $\partial\mathcal{M}_d$, for some integer $d > 1$.

Chapter 2

Prerequisites

We are going to make an overview of some fundamental concepts on the dynamics in one complex variable, which are found in the following references: [1],[7] [9] and [2].

2.1 Local dynamics

Let $U \subset \mathbb{C}$ be an nonempty open set and $f : U \to U$ be a holomorphic function. We define $f^0(z) = z$ and, inductively,

$$f^n(z) = f \circ f^{n-1}(z),$$

for every $n > 0$. Every f^n is an *iterate* of f. If $f(z) = z$, then z is a *fixed point* of f. If $f^n(z) = z$ for some $n > 0$, then z is a *periodic point* and its *period* is the minimal positive integer that satisfies this equation. Given a periodic point $z \in U$ of f, with period k, the *multiplier* λ of z is defined by

$$\lambda = (f^k)'(z).$$

A periodic point is classified in the following manner:

- *attracting*, if $0 < |\lambda| < 1$;
- *repelling*, if $|\lambda| > 1$;
- *superattracting*, if $|\lambda| = 0$;
- *indifferent*, if $\lambda = e^{2\pi i\theta}$, for some $\theta \in [0, 1)$;
  - *rationally indifferent* if $\theta \in \mathbb{Q}$;
  - *irrationally indifferent* if $\theta \in \mathbb{R} \backslash \mathbb{Q}$.

A rationally indifferent periodic point is also called *parabolic*.

Attracting and repelling periodic points. It is possible to show that a periodic point z_0 is attracting if, and only, if there exists an open neighborhood V of z_0 such that $\overline{f(V)} \subset V$ and, for all $z \in V$, the sequence $f^n(z)$ converges to z_0 as $n \to \infty$. We define the *basin of attraction* $\mathcal{A}(z_0)$ of z_0 by

$$\mathcal{A}(z_0) = \bigcup_{n=0}^{\infty} f^{-n}(V).$$

The basin of attraction $\mathcal{A}(z_0)$ consists of all the points $z \in U$ such that $f^n(z) \to z_0$ as $n \to \infty$. The *immediate basin of attraction* $\mathcal{A}_0(z_0)$ of z_0 is the connected component that contains z_0.

We say that two holomorphic functions $f : U \subseteq \mathbb{C} \to U$ and $g : V \subseteq \mathbb{C} \to V$ are *conjugated* if there exists a biholomorphic map $\varphi : U \to V$ such that $\varphi \circ f = g \circ \varphi$.

Theorem 2.1 (Attracting fixed point). *Assume U is a nonempty open subset of the complex plane. Let $f : U \to U$ be a holomorphic function and z_0 be a fixed point of f with multiplier λ satisfying $0 < |\lambda| < 1$. Then there exists a conformal map φ defined on a neighborhood of z_0 onto a neighborhood of 0 that conjugates f with the linear function $z \mapsto \lambda z$. This map φ is unique up to a multiplication by a nonzero constant.*

Proof. See [1, p. 31]. □

We may define the conjugation map for every $z \in \mathcal{A}(z_0)$ as

$$\phi(z) = \frac{\varphi(f^{\tilde{n}}(z))}{\lambda^{\tilde{n}}},$$

where $\tilde{n}$ is large enough so that $f^{\tilde{n}}(z)$ is in the neighborhood of definition of φ. A *critical point* $c \in U$ satisfies $f'(c) = 0$. It is possible to prove that the branch of the inverse φ^{-1} that maps 0 onto z_0 can be extended until we find a critical point of f or leave the domain of f.

A consequence of Theorem 2.1 is the existence of a conjugation between f and $z \mapsto \lambda z$ in a neighborhood of a repelling fixed point z_0. Since the multiplier λ of z_0 is nonzero, we can apply 2.1 to the local inverse f^{-1} and find a conjugation φ between f^{-1} and $z \mapsto \frac{1}{\lambda}z$, in a neighborhood of z_0. The map φ^{-1} will give a conjugation between f and $z \mapsto \lambda z$, in a neighborhood of z_0.

Theorem 2.2 (Koenigs 1884). *Let $f : U \to U$ be a holomorphic function and z_0 be a fixed point of f with multiplier λ satisfying $|\lambda| \neq 0, 1$. Then there exists a conformal map φ defined on a neighborhood of z_0 onto a neighborhood of 0 that conjugates f with the linear function $z \mapsto \lambda z$. This map φ is unique up to a multiplication by a nonzero constant.*

Suppose $f_c : U \to U$, with $c \in \mathbb{C}$, is a family of functions that depends analytically on c. Let z_0 be an attracting fixed point of f_{c_0}, with multiplier λ_0. Let $g : \mathbb{C} \times U \to \mathbb{C}$ be defined by

$$g(c, z) = f_c(z) - z.$$

We have

$$g(c_0, z_0) = 0 \quad \text{and} \quad \frac{\partial g}{\partial z}(c_0, z_0) \neq 0.$$

By the implicit mapping theorem, there exist a neighborhood V of c_0 and a function $\zeta : V \to \mathbb{C}$ depending analytically on c, such that $\zeta(c_0) = z_0$ and

$$g(c, \zeta(c)) = 0,$$

for all $c \in V$. Hence, for every $c \in V$ the image $\zeta(c)$ is a fixed point under f_c. The multiplier $\lambda(c)$ of $\zeta(c)$ depends analytically on c. We may restrict V, if necessary, so that $\zeta(c)$ is attracting. For every $c \in V$, there exists the conjugation map

$$\varphi_c(z) = \lim_{n \to \infty} \frac{f_c^n(z)}{\lambda(c)^n}.$$

Since the convergence of this sequence is uniform, we get that φ_c depends analytically on c. More precisely, there exists an analytic map $\Phi : V \times W \subset U \to V \times \mathbb{C}$, defined by

$$\Phi(c, z) = (c, \varphi_c(z)).$$

This fact holds for the repelling case as well.

2.1.1 Superattracting periodic points

Theorem 2.3 (Boettcher 1904). *Let $f : U \to U$ be a holomorphic function and z_0 be a superattracting fixed point under f, satisfying $f^{(i)}(z_0) = 0$ for all $i = 1, \dots, d-1$ and $f^{(d)}(z_0) \neq 0$. Then there exists a conformal map φ defined in a neighborhood of z_0 onto a neighborhood of 0 that conjugates f with the function $z \mapsto z^d$. This map φ is unique up to a multiplication by a $(d-1)$th root of unity.*

Proof. See [9, p. 90]. ◻

As in the attracting case, the idea of the proof is to define a sequence of functions that will converge to our conjugation map. This sequence φ_n is given by

$$\varphi_n(z) = f^n(z)^{d^{-n}}.$$

We can't easily extend the conjugation map to the basin of attraction. But it is possible to show that the inverse φ^{-1} can be extended analytically to a maximal disk

$\mathbb{D}_r$, with $0 < r \leqslant 1$. The map φ^{-1} also extends homeomorphically to $\partial\mathbb{D}_r$. If $r < 1$, then there is a critical point of f in $\varphi^{-1}(\partial\mathbb{D}_r)$ and if $r = 1$, then $\varphi^{-1} : \mathbb{D} \to \mathcal{A}(z_0)$ is a conformal isomorphism.

We can extend $g(z) = \log|\varphi(z)|$ to the whole basin of attraction, by setting, for all $z \in \mathcal{A}(z_0)$,

$$g(z) = \frac{g(f^{\tilde{n}}(z))}{d^{\tilde{n}}},$$

where $\tilde{n}$ is big enough so that $f^{\tilde{n}}(z)$ is in the neighborhood of definition of φ. The function g is harmonic, except for z_0 and its preimages, where it has logarithmic poles.

Indifferent periodic points. Given a parabolic fixed point z_0, there exists a conjugation between f and the translation $z \mapsto z + 1$. See [1, p. 35-41]. This conjugation is defined in attracting and repelling "petals", with z_0 contained in the boundary of these petals, as represented in the image below.

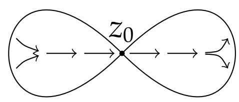

Figure 2.1: Representation of an attracting and a repelling petal for a parabolic fixed point z_0.

The parabolic fixed point has also a basin of attraction defined by the union of the preimages of the attracting petals.

Suppose that z_0 is a fixed point of f, with multiplier

$$\lambda = e^{2\pi i\theta}, \quad \text{with} \quad \theta \in \mathbb{R}\backslash\mathbb{Q}.$$

There are some values of θ for which there exists a conjugation between f and the irrational rotation $z \mapsto \lambda z$. According to Siegel, for θ Diophantine, this conjugation exists. See [1, p. 43]. When the conjugation exists, the domain of definition of the conjugation is called a *Siegel disk*.

2.2 Global theory

From now on we will work with holomorphic functions defined on the *Riemann sphere* $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Riemann sphere can be geometrically described by the stereographic projection.

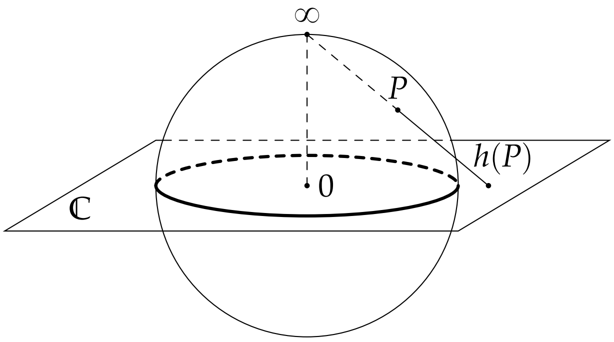

Figure 2.2: Riemann sphere and the stereographic projection h.

Since the stereographic projection yields a homemomorphism between $\hat{\mathbb{C}}$ and the unit sphere, it follows that $\hat{\mathbb{C}}$ is a compact space. The Riemann sphere is a compactification of the complex plane $\mathbb{C}$. To define differentiability at infinity, we can use the parametrization $z \mapsto 1/z$ that maps a neighborhood of ∞ onto a neighborhood of 0.

Julia sets and Fatou sets. It is possible to show that every holomorphic map defined in $\hat{\mathbb{C}}$ is a *rational map* $R = P/Q$, where P and Q are polynomials. The *degree* d of R is defined as the maximum between the degrees of P and Q. Unless otherwise stated, the degree is always assumed to be at least 2.

A family of functions $\mathcal{F} = (f_\lambda)_{\lambda \in \Lambda}$, with $f_\lambda : U \to U$ is *normal* if every sequence of functions $(f_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}$ admits a subsequence that converges on compact sets. Using Arzelá-Azcoli Theorem, a sequence of functions f_n defined in $\hat{\mathbb{C}}$ is normal if, and only if, it is equicontinuous. Another result from Montel states that if a family of functions omits the same three points of $\hat{\mathbb{C}}$, then it is a normal family.

Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with degree at least 2. The *Fatou set* F_R of R is the largest open set in $\hat{\mathbb{C}}$ where the sequence of iterates $(R^n)_{n \in \mathbb{N}}$ is normal. It follows that the sequence of iterates is equicontinuous. Then the basin of attraction of a given periodic point of R is contained in F_R.

The *Julia set* J_R of R is defined as $\hat{\mathbb{C}} \backslash F_R$. By definition, the Julia set is closed.

Proposition 2.1. *Let $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map with degree at least two. Then F_R and J_R are completely invariant:*

$$R(F_R) = F_R = R^{-1}(F_R) \quad \textit{and} \quad R(J_R) = J_R = R^{-1}(J_R).$$

Proof. See [1, p. 56]. □

Proposition 2.2. *For a rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, with degree $d \geqslant 2$, the Julia set J_R is nonempty.*

Proof. See [9, p. 46]. □

Proposition 2.3. *The Julia set of a rational map either has empty interior, or is equal to the whole Riemann sphere. The Julia set J_R is the closure of repelling periodic points of R.*

Proof. For the first statement, see [9, p. 48]. For the second statement, see [1, p. 63] □

Proposition 2.4. *Let R be a rational map with degree at least 2. Let $z \in J_R$. Then*

$$\overline{\bigcup_{n \geqslant 0} R^{-n}(z)} = J_R.$$

Proof. See [1, p. 57]. □

2.2.1 Critical points

Theorem 2.4. *If z_0 is an attracting periodic point of a rational function R with degree at least 2, then the immediate basin of attraction $\mathcal{A}_0(z_0)$ contains at least one critical point.*

Proof. See [9, p. 59]. □

Example 2.1. Consider the quadratic family $f_c(z) = z^2 + c$. It can be shown that if $c \in D(-1, \frac{1}{4})$, the disk centered at -1, with radius $1/4$, then there exists an attracting cycle of period 2. The periodic points are given by

$$z_1 = \frac{-1 + \sqrt{1 - 4(c+1)}}{2} \quad \text{e} \quad z_2 = \frac{-1 - \sqrt{1 - 4(c+1)}}{2}.$$

By Theorem 2.4,

$$\lim_{n \to \infty} f_c^{2n}(0) \cdot f_c^{2n+1}(0) = z_1 \cdot z_2 = c + 1.$$

Theorem 2.5. *If z_0 is a parabolic periodic point of a rational function R with degree at least 2, then each immediate basin of attraction associated with the cycle of z_0 contains a critical point.*

Proof. See [1, p. 60]. □

Example 2.2. The rational function given by

$$h(z) := \frac{3z^2 + 1}{z^2 + 3}$$

has a parabolic fixed point at 1, with multiplier $\lambda = 1$. The function h has two critical points at 0 and ∞. The fixed point 1 has two attracting petals, each invariant by forward iteration under h. One petal is in the direction defined by the vector -1 and the other in the direction defined by 1. The points in $\mathbb{D}$ are mapped into $\mathbb{D}$, so the critical point 0 must converge to -1 through the left side, while ∞ converges to -1 through the right.

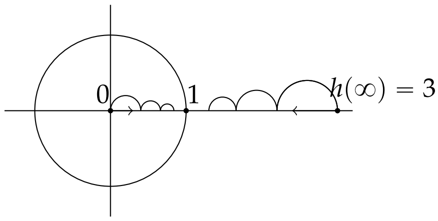

Figure 2.3: Representation of the convergence of the critical points

The postcritical set of a rational function with degree at least 2 is by definition the closure of the set $\{f^n(c) : n > 0, f'(c) = 0\}$.

Theorem 2.6. *Let R be a rational function with degree at least 2. If U is a Siegel disk, then the postcritical set of R contains ∂U.*

Proof. See [1, p. 82]. □

Classification of periodic components. The image of any connected component of the Fatou set F_R is a connected component of F_R. Let U be a connected component of the Fatou set. We have the following possibilities:

- If $R(U) = U$, then U is a *fixed component* of F_R;
- If $R^n(U) = U$, then U is a *periodic component* of F_R;
- If $R^p(R^l(U)) = R^l(U)$, then U a *preperiodic component* of F_R;
- If $R^n(U) \neq U$ for all $n \in \mathbb{N}$, then U is a *wondering domain*.

Theorem 2.7 (Sullivan). *A rational map with degree at least 2 has no wondering domains.*

A *Herman ring* is a connected component of the Fatou set conformally isomorphic to a annulus

$$A(R) = \{z \in \mathbb{C} \mid 1 < |z| < R\},$$

such that f or some iterate of f is conjugated to a irrational rotation on $A(R)$. There are no Herman rings for polynomials.

Theorem 2.8. *Suppose U is a periodic connected component of F_R for a rational map R. Then either*

1. *U contains a attracting periodic point;*
2. *U is parabolic;*
3. *U is a Siegel disk;*
4. *U is a Herman ring.*

Proof. See [9, p. 167]. □

Polynomials. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial of degree $d \geqslant 2$. Using the parametrization $z \mapsto 1/z$ it is possible to show that ∞ is a superattracting fixed point of f, with local degree d. Let $\mathcal{A}(\infty)$ denote the basin of attraction of infinity. The set $K_f = \hat{\mathbb{C}} \backslash \mathcal{A}(\infty)$ is called the *filled-in Julia set*. Note that K_f is closed by definition (it is actually a full compact set, i.e., its complement is connected.).

Proposition 2.5. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a polynomial, then $\partial K_f = J_f = \partial \mathcal{A}(\infty)$.*

Proof. See [1, p. 65]. □

Since ∞ is a superattracting fixed point of f, there exists a map B that conjugates f with $z \mapsto z^d$ in a neighborhood of infinity. This map B is called the Böttcher map. As we have seen, we can extend B through $\mathcal{A}(\infty)$ until we find a critical point of f. If $\mathcal{A}(\infty)$ has no finite critical points, then the extended map $\hat{\mathbb{C}} \backslash K_f \to \hat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ is a conformal isomorphism. It follows that $\hat{\mathbb{C}} \backslash K_f$ is simply connected, which implies that K_f is connected. The reciprocal is true.

Theorem 2.9 (Fatou). *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a polynomial. Then K_f (consequently J_f) is connected if, and only if, all the finite critical points are in K_f. Otherwise K_f has uncountably many connected components.*

Proof. See [1, p. 66-67]. □

The quadratic family. We shall study the quadratic family $f_c(z) = z^2 + c$. We will now use the notation $J_c := J_{f_c}$ and $K_c := K_{f_c}$. The only finite critical point of f_c is 0.

We define the *locus of connectedness* $\mathcal{M}$ as the set of all the parameters $c \in \mathbb{C}$ for which the filled-in Julia set K_c is connected. Equivalently,

$$\mathcal{M} = \{c \in \mathbb{C} \mid 0 \in K_c\}.$$

We call $\mathcal{M}$ the *Mandelbrot set*. The set $\mathcal{M}$ is a compact subset of $\mathbb{C}$.

Chapter 3

Conformal metrics and subhyperbolic maps

3.1 Conformal metrics

Definition 3.1. A *Riemannian metric ds* defined in a open subset of $U \subseteq \mathbb{C}$ is given by an expression of the form

$$ds^2 = g_{11}dx^2 + g_{12}dxdy + g_{22}dy^2,$$

where $[g_{kj}]$ is a positive definite matrix depending smoothly on $z \in U$.

A Riemannian metric is said to be a *conformal metric* if $g_{11} = g_{22}$ and $g_{12} = 0$.

In other words we can define a smooth map ρ positive and nonsingular so that the conformal metric is given by

$$ds = \rho(z)|dz|,$$

where z is a local uniformizing parameter.

A conformal metric is said to be *invariant* under a conformal automorphism $f : U \to U$ if

$$\rho(w)|dw| = \rho(z)|dz|$$

whenever $w = f(z)$. Equivalently,

$$|f'(z)|\frac{\rho(f(z))}{\rho(z)} = 1,$$

for every $z \in U$.

Example 3.1. Consider the open disk $\mathbb{D}$. The conformal automorphisms of $\mathbb{D}$ are of the form

$$\varphi(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z},$$

where $a \in \mathbb{D}$. The conformal metric

$$ds = \frac{2|dz|}{1-|z|^2}$$

is invariant under any conformal automorphism of $\mathbb{D}$. This metric, with density function

$$\rho_{\mathbb{D}}(z) = \frac{2}{1-|z|^2},$$

is called the *hyperbolic metric* of $\mathbb{D}$.

Riemann Surfaces. A holomorphic map $p : U \to V$ between two Riemann surfaces is called a *covering map* if for every $z \in V$ there is an open neighborhood W of z, such that every connected component of $p^{-1}(W)$ is mapped onto W by a conformal isomorphism.

If there exists a covering map $p : \mathbb{D} \to U$, we say that U is a *hyperbolic Riemann surface.*

Let U and V be two hyperbolic Riemann surfaces, with hyperbolic metrics $\rho_U|dz|$ and $\rho_V|dz|$, respectively. We say that a holomorphic function $f : U \to V$ does not increase the hyperbolic metric if

$$|f'(z)| \cdot \frac{\rho_V(f(z))}{\rho_U(z)} \leqslant 1.$$

The map f is a *local isometry* if the equality holds.

Theorem 3.1 (Schwarz-Pick). *Every map $f : U \to V$ between two hyperbolic Riemann surfaces does not increase the hyperbolic metric. The map f is a local isometry if, and only if, f is a covering map.*

Example 3.2. The half plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ is conformaly isomorphic to $\mathbb{D}$ by the map

$$\psi(z) = \frac{z-i}{z+i}.$$

By Schwarz-Pick's Theorem it is possible to show that

$$\rho_{\mathbb{H}}(z) = \frac{1}{y}.$$

The map $z \mapsto e^{iz}$ is a covering map from $\mathbb{H}$ onto the punctured disk $\mathbb{D}^* := \mathbb{D} - \{0\}$. Indeed, every strip of the form

$$\{z = x + iy \in \mathbb{H} \mid 2(k-1)\pi < x < 2k\pi\}$$

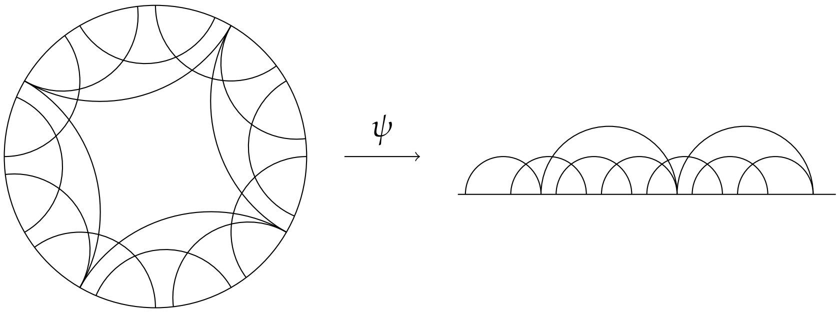

Figure 3.1: Straight lines in the hyperbolic disk and in the hyperbolic upper half-plane.

is mapped homeomorphically onto $\mathbb{D}^*\backslash\mathbb{R}_+$. Using, Schwarz-Pick's Theorem,

$$\rho_{\mathbb{D}*}(z) = \frac{1}{|z|\log(1/|z|)}.$$

Another hyperbolic metric that can be computed using Schwarz-Pick's Theorem is the hyperbolic metric of the annulus $A(R) = \{z \in \mathbb{C} \mid 1 < |z| < R\}$. The map $z \mapsto z^{\frac{\log R}{\pi i}}$ is a covering map from $\mathbb{H}$ onto $A(R)$. Indeed, let $z = re^{i\theta}$, with $\theta \in (0, \pi)$. Then

$$\left|z^{\frac{\log R}{\pi i}}\right| = R^{\frac{\theta}{\pi}}.$$

We can compute the hyperbolic metric of $A(R)$:

$$\rho_{A(R)}(z) = \frac{\pi/\log R}{\sin(\pi\log|z|/\log R)|z|}.$$

Example 3.3. Using the stereographic projection, we can define the spherical metric, given by

$$ds = \frac{2|dz|}{1+|z|^2}.$$

The set of conformal automorphisms of $\hat{\mathbb{C}}$ is given by the Möbius transformations

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The spherical metric is not invariant under every Möbius transformation, but it is invariant under $z \mapsto 1/z$.

3.2 Hyperbolic and subhyperbolic maps

Definition 3.2. A rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with degree at least 2 is *hyperbolic* if there exists a conformal metric $\rho(z)|dz|$, defined in a neighborhood of the Julia set J_R, and $k > 1$ satisfying

$$|R'(z)|\frac{\rho(f(z))}{\rho(z)} \geqslant k,$$

for every $z \in J_R$.

We also say that R is *expanding* on J_R for some conformal metric ρ.

Theorem 3.2. *A rational map R of degree $d \geqslant 2$ is hyperbolic if and only if the orbit of every critical point of R converges to an attracting cycle.*

Proof. See [9, p. 206] □

Example 3.4. Let f_c be a function in the quadratic family that admits an attracting periodic point other than ∞. It follows that f_c is hyperbolic. For every $c \notin \mathcal{M}$, the map f_c is hyperbolic.

Definition 3.3. A conformal metric on a Riemann surface, given by $\rho(z)|dz|$ for a local uniformizing parameter z, is an *orbifold metric* if it is smooth and nonzero except for a locally finite set of points $a_1, a_2, \ldots$, where ρ blows up in such a way that for each a_i, there is a number $\nu_i \geqslant 2$ such that if we set $z(w) = a_i + w^{\nu_i}$, the induced metric

$$\rho(z(w)) \left|\frac{dz}{dw}\right| \cdot |dw| = \gamma(w)|dw|$$

is smooth and nonsingular in some neighborhood of 0.

Remark. From the definition, in a neighborhood of 0, we have

$$\rho(z(w))\nu_i|w^{\nu_i - 1}| = \gamma(w),$$

where $w \mapsto \gamma(w)$ is smooth and $\gamma(0) > 0$. Then,

$$\rho(z) = \frac{\gamma(\sqrt[\nu_i]{z - a_i})}{\nu_i|z - a_i|^{\frac{\nu_i - 1}{\nu_i}}},$$

in a neighborhood of a_i. Set

$$\beta := \frac{\nu_i - 1}{\nu_i} < 1.$$

Definition 3.4. A rational map $R : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with degree at least two is said to be *subhyperbolic* if there exist $\lambda > 1$ and an orbifold metric ρ defined on a neighborhood of J_R, with finitely many singularities in the Julia set, such that $\|R'(z)\|_\rho > \lambda$, whenever $z \in J_R$.

Theorem 3.3. *A rational map R with degree at least 2 is subhyperbolic if, and only if, every critical point in the Julia set is preperiodic and every critical point in the Fatou set converges to an attracting cycle.*

Proof. See [9, p. 211] □

3.2.1 Misiurewicz points

A parameter $c \in \mathcal{M}$ is a *Misiurewicz point* if the orbit of 0 under $f_c(z) = z^2 + c$ is strictly preperiodic. There are minimal integers p, l such that

$$f_c^p((f_c^l(c))) = f_c^l(c).$$

Since 0 is a critical point, it follows that $l > 1$. Misiurewicz parameters are in $\partial\mathcal{M}$ and form a dense subset of the boundary of the Mandelbrot set. A simple example of a Misiurewicz parameter for the quadratic family is the value $c = -2$. Indeed, the orbit of 0 under f_{-2} is $0 \mapsto -2 \mapsto 2 \mapsto 2$.

Example 3.5. It follows directly from Theorems 3.2 and 3.3 that every hyperbolic rational function is subhyperbolic. The converse statement is not true. If $c \in \partial\mathcal{M}$ is a Misiurewicz point, then f_c is subhyperbolic but is not hyperbolic.

Proposition 3.1. *Let c_0 be a Misiurewicz point in the quadratic family. Then*

1. *The cycle to which* 0 *is eventually mapped is repelling.*

2. *The filled-in Julia set K_{c_0} has no interior.*

Proof. Since the orbit of the only finite critical point is preperiodic, it follows that f_{c_0} is subhyperbolic. For all $z \in J_{c_0}$ except for a finite number of points, we have

$$|f'_{c_0}(z)|\frac{\rho(f_{c_0}(z))}{\rho(z)} \geqslant k > 1,$$

for $\rho(z)|dz|$ an orbifold metric. Set $z_n := f_{c_0}^n(z)$. Then

$$|(f_{c_0}^n)'(z)|\frac{\rho(f_{c_0}^n(z))}{\rho(z)} = \prod_{i=0}^{n-1} |f'_{c_0}(z_i)|\frac{\rho(f_{c_0}(z_i))}{\rho(z_i)} \geqslant k^n > 1.$$

Let $\alpha = f_{c_0}^l(c_0)$ be the first periodic point of period p in the orbit of 0. The metric $\rho(z)|dz|$ is constructed in such a way that the postcritical points are the exceptional points. So we have that α is approximated by a sequence of points $\alpha_m \in J_{c_0}$ that satisfy

$$|(f_{c_0}^p)'(\alpha_m)|\frac{\rho(f_{c_0}^p(\alpha_m))}{\rho(\alpha_m)} \geqslant k > 1. \tag{3.1}$$

Using the remark above, for m sufficiently large,

$$|(f_{c_0}^p)'(\alpha_m)|\frac{\rho(f_{c_0}^p(\alpha_m))}{\rho(\alpha_m)} = |(f_{c_0}^p)'(\alpha_m)|\frac{\gamma\left(\sqrt[v_i]{f_{c_0}^p(\alpha_m) - \alpha}\right)}{\gamma(\sqrt[v_i]{\alpha_m - \alpha})}\left|\frac{\alpha_m - \alpha}{f_{c_0}^p(\alpha_m) - \alpha}\right|^{\beta} \xrightarrow[m\to\infty]{} |(f_{c_0}^p)'(\alpha)|^{1-\beta} \tag{3.2}$$

Combining (3.1) with (3.2) we get

$$|(f_{c_0}^{p})'(\alpha)|^{1-\beta} \geqslant k > 1.$$

Thus α is repelling.

To prove that K_{c_0} has no interior, we just need to show that there can't be a Fatou component other than $\mathcal{A}(\infty)$. Using the Theorem 2.8 of classification of Fatou components, we conclude that another Fatou component either contains an attracting periodic point, a parabolic periodic point or a Siegel disk. But there cannot be an attracting periodic point besides ∞, or a parabolic periodic point because in these cases, from Theorems 2.4 and 2.5, we would have the orbit of 0 converging to the periodic point. Since the eventual cycle of 0 is repelling, this cannot happen. There cannot be a Siegel disk, since the postcritical points make a finite set. The postcritical set does not contain the boundary of the disk (Theorem 2.6). We conclude that

$$K_{c_0} = J_{c_0}.$$

□

Chapter 4

Transversality property

The main goal of this section is Theorem 4.3 (transversality), which is a key property for establishing the similiarity between the Mandelbrot set and the Julia set.

Proper maps. Recall that a Riemann surface is a connected complex manifold with dimension 1.

Definition 4.1. *Let X and Y be Riemann surfaces. A nonconstant holomorphic map $f : X \to Y$ is* proper *if the pre-image of any compact subset of Y is a compact subset of X.*

If $f : X \to Y$ is proper then the pre-image of any point in Y is a finite subset of X (otherwise f would be constant, by the Identity Theorem). The set of critical points $B = \{c \in X : f'(c) = 0\}$ is locally finite. The image $R = f(B)$ is the set of ramification points. Let $B' = f^{-1}(R)$. It is easy to show that

$$f : X - B' \to Y - R$$

is a covering map between connected manifolds with finite degree d. For this reason, f may also be called a *d-fold branched covering.*

4.1 Green's function

Let $f_c(z) = z^2 + c$. Using Theorem 2.3, we can define the Boettcher map B_c in a neighborhood of infinity. Let V be the maximum domain of the extension of B_c. If $c \in \mathcal{M}$, then $V = \hat{\mathbb{C}} \backslash K_c$ and $B_c : \hat{\mathbb{C}} \backslash K_c \to \hat{\mathbb{C}} \backslash \overline{\mathbb{D}}$ will be a conformal isomorphism. On the contrary, if $c \in \mathbb{C} \backslash \mathcal{M}$ there exists a real number $\rho > 1$ such that $B_c : V \to \hat{\mathbb{C}} \backslash \overline{\mathbb{D}}_\rho$ is a conformal isomorphism, satisfying $0 \in B^{-1}(\partial \mathbb{D}_\rho)$.

If $z \notin V$, let $\tilde{n}$ be large enough so that $f_c^{\tilde{n}}(z) \in V$. We can define the *Green's function* $G_c : \mathbb{C} \backslash K_f \to \mathbb{R}^+$ by

$$G_c(z) = \log |B_c(z)|,$$

if $z \in V$ and

$$G_c(z) = \frac{\log |B_c(f_c^{\tilde{n}}(z))|}{2^{\tilde{n}}},$$

if $z \notin V$. The Green's function has some properties.

1. G_c is positive harmonic;

2. $G_c(f_c(z)) = 2G_c(z)$;

3. $G_c(z) \to 0$ when $d(z, K_c) \to 0$.

To prove 3, suppose that $c \in \mathcal{M}$. (The case $c \in \mathbb{C} \backslash \mathcal{M}$ is analogous). We will use the fact that

$$B_c(z) = \lim_{n \to \infty} f_c^n(z)^{2^{-n}}.$$

Then

$$G_c(z) = \lim_{n \to \infty} \frac{1}{2^n} \log |f_c^n(z)|.$$

Since $d(z, K_c) \to 0$, the orbit of z is getting closer to a bounded orbit. Hence $\log |f_c^n(z)|$ will be dominated by $1/2^n$ and $G_c(z) \to 0$.

Proposition 4.1. *A point $z \in \mathbb{C} \backslash K_f$ is critical for G_c if, and only if, $f_c^n(z)$ is critical for f_c, for some $n \geqslant 0$.*

Proof. Suppose $f_c^n(z)$ is a critical point of f_c for some $n \geqslant 0$. We have

$$G_c(f_c^{n+1}(c)) = 2^{n+1} G_c(z).$$

Then

$$G_c'(z) = \frac{1}{2^{n+1}} G_c'(f_c^{n+1}(z)) \cdot f_c'(f_c^n(z)) \cdot (f_c^n(z))' = 0.$$

Conversely, suppose z is critical of G_c. Then

$$G_c'(f_c^n(z)) = 0,$$

for every $n \geqslant 0$. Every point of the sequence $f^n(z)$ cannot be a critical of G_c, because the sequence will be eventually contained in V, where B_c is a conformal isomorphism. Hence, for some n, the point $f_c^n(z)$ must be critical for f_c.

□

This proposition gives us explicitly a domain V where B_c is defined, when $c \in \mathbb{C} \backslash \mathcal{M}$. The map $B_c : \mathbb{C} \backslash L_c \to \mathbb{C} \backslash \overline{\mathbb{D}}_\rho$, where $L_c = \{z \in \mathbb{C} \mid G_c(z) \leqslant G_c(0)\}$ and $\rho = e^{G_c(0)}$, is a conformal isomorphism.

We want to extend G_c continuously to $\mathbb{C}$. Since $G_c(z) \to 0$ as $d(z, K_c) \to 0$, we define $G_c(z) = 0$ for $z \in K_c$.

Lemma 4.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a monic polynomial of degree d given by*

$$f(z) = z^d + a_{d-1} z^{d-1} + \cdots + a_0.$$

Let

$$R^*(f) := 1 + |a_{d-1}| + \cdots + |a_0|.$$

If $z_0 \in \mathbb{C}$ satisfies $|z_0| > R^*(f)$, then the orbit of z_0 under f tends to infinity.

Proof. We are going to prove that

$$f^n(z_0) \to \infty \quad \text{if} \quad n \to \infty.$$

We have

$$\begin{aligned} |f(z_0)| &= |z_0|^d \left| 1 + \frac{a_{d-1}}{z_0} + \cdots + \frac{a_0}{z_0^d} \right| \\ &\geqslant |z_0|^d \left(1 - \frac{|a_{d-1}|}{|z_0|} - \cdots - \frac{|a_0|}{|z_0|^d} \right) \\ &\geqslant |z_0|^d \left(1 - \frac{|a_{d-1}| + \cdots + |a_0|}{|z_0|} \right) \\ &> \frac{|z_0|^d}{R^*(f)}. \end{aligned}$$

Inductively, we come to

$$|f^n(z_0)| > \frac{|z_0|^{d^n}}{R^*(f)^{(n-1)\cdot d+1}} \to \infty \quad \text{when} \quad n \to \infty.$$

□

Proposition 4.2. *Let*

$$\mathcal{K} := \{(c, z) \mid z \in K_c\}.$$

Then $\mathcal{K}$ is closed and the map $(c, z) \mapsto G_c(z)$ is continuous.

Proof. For every $c \in \mathbb{C}$ let $R^*(c) = 1 + |c|$ and

$$V_1 := \{(c, z) \mid R^*(c) < |z|\}.$$

Define the map $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $F(c,z) = (c, f_c(z))$. We have

$$(\mathbb{C} \times \mathbb{C})/\mathcal{K} = \bigcup_{n=0}^{\infty} F^{-n}(V_1).$$

In fact, suppose $(c_0, z_0) \in (\mathbb{C} \times \mathbb{C})/\mathcal{K}$. This means $z_0 \notin K_{c_0}$. For n big enough,

$$|f_{c_0}^n(z_0)| > R^*(c_0).$$

Hence

$$F^n(c_0, z_0) = (c_0, f_{c_0}^n(z_0)) \in V_1.$$

Conversely, suppose $F^n(c_0, z_0) \in V_1$ for some $n > 0$. From Lemma 4.1, the orbit of $f_{c_0}^n(z_0)$ and, consequently, the orbit of z_0 tends to infinity. Then $z_0 \notin K_c$. Thus, since V_1 is open and F is continuous, then $\mathcal{K}$ is closed.

Let V_0 be the set

$$V_0 := \left\{(c,z) \mid R^*(c)^2 < |z|\right\}$$

and define $\phi : V_1 \to \mathbb{C} \times \mathbb{C}$ by $\phi(c,z) = (c, B_c(z))$. If $(c,w) \in V_0$, then $|w| > R^*(c)^2$. There exists z, with $|z| > R^*(c)$ such that $z^2 = w$. Then

$$|B_c^{-1}(w)| = |B_c^{-1}(z^2)| = |f_c(B_c^{-1}(z))| > R^*(c).$$

Hence there is $V \subset V_1$ such that ϕ is an isomorphism between V and V_0. The map $(c,z) \mapsto G_c(z)$ is continuous in V_1, because if $(c,z) \in V_1$, then $G_c(z) > G_c(0)$. Hence the Böttcher map B_c is well defined and depends analytically on c. The continuity holds for each preimage $F^{-n}(V_1)$ as well, since

$$G_c(z) = \frac{G(f_c^n(z))}{2^n}.$$

We just need to prove continuity on $\mathcal{K}$. If $(c,z) \in \mathcal{K}$, then $G_c(z) = 0$. Then we have to prove that for every $\varepsilon > 0$ the set

$$W_\varepsilon := \{(c,z) \mid G_c(z) < \varepsilon\}$$

is an open set. It is sufficient to prove that, for any bounded open subset $\Lambda \subset \mathbb{C}$, the intersection $W_\varepsilon \cap (\Lambda \times \mathbb{C})$ is open in $\Lambda \times \mathbb{C}$. We can assume $\varepsilon < 1$, since we are interested in neighborhoods of $\mathcal{K}$. Let $N > 0$ be big enough such that

$$2^N \varepsilon > \sup_{c \in \Lambda} R^*(c)^2. \tag{4.1}$$

Then

$$\Lambda \times \mathbb{C} \backslash (W_\varepsilon \cap (\Lambda \times \mathbb{C})) = (\Lambda \times \mathbb{C}) \cap W_\varepsilon^c,$$

where $W_\varepsilon^c := \{(c,z) \mid G_c(z) \geqslant \varepsilon\}$. We have

$$(\Lambda \times \mathbb{C}) \cap W_\varepsilon^c = F^{-N}((\Lambda \times \mathbb{C}) \cap W_{2^N\varepsilon}^c).$$

Indeed, if $(c,z) \in (\Lambda \times \mathbb{C}) \cap W_\varepsilon^c$, then

$$G_c(f_c^N(z)) = 2^N G_c(z) \geqslant 2^N \varepsilon.$$

Let

$$(c,z) \in (\Lambda \times \mathbb{C}) \cap W_{2^N\varepsilon}^c.$$

Since $G_c(z) \geqslant 2^N\varepsilon > R^*(c)^2$, we have $|B_c(z)| \geqslant 2^N\varepsilon$. Hence

$$F^{-N}((\Lambda \times \mathbb{C}) \cap W_{2^N\varepsilon}^c) \subset F^{-N}(\phi^{-1}(\{(c,z) \mid 2^N\varepsilon \leqslant |z|\})).$$

On the other hand if (c,z) satisfies $|f_c^N(B_c(z))| \geqslant 2^N\varepsilon$, then

$$G_c(f_c^N(z)) = 2^N G_c(z) \geqslant 2^N \varepsilon.$$

We conclude that

$$\Lambda \times \mathbb{C} \backslash (W_\varepsilon \cap (\Lambda \times \mathbb{C})) = F^{-N}(\phi^{-1}(\{(c,z) \mid 2^N\varepsilon \leqslant |z|\})),$$

which is a closed set.

□

The Böttcher map B_c is well defined for every z such that $G_c(z) > G_c(0)$. Then for every $c \in \mathbb{C}\backslash\mathcal{M}$, set

$$\Psi(c) = B_c(c).$$

The map Ψ is well defined, since $G_c(c) = 2G_c(0) > G_c(0)$.

Theorem 4.1. *The map $\Psi : \hat{\mathbb{C}}\backslash\mathcal{M} \to \hat{\mathbb{C}}\backslash\overline{\mathbb{D}}$ is a conformal isomorphism.*

Proof. Ψ is holomorphic because

$$L = \{(c,z) \mid z \in L_c\} = \{(c,z) \mid G_c(z) \leqslant G_c(0)\}$$

is closed and the map $(c,z) \mapsto B_c(z)$ is holomorphic in $\mathbb{C}^2\backslash L$.

We have that

$$\begin{aligned} f_c^n(c) &= (f_c^{n-1}(c)^2)\left(1 + \frac{c}{(f_c^{n-1}(c))^2}\right) \\ &= (f_c^{n-2}(c))^{2^2}\left(1 + \frac{c}{(f_c^{n-2}(c))^2}\right)^2\left(1 + \frac{c}{(f_c^{n-1}(c))^2}\right). \end{aligned}$$

Continuing the argument inductively, we get

$$f_c^n(c) = c^{2^n} \cdot \left(1 + \frac{c}{c^2}\right)^{2^{n-1}} \cdot \ldots \cdot \left(1 + \frac{c}{(f_c^{n-1}(c))^2}\right).$$

Recall that

$$B_c(c) = \lim_{n\to\infty} f_c^n(c)^{2^{-n}}.$$

Then

$$\frac{\Psi(c)}{c} = \prod_{n=0}^{\infty} \left(1 + \frac{c}{(f_c^n(c))^2}\right)^{2^{-n-1}}. \tag{4.2}$$

For $|c| \geqslant 4$ the sequence of iterates of the critical point tends uniformly to infinity. So the infinite product converges uniformly. Moreover, each factor of the product tends to 1 as c goes to infinity. Hence, we have $\Psi(c)/c \to 1$, if $c \to \infty$ and Ψ has a simple pole at ∞. Thus, we can extend Ψ to $\hat{\mathbb{C}}\backslash\mathcal{M}$ by setting $\Psi(\infty) = \infty$.

This extension is proper, since $c \to c_0 \in \partial\mathcal{M}$ implies $G_c(c) \to G_{c_0}(c_0) = 0$, which gives us $\Psi \to \partial\mathbb{D}$. The proper map Ψ has a degree. Since $\Psi^{-1}(\infty) = \infty$ with multiplicity 1, the degree of Ψ is 1 and we have the desired result. □

Corollary 4.2. *The Mandelbrot set $\mathcal{M}$ is connected.*

4.2 External rays

Recall that for $c \in \mathcal{M}$, the filled-in Julia set K_c is connected and the Böttcher map $B_c : \mathbb{C}\backslash K_c \to \mathbb{C}\backslash\overline{\mathbb{D}}$ is a conformal isomorphism. For $\theta \in [0, 1)$, the *external ray* $\mathcal{R}(K_c, \theta)$ of K_c with argument θ is defined by

$$\mathcal{R}(K_c, \theta) = B_c^{-1}(\{R \cdot e^{2\pi i\theta}\}_{R>1}).$$

We say that $\mathcal{R}(K_c, \theta)$ lands at $a \in K_c$ if the limit below exists

$$\lim_{R\to 1^+} B_c^{-1}(R \cdot e^{2\pi i\theta}) = a.$$

In this case we also say that a has an *external argument* θ.

Using the fact that B_c conjugates f_c with $z \mapsto z^2$, if $z \in \mathcal{R}(K_c, \theta)$, we have $f_c(z) \in \mathcal{R}(K_c, 2\theta)$.

Let c_0 be a Misiurewicz point. Let l, p be the minimal integers that satisfy

$$f_{c_0}^p((f_{c_0}^l(c_0))) = f_{c_0}^l(c_0).$$

We already know, from Proposition 3.1, that the periodic point $f_{c_0}^l(c_0)$ is repelling. As we did before, define an implicit function $\alpha_0 : W \to \mathbb{C}$, from a neighborhood W of c_0

into a neighborhood of the periodic point in such a way that for all $c \in W$, the image $\alpha_0(c)$ is a repelling periodic point of period p and

$$\alpha_0(c_0) = f_{c_0}^l(c_0).$$

Theorem 4.3 (Transversality). *The function $c \mapsto f_c^l(c) - \alpha_0(c)$ has a simple root at c_0.*

Proof. From Proposition 3.1, we have $K_{c_0} = J_{c_0}$, which is connected. The Boettcher map B_c is defined at every point $z \in \mathbb{C} \backslash K_c$. Then, every point in K_{c_0} has an external argument. Let θ be an external argument of c_0 and let $\phi \equiv 2^l\theta \bmod 1$. The ray $\mathcal{R}(K_{c_0}, \phi)$ lands at $\alpha_0(c_0)$. Indeed

$$\lim_{R\to 1^+} B_{c_0}^{-1}(R \cdot e^{2\pi i\phi}) = \lim_{R\to 1^+} f_{c_0}^l(B_{c_0}^{-1}(R \cdot e^{2\pi i\theta})) = \alpha_0(c_0).$$

Consider the rays $\mathcal{R}(K_c, \phi)$ for $c \in W$. Close to $\alpha_0(c)$, we can use the inverse mapping theorem to define a local inverse f_c^{-p} in a neighborhood of $\alpha_0(c)$. Then

$$\lim_{R\to 1^+} f_c^{-p} \circ B_c^{-1}(R \cdot e^{2\pi i\phi}) = \lim_{R\to 1^+} B_c^{-1}(R \cdot e^{2\pi i\phi/2^p}) = \lim_{R\to 1^+} B_c^{-1}(R \cdot e^{2\pi i\phi}). \tag{4.3}$$

On the other hand, let φ_c be the Koenigs linearization map defined in a neighborhood of $\alpha_0(c)$, conjugating f_c^p with $z \mapsto \lambda(c)z$. We have

$$\lim_{R\to 1^+} f_c^{-p} \circ B_c^{-1}(R \cdot e^{2\pi i\phi}) = \varphi_c^{-1}\left(\frac{1}{\lambda(c)} \cdot \varphi_c\left(\lim_{R\to 1^+} B_c^{-1}(R \cdot e^{2\pi i\phi})\right)\right). \tag{4.4}$$

From (4.3) and (4.4), the ray $\mathcal{R}(K_c, \phi)$ lands at a point that fixes the composition $z \mapsto \varphi_c^{-1}(\frac{1}{\lambda(c)}\varphi_c(z))$. Then it must be $\alpha_0(c)$.

For $t \geqslant 0$, let $\alpha_t(c)$ to be the point in $\mathcal{R}(K_c, \phi)$ that satisfies

$$G_c(\alpha_t(c)) = t.$$

Then

$$B_c(\alpha_t(c)) = e^t \cdot e^{2\pi i\phi}. \tag{4.5}$$

The map $(c, t) \mapsto \alpha_t(c)$ depends continuously on t and c and depends analytically on c, for each fixed t. The equality (4.5) and the fact that the rays $\mathcal{R}(K_c, \phi)$ land at $\alpha_0(c)$ also gives us that $\alpha_t(c)$ tends to $\alpha_0(c)$ as $t \to 0$.

We cannot have $\alpha_0(c) - f_c^l(c) = 0$ for every $c \in W$, or we would have $f_c^{l+p} \equiv f_c^l$. For c close to c_0, there is $k \geqslant 1$ and $a_k \neq 0$ such that

$$\alpha_0(c) - f_c^l(c) = a_k(c - c_0)^k + \mathcal{O}(|c - c_0|^{k+1}).$$

For small values of t, the equation

$$\alpha_t(c) - f_c^l(c) = 0 \tag{4.6}$$

has k solutions $c_1(t), ..., c_k(t)$, counting with multiplicity and these solutions depend continuously on t. They also tend to c_0 as $t \to 0$. In particular, by definition of $\alpha_t(c)$, it follows that $f^l_{c_i(t)}(c_i(t)) \in \mathcal{R}(K_{c_i(t)}, \phi)$. Since $(f^l_{c_0})'(c_0) \neq 0$, we can assume that $(f^l_{c_i(t)})'(c_i(t)) \neq 0$ and apply the appropriate branch of $f^{-l}_{c_i(t)}$, so that we get $c_i(t) \in \mathcal{R}(K_{c_i(t)}, \theta)$.

We have

$$B_{c_i(t)}(f^l_{c_i(t)}(c_i(t))) = B_{c_i(t)}(\alpha_t(c_i(t))) = e^t \cdot e^{2\pi i\phi}.$$

Since $c_i(t) \in \mathcal{R}(K_{c_i(t)}, \theta)$, we can apply the appropriate branch of the 2^lth root to get

$$\Psi(c_i(t)) = B_{c_i(t)}(c_i(t)) = e^{t/2^l} \cdot e^{2\pi i\theta}.$$

Since Ψ is one-to-one, the solutions $c_i(t)$ coincide. Call the common value $c(t)$. Then for c close to $c(t)$

$$\alpha_t(c) - f^l_c(c) = O(|c - c(t)|^k).$$

For values of z close to $\alpha_t(c)$, we have

$$B_c(z) = e^t \cdot e^{2\pi i\phi} + O(|z - \alpha_t(c)|).$$

Thus

$$\begin{aligned}\Psi(c)^{2^l} &= B_c(f^l_c(c)) \\ &= e^t \cdot e^{2\pi i\phi} + O(|f^l_c(c) - \alpha_t(c)|) \\ &= e^t \cdot e^{2\pi i\phi} + O(|c - c(t)|^k).\end{aligned}$$

Then Ψ is a conformal isomorphism. Hence the map $\Psi(c)^{2^l}$ has non-zero derivative at $c = c(t)$. We conclude that $k = 1$ and we have the desired result.

□

Chapter 5

Similarity at Misiurewicz points

5.1 Hausdorff- Chabauty distance and similarity

Definition 5.1 (Hausdorff distance)**.** Let A and B be two compact subsets of $\mathbb{C}$. The *semi-distance* from A to B is given by

$$\delta(A, B) = \sup_{x \in A} d(x, B).$$

and the *Hausdorff distance* between A and B is defined as

$$d(A, B) = \sup(\delta(A, B), \delta(B, A)).$$

The *Hausdorff distance* is an useful tool to compare compact subsets of $\mathbb{C}$. But, for what we are aiming for, we need to be able to compare two closed subsets of $\mathbb{C}$.

Let A be a closed subset of $\mathbb{C}$ and $r > 0$. We define the compact subset

$$A_r := \left(A \cap \overline{\mathbb{D}}_r\right) \cup \partial \mathbb{D}_r.$$

The disc $\mathbb{D}_r$ centered in 0, with radius r, works as a window, through which we will compare sets around 0. If we want to compare a set A around any given point $a \in A$ we use a translation $\tau_{-a} : z \to z - a$ and analyse the compact set $(\tau_{-a} A)_r$.

Definition 5.2 (Hausdorff-Chabauty distance). Let A, B be two closed subsets of $\mathbb{C}$ and $r > 0$. We define the *Hausdorff-Chabauty distance* between A and B to be

$$d_r(A, B) = d(A_r, B_r)$$

where d is the *Hausdorff distance*.

Lemma 5.1. *Let A, B be two closed subsets of $\mathbb{C}$ and fix $r > 0$. We have*

$$d(a, B_r) \leqslant d(a, B) \leqslant d(a, B \cap \overline{\mathbb{D}}_r), \quad \textit{for every} \quad a \in A \cap \overline{\mathbb{D}}_r \tag{5.1}$$

and

$$\delta(A_r, B_r) \leqslant \sup_{a \in A \cap \overline{\mathbb{D}}_r} d(a, B) \leqslant d(A \cap \overline{\mathbb{D}}_r, B \cap \overline{\mathbb{D}}_r) \tag{5.2}$$

.

Proof. The geometric idea for (5.1) is pretty simple. We can see on the figure below an example where the inequalities hold in a strict fashion.

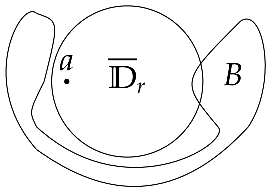

Figure 5.1: Example of restrict inequalities.

Let $a \in A \cap \overline{\mathbb{D}}_r$. We have

$$d(a, B_r) = \inf_{x \in B_r} d(a, x).$$

We know that $B_r = \left(B \cap \overline{\mathbb{D}}_r\right) \cup \partial\mathbb{D}_r$. As $a \in \overline{\mathbb{D}}_r$, it follows that $d(a, \partial\mathbb{D}_r) \leqslant d(a, B \backslash \overline{\mathbb{D}}_r)$. Thus,

$$\inf_{x \in B_r} d(a, x) \leqslant \inf_{x \in (B \cap \overline{\mathbb{D}}_r) \cup (B \backslash \overline{\mathbb{D}}_r)} d(a, x) = d(a, B).$$

For the second inequality, note that $B \cap \overline{\mathbb{D}}_r \subseteq B$. Then,

$$d(a, B) \leqslant d(a, B \cap \overline{\mathbb{D}}_r).$$

The inequalities in (5.2) are solved by first noting that $d(x, B_r) = 0$, for every $x \in \partial\mathbb{D}_r$. Hence

$$\begin{aligned}
\delta(A_r, B_r) &= \sup_{a \in A_r} d(a, B_r) \\
&= \sup_{a \in A \cap \overline{\mathbb{D}}_r} d(a, B_r) \\
&\leqslant \sup_{a \in A \cap \overline{\mathbb{D}}_r} d(a, B).
\end{aligned}$$

For (5.2), we just use (5.1):

$$\sup_{a \in A \cap \overline{\mathbb{D}}_r} d(a, B) \leqslant \sup_{a \in A \cap \overline{\mathbb{D}}_r} d(a, B \cap \overline{\mathbb{D}}_r) = \delta(A \cap \overline{\mathbb{D}}_r, B \cap \overline{\mathbb{D}}_r).$$

Since

$$d(A \cap \overline{\mathbb{D}}_r, B \cap \overline{\mathbb{D}}_r) = \max\left\{\delta(A \cap \overline{\mathbb{D}}_r, B \cap \overline{\mathbb{D}}_r), \delta(B \cap \overline{\mathbb{D}}_r, A \cap \overline{\mathbb{D}}_r)\right\},$$

we have the desired result.

$\square$

For the Definitions 5.3 and 5.4, assume $\varrho = |\varrho|e^{i\theta}$, with $|\varrho| > 1$ and $0 \leqslant \theta < 2\pi$.

Definition 5.3. A closed subset $B \subseteq \mathbb{C}$ is *ϱ-self-similar* about 0 if there exists $r > 0$ such that

$$(\varrho B)_r = B_r.$$

We also say that B is self-similar about 0 with scale ϱ.

Definition 5.4. A closed subset $A \subseteq \mathbb{C}$ is *asymptotically ϱ-self-similar* about a point $x \in A$ if there is $r > 0$ and a closed set B such that

$$(\varrho^n \tau_{-x} A)_r \to B_r \quad \text{while} \quad n \to \infty$$

for the *Hausdorff distance*. The set B is called the limit model of A at x.

Definition 5.5. Two closed sets $A, B \subseteq \mathbb{C}$ are *asymptotically similar* about 0 if there is $r > 0$ such that

$$\lim_{t \in \mathbb{C},\, t \to \infty} d_r(tA, tB) = 0.$$

Proposition 5.1. *Let $A \subseteq \mathbb{C}$ be a closed set asymptotically ϱ-self-similar about $x \in A$ with limit model $B \subseteq \mathbb{C}$. Then B is ϱ-self-similar.*

Proof. We know that for some $r > 0$

$$\left(\varrho^n \tau_{-x} A \cap \overline{\mathbb{D}}_r\right) \cup \partial\mathbb{D}_r \to \left(B \cap \overline{\mathbb{D}}_r\right) \cup \partial\mathbb{D}_r \quad \text{while} \quad n \to \infty.$$

By intersecting both sides of the limit with $\overline{\mathbb{D}}_s$, with $0 < s \leqslant r$, we get

$$\varrho^n \tau_{-x} A \cap \overline{\mathbb{D}}_s \to B \cap \overline{\mathbb{D}}_s \quad \text{while} \quad n \to \infty$$

Since $|\varrho| > 1$

$$\begin{aligned}
(\varrho B)_r &= \left(\varrho B \cap \overline{\mathbb{D}}_r\right) \cup \partial\mathbb{D}_r \\
&= \varrho\left(\left(B \cap \overline{\mathbb{D}}_{r/|\varrho|}\right) \cup \partial\mathbb{D}_{r/|\varrho|}\right) \\
&= \varrho\left(\lim_{n\to\infty}[\left(\varrho^n \tau_{-x} A \cap \overline{\mathbb{D}}_{r/|\rho|}\right) \cup \partial\mathbb{D}_{r/|\rho|}]\right) \\
&= \lim_{n\to\infty}\left[\left(\varrho^{n+1}\tau_{-x} A \cap \overline{\mathbb{D}}_r\right) \cup \partial\mathbb{D}_r\right] \\
&= B_r.
\end{aligned}$$

$\square$

Example 5.1. The disk $\overline{\mathbb{D}}_r$ is self-similar, for every $0 < s \leqslant r$. A line is self-similar, for every $r > 0$ and a segment is self-similar, for $0 < r \leqslant l/2$, where l is the length of the segment.

Example 5.2. A regular curve is asymptotically similar to its tangent line about any given point.

Example 5.3. The spiral $S(\lambda) = \left\{e^{\lambda x} \mid x \in \mathbb{R}\right\}$ is e^{λ}-self-similar about 0 if $|e^{\lambda}| > 1$, for every $r > 0$. Indeed, this fact comes directly from the properties of the exponential. If $z \in e^{\lambda} S(\lambda)$, then $z = e^{\lambda} \cdot e^{\lambda x}$, for some $x \in \mathbb{R}$, which implies that $z = e^{\lambda(x+1)} \in S(\lambda)$. Reciprocally, if $z \in S(\lambda)$, then $z = e^{\lambda x}$, for some $x \in \mathbb{R}$. Then $z = e^{\lambda} \cdot e^{\lambda(x-1)} \in e^{\lambda} S(\lambda)$. Note that the argument for the self-similarity of $S(\lambda)$ is independent of the disk used.

Example 5.4. The Cantor set C is self-similar about any rational point. There is a bijection $\sigma : C \to \Sigma$ between C and the space of sequences $\Sigma := \{(s_i)_{i\in\mathbb{N}} \mid s_i \in \{0, 2\}, \forall i \in \mathbb{N}\}$, where σ is known as the itinerary map. Also, a point $x \in C$ is rational if and only if the itinerary of x is eventually periodic, i.e if $\sigma(x) = (t_1, t_2, ...)$ there is l and k such that $t_n = t_{n+k}$ for every $n \geqslant l$. The number k is minimal and is called the eventual period of x.

Proposition 5.2. *The Cantor set C is 3^p-self-similar about $x \in C$ if and only if x is rational and the eventual period of x divides p.*

Proof. Let $x \in C$ be a rational number, let $l, k \in \mathbb{N}$ be as in the example and let $(t_i)_{i\in\mathbb{N}}$ be the itinerary of x, such that the eventual period k divides p. We can recover x from its itinerary by the ternary expansion

$$x = \sum_{i=1}^{\infty} \frac{t_i}{3^i}.$$

Now let $r > 0$ be small enough such that for every point $y \in C \cap [x - r, x + r]$, the itinerary of y is equal to the itinerary of x in the first l entries. If $w \in (\tau_{-x}C)_r$, then $w = y - x$, where $y \in C \cap [x - r, x + r]$, with $\sigma(y) = (s_i)_{i\in\mathbb{N}}$. Hence,

$$w = \sum_{i=l+1}^{\infty} \frac{s_i - t_i}{3^i} = 3^p \left(\sum_{i=l+1}^{\infty} \frac{s_i - t_i}{3^{i+p}} \right).$$

But then, as x is rational and its eventual period divides p, it follows that $w = 3^p(a - x)$, where $a \in C \cap [x - r, x + r]$ has the first $l + p$ entries of its itinerary equal to x. We have $w \in 3^p(\tau_{-x}C)_r$. On the other hand, if $z \in 3^p(\tau_{-x}C)_r$, then

$$z = \sum_{i=1}^{\infty} \frac{s_i - t_i}{3^{i-p}}.$$

We can make r small enough so that the first p entries of $(s_i)_{i\in\mathbb{N}}$ are equal to the first p entries of the sequence of x. Thus, we have $z \in (\tau_{-x}C)_r$.

Reciprocally, suppose the eventual period of x, if it exists, doesn't divide p and let $r > 0$ and $\sigma(x) = (t_i)_{i\in\mathbb{N}}$. Let $a_k \in C$ be a point, with itinerary

$$\sigma(a_k) = (t_1, ..., t_{k-1}, s_k, t_{k+1}, ..., t_{k+p-1}, s_{k+p}, t_{k+p+1}, ...),$$

where $s_j \neq t_j$ for $j = k, k+p$. We can take k big enough such that $a_k \in [x-r, x+r]$ and $t_k \neq t_{k+p}$ since p isn't an eventual period of x. Let $w := a_k - x$. It follows that $w \in (\tau_{-x}C)_r$ and

$$w = \frac{s_k - t_k}{3^k} + \frac{s_{k+p} - t_{k+p}}{3^{k+p}}.$$

This leaves us with two cases: if $t_k = 0$, then

$$w = \frac{2}{3^k} - \frac{2}{3^{k+p}} = 3^p\left(\frac{2}{3^{k+p}} - \frac{2}{3^{k+2p}}\right).$$

But if $w \in 3^p(\tau_{-x}C)_r$, then $t_{k+p} = 0$, which is a contradiction because we assumed $t_k \neq t_{k+p}$. A similar analysis will give us a contradiction for the case $t_k = 2$.

□

Proposition 5.3. *Let $A, B \subseteq \mathbb{C}$ be closed sets and suppose there are $r > 0$ and $\varrho \in \mathbb{C}$, with $|\varrho| > 1$ such that*

$$\lim_{n\to\infty} d_r(\varrho^n A, \varrho^n B) = 0.$$

Then

$$\lim_{t\to\infty, t\in\mathbb{C}} d_r(tA, tB) = 0.$$

In other words A and B are asymptotically similar about 0.

Proof. Let $\varepsilon > 0$. There is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$

$$\delta((\varrho^n A)_r, (\varrho^n B)_r)) < \frac{\varepsilon}{|\varrho|}. \tag{5.3}$$

For any $t \in \mathbb{C}$ such that $|t| > |\varrho|^{n_0}$, there is $n \geqslant n_0$ that satisfies $|\varrho|^n \leqslant |t| < |\varrho|^{n+1}$. If $x \in (tA) \cap \overline{\mathbb{D}}_r$, we have

$$x = ta = \frac{t}{\varrho^n} \cdot \varrho^n a, \quad \text{for some } a \in A.$$

Then

$$|\varrho^n a| = |x| \cdot \frac{|\varrho|^n}{|t|}.$$

Since $x \in \overline{\mathbb{D}}_r$ and $|t| \geqslant |\varrho|^n$, we have $|\varrho^n a| \leqslant r$. Hence, $\varrho^n a \in \varrho^n A \cap \overline{\mathbb{D}}_r$. From (5.3), we get

$$d(\varrho^n a, \varrho^n B) < \frac{\varepsilon}{|\varrho|} \tag{5.4}$$

Also,

$$\frac{t}{\varrho^n}(\varrho^n B)_r = \left(\frac{t}{\varrho^n}\varrho^n B \cap \overline{\mathbb{D}}_{r\cdot|t/\varrho^n|}\right) \cup \partial\mathbb{D}_{r\cdot|t/\varrho^n|} \tag{5.5}$$

Since $|t| \geqslant |\varrho|^n$,

$$\begin{gathered}\overline{\mathbb{D}}_r \cap \overline{\mathbb{D}}_{r\cdot|t/\varrho^n|} = \overline{\mathbb{D}}_r \\ \text{and} \\ \overline{\mathbb{D}}_r \cap \partial\mathbb{D}_{r\cdot|t/\varrho^n|} = \varnothing \quad \text{or} \quad \overline{\mathbb{D}}_r \cap \partial\mathbb{D}_{r\cdot|t/\varrho^n|} = \partial\mathbb{D}_r.\end{gathered} \tag{5.6}$$

Hence, using (5.5) and (5.6)

$$\frac{t}{\varrho^n}(\varrho^n B)_r \cap \overline{\mathbb{D}}_r = \left(\frac{t}{\varrho^n}\varrho^n B\right) \cap \overline{\mathbb{D}}_r$$

or

$$\frac{t}{\varrho^n}(\varrho^n B)_r \cap \overline{\mathbb{D}}_r = \left(\frac{t}{\varrho^n}\varrho^n B\right)_r.$$

We have (in any case)

$$\left(\frac{t}{\varrho^n}(\varrho^n B)_r\right)_r = \left(\frac{t}{\varrho^n}\varrho^n B\right)_r. \tag{5.7}$$

Now, we may prove that given $x = ta \in (tA) \cap \overline{\mathbb{D}}_r$, for $|t| > |\varrho|^{n_0}$, we have

$$d(x, (tB)_r) < \varepsilon.$$

We begin by noting that (5.7) implies

$$d(x, (tB)_r) = d\left(\frac{t}{\varrho^n}\cdot\varrho^n a, \left(\frac{t}{\varrho^n}\varrho^n B\right)_r\right) = d\left(\frac{t}{\varrho^n}\cdot\varrho^n a, \left(\frac{t}{\varrho^n}(\varrho^n B)_r\right)_r\right).$$

From Lemma 5.1:

$$d\left(\frac{t}{\varrho^n}\cdot\varrho^n a, \left(\frac{t}{\varrho^n}(\varrho^n B)_r\right)_r\right) \leqslant d\left(\frac{t}{\varrho^n}\cdot\varrho^n a, \frac{t}{\varrho^n}(\varrho^n B)_r\right).$$

It follows that

$$\begin{aligned} d(x, (tB)_r) &= d\left(\frac{t}{\varrho^n}\cdot\varrho^n a, \left(\frac{t}{\varrho^n}(\varrho^n B)_r\right)_r\right) \\ &\leqslant d\left(\frac{t}{\varrho^n}\cdot\varrho^n a, \frac{t}{\varrho^n}(\varrho^n B)_r\right) \\ &= \frac{|t|}{|\varrho|^n}\cdot d(\varrho^n a, (\varrho^n B)_r). \end{aligned}$$

Using (5.4) and since $|t| < |\varrho|^{n+1}$, we get

$$d(x, (tB)_r) < \frac{|t|}{|\varrho|^{n+1}}\cdot\varepsilon < \varepsilon$$

By Lemma 5.1:

$$\delta((tA)_r, (tB)_r) \leqslant \sup_{x \in (tA) \cap \overline{\mathbb{D}}_r} d(x, (tB)_r) < \varepsilon.$$

This proves that

$$\delta((tA)_r, (tB)_r) \to 0 \quad \text{while} \quad t \to \infty.$$

The proof of this fact for $\delta((tB)_r, (tA)_r)$ is analogous. Thus, we have the desired result. □

Corollary 5.1. *Let $A \subseteq \mathbb{C}$ be a asymptotically self-similar set with limit model $B \subseteq \mathbb{C}$. Then A is asymptotically similar to B.*

Proof. We know, from Proposition 5.1, that B is ϱ-self-similar. Let $r > 0$ such that

$$(\varrho B)_r = B_r.$$

It follows that

$$(\varrho^n B)_r = B_r. \tag{5.8}$$

Indeed,

$$\begin{aligned} (\varrho^n B)_r &= \varrho^{n-1} (\varrho B)_{r/|\varrho|^{n-1}} \\ &= \varrho^{n-1} B_{r/|\varrho|^{n-1}} \\ &= (\varrho^{n-1} B)_r. \end{aligned}$$

Continuing this argument inductively, we get (5.8). Since B is the limit model of A,

$$\lim_{n \to \infty} d_r(\varrho^n A, \varrho^n B) = \lim_{n \to \infty} d_r(\varrho^n A, B) = 0.$$

By Proposition 5.3 we have the desired result. □

Proposition 5.4. *Let $U, V \subseteq \mathbb{R}^2$ be neighborhoods of $x, y \in \mathbb{R}^2$, respectively, and let $f : U \to V$ be a C^1-diffeomorphism such that $f(x) = y$. If $A \subseteq U$ is a closed set that contains x, there is $r > 0$ such that*

$$\lim_{t \in \mathbb{C}, t \to \infty} d_r(t(\tau_{-y} f(A)), tDf(x)(\tau_{-x} A)) = 0.$$

Proof. Fix $r > 0$. Since f is differentiable at x, for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < \|x - h\| < \delta$, then

$$\frac{\|f(x) - f(h) - Df(x)\,(x - h)\|}{\|x - h\|} = \frac{\|y - f(h) - Df(x)(x - h)\|}{\|x - h\|} < \varepsilon.$$

Let $a \in tDf(x)(\tau_{-x}A) \cap \overline{\mathbb{D}}_r$. There is $h \in A$ such that $a = tDf(x)(h-x)$. First, note that, for a fixed $t \in \mathbb{C}$,

$$d(tDf(x)(h-x), t(\tau_{-y}f(A))) \leqslant |t| \cdot \left\| f(h) - y - Df(x)(h-x) \right\|.$$

Indeed, we have

$$\begin{aligned} d(tDf(x)(h-x), t\tau_{-y}f(A)) &= \inf_{\zeta \in f(A)} d(tDf(x)(h-x), t(\zeta - y)) \\ &\leqslant d(tDf(x)(h-x), t(f(h)-y)) \\ &= |t| \cdot \left\| f(h) - y - Df(x)(h-x) \right\|. \end{aligned}$$

Since f is a C^1-diffeomorphism, it follows that

$$h - x = Df(x)^{-1}\left(\frac{a}{t}\right).$$

Then

$$\|x - h\| \leqslant \left\| Df(x)^{-1} \right\| \frac{\|a\|}{|t|} \leqslant \left\| Df(x)^{-1} \right\| \frac{r}{|t|}. \tag{5.9}$$

The last inequality in (5.9) is due to the fact that $a \in \overline{\mathbb{D}}_r$. Let $c := \left\| Df(x)^{-1} \right\| \cdot r$. If $|t| > c/\delta$, then for any

$$a = tDf(x)(h-x) \in tDf(x)(\tau_{-x}A) \cap \overline{\mathbb{D}}_r,$$

equation (5.9) gives us

$$\|x - h\| \leqslant \frac{c}{|t|} < \delta.$$

Hence,

$$\begin{aligned} d(tDf(x)(h-x), t(\tau_{-y}f(A))) &\leqslant |t| \cdot \left\| f(h) - y - Df(x)(h-x) \right\| \\ &\leqslant c \cdot \frac{\left\| y - f(h) - Df(x)(x-h) \right\|}{\|x-h\|} \\ &< c \cdot \varepsilon. \end{aligned} \tag{5.10}$$

From lemma 5.1,

$$\delta((tDf(x)\tau_{-x}A)_r, (t(\tau_{-y}f(A)))_r) \leqslant \sup_{a \in tDf(x)(\tau_{-x}A) \cap \overline{\mathbb{D}}_r} d(a, t(\tau_{-y}f(A))) < c \cdot \varepsilon.$$

We need to prove that

$$\delta((t(\tau_{-y}f(A)))_r, (tDf(x)\tau_{-x}A)_r) \to 0 \quad \text{while} \quad t \to \infty.$$

Let $a \in t(\tau_{-y}f(A)) \cap \overline{\mathbb{D}}_r$. Then $a = t(f(h) - y)$ for some $h \in A$. As f is a C^1-diffeomorphism, we have

$$h = f^{-1}\left(\frac{a}{t} + y\right).$$

By the mean value inequality

$$\begin{aligned}\|x-h\| &= \left\| f^{-1}(y) - f^{-1}\left(\frac{a}{t}+y\right)\right\| \\ &\leqslant \sup_{z\in[x,h]} \left\|Df(z)^{-1}\right\| \frac{\|a\|}{|t|} \\ &\leqslant \sup_{z\in[x,h]} \left\|Df(z)^{-1}\right\| \frac{r}{|t|}.\end{aligned}$$

Define $C := \sup_{z\in[x,h]} \left\|Df(z)^{-1}\right\| \cdot r$. If $|t| > C/\delta$, then $\|x-h\| < \delta$. Following the same steps as in (5.10), we have the desired result.

□

Remark. This result can be adapted to the context of a regular curve in order to prove the statement in the Example 5.2.

Proposition 5.5. *Let f be as in Proposition 5.4 and A a closed set such that $f(A)$ is asymptotically ϱ-self-similar about y, with limit model B. If $Df(x)$ is $\mathbb{C}$-linear, then A is also asymptotically ϱ-self-similar about x, with limit model $Df(x)^{-1}B$. In other words, there is $r' > 0$ such that*

$$\lim_{n\to\infty} d_{r'}\left(\varrho^n \tau_{-x} A, Df(x)^{-1}B\right) = 0.$$

Proof. We already know that $Df(x)\,A$ and $f(A)$ are similar. That is, for some $r > 0$

$$\lim_{n\to\infty} d_r(\varrho^n Df(x)(\tau_{-x}A), \varrho^n \tau_{-y} f(A)) = 0.$$

Note that the definition of similarity between sets states that t tends to infinity in any direction. Particularly, we can choose the sequence $(\varrho^n)_{n\in\mathbb{N}}$ that tends to infinity as $n \to \infty$. Since $f(A)$ is asymptotically ϱ-self-similar about y, we have

$$\lim_{n\to\infty} d_r(\varrho^n \tau_{-y} f(A), B) = 0.$$

Then,

$$\lim_{n\to\infty} d_r(\varrho^n Df(x)(\tau_{-x}A), B) \leqslant \lim_{n\to\infty} \left[d_r(\varrho^n \tau_{-y} f(A), B) + d_r(\varrho^n Df(x)(\tau_{-x}A), \varrho^n \tau_{-y} f(A))\right] = 0$$

Hence, $Df(x)(\tau_{-x}A)$ is also asymptotically ϱ-self-similar.

Since $Df(x)$ is $\mathbb{C}$-linear, it follows that

$$\varrho^n Df(x)(\tau_{-x}A) = Df(x)(\varrho^n \tau_{-x}A).$$

Now, let $r' > 0$ be small enough such that $Df(x)\,\mathbb{D}_{r'} \subset \mathbb{D}_r$. We have

$$\begin{aligned}
\left(Df(x)^{-1}B \cap \overline{\mathbb{D}}_{r'}\right) \cup \partial\mathbb{D}_{r'} &= Df(x)^{-1}\left[\left(B \cap Df(x)\,\overline{\mathbb{D}}_{r'}\right) \cup Df(x)\partial\mathbb{D}_{r'}\right] \\
&= Df(x)^{-1}\left[\left(B_r \cap Df(x)\,\overline{\mathbb{D}}_{r'}\right) \cup Df(x)\partial\mathbb{D}_{r'}\right] \\
&= Df(x)^{-1}\left[\left(\lim_{n\to\infty}(\varrho^n Df(x)(\tau_{-x}A))_r \cap Df(x)\,\overline{\mathbb{D}}_{r'}\right) \cup Df(x)\partial\mathbb{D}_{r'}\right] \\
&= Df(x)^{-1}\left[\left(\lim_{n\to\infty} Df(x)(\varrho^n\tau_{-x}A) \cap Df(x)\,\overline{\mathbb{D}}_{r'}\right) \cup Df(x)\partial\mathbb{D}_{r'}\right] \\
&= \lim_{n\to\infty}\left(\varrho^n\tau_{-x}A \cap \overline{\mathbb{D}}_{r'}\right) \cup \partial\mathbb{D}_{r'}.
\end{aligned}$$

□

Proposition 5.6. *Let $X \subseteq \mathbb{C}$ be a ϱ-self-similar set about 0. Define Y to be*

$$Y := \left\{z \in \mathbb{C} \mid z^d \in X\right\}.$$

Then the set Y is also ϱ-self-similar about 0. In fact this result holds for every scale ρ such that $\rho^d = \varrho$.

Proof. Fix $r > 0$ such that $(\varrho X)_{r^d} = X_{r^d}$. Particularly,

$$\varrho X \cap \mathbb{D}_{r^d} = X \cap \mathbb{D}_{r^d}.$$

Indeed this equality isn't necessarily true for the closed disk, because in the definition of self-similarity we include a union with the boundary of the disk. But, when we are dealing with just the open disk, this equality holds. Note, from the definition of Y, that

$$Y \cap \mathbb{D}_r = \left\{z \in \mathbb{C} \mid z^d \in X \cap \mathbb{D}_{r^d}\right\}.$$

Now, let $z \in \rho Y \cap \mathbb{D}_r$, where $\rho^d = \varrho$. Then, $z = \rho y$, where $y \in Y \cap \mathbb{D}_{r/|\rho|}$. This implies that $y^d \in X \cap \mathbb{D}_{r^d/|\rho^d|}$. But then, since X is ϱ-self-similar, it follows that

$$\varrho y^d \in \varrho X \cap \mathbb{D}_{r^d} = X \cap \mathbb{D}_{r^d}.$$

Hence,

$$\varrho y^d = (\rho y)^d \in X \cap \mathbb{D}_{r^d} \Rightarrow z = \rho y \in Y \cap \mathbb{D}_r.$$

Thus, the set $\rho Y \cap \mathbb{D}_r \subseteq Y \cap \mathbb{D}_r$.

On the other hand, let $y \in Y \cap \mathbb{D}_r$. Well there is $z \in \mathbb{C}$ such that $y = \rho z$. Which implies that

$$\varrho z^d = (\rho z)^d \in X \cap \mathbb{D}_{r^d} = \varrho X \cap \mathbb{D}_{r^d}.$$

But then,

$$\varrho z^d \in \varrho X \cap \mathbb{D}_{r^d} \Rightarrow z^d \in X \cap \mathbb{D}_{r^d/|\rho^d|}.$$

It follows that $z \in Y \cap \mathbb{D}_{r/|\rho|}$. Thus,

$$y = \rho z \in \rho Y \cap \mathbb{D}_r$$

and then, the set $\rho Y \cap \mathbb{D}_r = Y \cap \mathbb{D}_r$. We conclude that $(\rho Y)_r = Y_r$.

□

5.2 Self-similarity at points of the Julia set

Before tackling the asymptotic self-similarity of the Julia set at a Misiurewicz point, we must prove the following lemma:

Lemma 5.2. *Let $U, V \subseteq \mathbb{C}$ be neighborhoods of $x, y \in \mathbb{C}$ respectively and let $g : U \to V$ a holomorphic function such that $g(x) = y$ and $g'(x) = g''(x) = ... = g^{(d-1)}(x) = 0$, with $g^{(d)}(x) \neq 0$. Suppose there exists a neighborhood V_y of y, a real number $s > 0$ and a conformal isomorphism $\varphi_y : V_y \to \overline{\mathbb{D}}_{s^d}$, with $\varphi_y(y) = 0$. Then there exists a neighborhood U_x of x and a conformal isomorphism $\varphi_x : U_x \to \overline{\mathbb{D}}_s$ satisfying*

$$\varphi_y \circ g \circ \varphi_x^{-1}(z) = z^d.$$

Proof. Since g has local degree d, let $U_x = D(x, r)$ be the disk centered at x, with radius $r > 0$ small enough, such that $g(U_x) \subseteq V_y$ and where the expansion in power series of $h := \varphi_y \circ g$ is defined:

$$h(z) = a(z - x)^d + b(z - x)^{d+1} + ...$$

We can also make U_x small enough, so that we have $g'(z) \neq 0$ for all $z \in U_x - \{x\}$. Then, the map $h : U_x - \{x\} \to \mathbb{D}_{s^d} - \{0\}$ is a covering map with d branches. Indeed, let $z_0 \in U_x$. It follows that

$$z_0 - x = \rho \cdot e^{i\theta}, \quad \text{with} \quad 0 < \rho < r.$$

Then, the image $h(z_0)$ has exactly d pre-images $z_k \in U_x$ such that

$$z_k - x = \rho \cdot e^{i\theta_k}, \quad \text{where} \quad \theta_k = \frac{\theta \cdot d + 2\pi \cdot k}{d}$$

for $k = 0, .., d - 1$. Since, for every z_k, we have $g'(z_k) \neq 0$, the inverse mapping theorem gives us that there exists neighborhoods $\tilde{U}_k$ and $\tilde{V}_k$ of z_k and $h(z_k)$, respectively, such that $h : \tilde{U}_k \to \tilde{V}_k$ is a biholomorphism. Analogously, the function $p : \mathbb{D}_s - \{0\} \to \mathbb{D}_{s^d} - \{0\}$, given by $p(z) = z^d$ is a covering map, with d branches.

Let $\pi_1(\mathbb{D}_{s^d} - \{0\}) \cong \mathbb{Z}$ be the fundamental group of $\mathbb{D}_{s^d} - \{0\}$. Since h and p are covering maps, with d branches, it follows that

$$h_*(\pi_1(U_x - \{x\})) \cong d\mathbb{Z} \cong p_*(\pi_1(\mathbb{D}_s - \{0\})),$$

where h_* and p_* are the group homomorphisms induced by h and p, respectively. This isomorphism between groups holds, because the index of the subgroup defined by the image of a homomorphism induced by a covering map is equal to the number of branches.

Hence, by the lifting theorem, there exists a conformal isomorphism $\varphi_x : U_x - \{x\} \to D_s - \{0\}$, satisfying

$$p = h \circ \varphi_x^{-1}.$$

Since

$$\lim_{z \to x} \varphi_x(z) = 0,$$

we can extend φ_x analytically to x, by declaring $\varphi_x(x) = 0$.

□

Theorem 5.2. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map and $A \subseteq \mathbb{C}$ be a closed subset, completely invariant under f. If $x \in A$ is an eventually repelling periodic point for f, then A is asymptotically ϱ-self-similar about x, where ϱ is the multiplier of the eventual cycle of x.*

Moreover, there exist a conformal mapping ϕ defined in a closed neighborhood $\overline{U}$ of x such that the limit model of A is the set

$$B := \frac{1}{\phi'(x)} \phi(A \cap \overline{U}).$$

Furthermore, if x is periodic, then ϕ can be chosen to satisfy $\phi'(x) = 1$. And if x is eventually periodic satisfying

$$f^p(f^l(x)) = f^l(x) \quad \text{with} \quad (f^l)'(x) \neq 0,$$

then A is asymptotically ϱ-self-similar about $f^l(x)$, with the same limit model as x up to multiplication by $(f^l)'(x)$.

Proof. By definition, since A is completely invariant under f,

$$f(A) = A = f^{-1}(A).$$

It follows that, for every subset $U \subseteq \mathbb{C}$,

$$f(A \cap U) = A \cap f(U) \quad \text{and} \quad f^{-1}(A \cap f(U)) \cap U = A \cap U. \tag{5.11}$$

Indeed,

$$f(A \cap U) \subseteq f(A) \cap f(U) = A \cap f(U).$$

On the other hand, if $y \in A \cap f(U)$, then $y = f(x) \in A$, where $x \in U$. But, since A is completely invariant, we have $x \in A$. Hence, the image $y \in f(A \cap U)$. Thus,

$$f(A \cap U) = A \cap f(U).$$

For the second equality in (5.11), since $U \subseteq f^{-1}(f(U))$,

$$f^{-1}(A \cap f(U)) \cap U = f^{-1}(A) \cap f^{-1}(f(U)) \cap U = A \cap U.$$

Now, let $x \in A$ be an eventually repelling periodic point for f and let l, p be the minimal integers such that

$$f^p(f^l(x)) = f^l(x).$$

Then, the periodic point $y := f^l(x)$ is repelling. Which means that $\varrho := (f^p)'(y)$ satisfies $|\varrho| > 1$. By König's theorem, there are neighborhoods V and V_0 of y and 0 respectively and a conformal mapping $\varphi : V \to V_0$ that satisfies

$$\varphi \circ f^p(z) = \varrho \varphi(z),$$

for every $z \in V$. Moreover, we have $\varphi(y) = 0$ and $\varphi'(y) = 1$.

Let $r > 0$ be small enough such that $\overline{\mathbb{D}}_r \subseteq V_0$. Set

$$U := \varphi^{-1}(\mathbb{D}_{r/|\varrho|})$$

Since f^p is continuous and φ is conformal,

$$f^p(\overline{U}) \subseteq \overline{f^p(U)} \quad \text{and} \quad \varphi^{-1}(\overline{\mathbb{D}}_r) = \overline{\varphi^{-1}(\mathbb{D}_r)}.$$

Also, we have

$$f^p(\varphi^{-1}(z)) = \varphi^{-1}(\varrho z),$$

for every $z \in V_0$. Then,

$$\begin{aligned} f^p(\overline{U}) \subseteq \overline{f^p(U)} &= \overline{f^p(\varphi^{-1}(\mathbb{D}_{r/|\varrho|}))} \\ &= \overline{\varphi^{-1}(\varrho \mathbb{D}_{r/|\varrho|})} \\ &= \overline{\varphi^{-1}(\mathbb{D}_r)} \\ &= \varphi^{-1}(\overline{\mathbb{D}}_r). \end{aligned} \tag{5.12}$$

Since $\overline{\mathbb{D}}_r \subseteq V_0$, it follows that $f^p(\overline{U}) \subseteq V$. By the definition of U and the fact that $f^p(U) = \varphi^{-1}(\mathbb{D}_r)$, we have $U \subseteq f^p(\overline{U})$. Hence, from (5.11),

$$f^p(A \cap \overline{U}) \cap U = A \cap f^p(\overline{U}) \cap U = A \cap U.$$

It follows that

$$\varphi(f^p(A \cap \overline{U}) \cap U) = \varphi(A \cap U) = \varphi(A \cap \overline{U} \cap U).$$

Since φ is bijective, it preserves intersection:

$$\varphi(f^p(A \cap \overline{U})) \cap \varphi(U) = \varphi(A \cap \overline{U}) \cap \varphi(U).$$

The conjugation implies

$$\varrho \varphi(A \cap \overline{U}) \cap \mathbb{D}_{r/|\varrho|} = \varphi(A \cap \overline{U}) \cap \mathbb{D}_{r/|\varrho|}.$$

Thus, the set $\varphi(A \cap \overline{U})$ is ϱ-self-similar about 0. In particular it is asymptotically ϱ-self-similar about 0, with limit model equals to itself. We can apply Proposition 5.5 to $\varphi : V \to V_0$ and conclude that A is asymptotically ϱ-self-similar about y, with limit model

$$\frac{1}{\varphi'(y)}\varphi(A \cap \overline{U}) = \varphi(A \cap \overline{U}).$$

This already proves that if x is periodic, then A is asymptotically ϱ-self-similar about x. Now, we need to prove it for the case x strictly eventually periodic.

First assume that $(f^l)'(x) \neq 0$. By the inverse mapping theorem, there are $\overline{W}_x$ and $\overline{W}_y$ neighborhoods of x and y respectively, such that $f^l : \overline{W}_x \to \overline{W}_y$ is a biholomorphism. We can make W_x small enough such that $\overline{W}_y \subseteq U$. Then, since f^l is a biholomorphism,

$$f^l(A \cap \overline{W}_x) = A \cap \overline{W}_y.$$

Since A is already asymptotically ϱ-self-similar about y,we can apply proposition 5.5 again for f^l to conclude that A is asymptotically ϱ-self-similar about x, with limit model

$$\frac{1}{(f^l)'(x)}\varphi(A \cap \overline{U}).$$

Note the limit model is the same up to multiplication by $(f^l)'(x)$. Actually, by choosing $r' > 0$ such that

$$\overline{\mathbb{D}}_{r'} \subseteq \varphi \circ f^l(\overline{W}_x),$$

we can follow an analogous argument as above to prove

$$\frac{1}{(f^l)'(x)}\varphi \circ f^l(A \cap \overline{W}_x)$$

is also a limit model.

Finally, we will see now the case $(f^l)'(x) = 0$. Let d be an integer such that $(f^l)^{(i)}(x) = 0$, for $i = 1, ..., d-1$ and $(f^l)^{(d)}(x) \neq 0$. From Lemma 5.2, there exist $s > 0$, neighborhoods $V_y \subseteq V$ and W_x of y and x respectively, and $\varphi_x : U_x \to \mathbb{D}_s$ a conformal mapping, such that

$$\varphi \circ f^l \circ \varphi_x^{-1} = z^d, \tag{5.13}$$

for every $z \in \mathbb{D}_s$. Let $s' < s$ and call $K := \varphi_x^{-1}(\overline{\mathbb{D}}_{s'})$. Also set

$$X := \varphi \circ f^l(A \cap K) \quad \text{and} \quad Y := \left\{z \in \mathbb{C} \mid z^d \in X\right\}.$$

The set X is ϱ-self-similar about 0, because of Proposition 5.5 and the fact that

$$\varphi^{-1}(X) = A \cap f^l(K),$$

which we know is ϱ-self-similar about y. Then, from Proposition 5.6, the set Y is also ϱ-self-similar about 0. From (5.13), we have

$$X = (\varphi_x(A \cap K))^d.$$

Then $Y = \varphi_x(A \cap K)$. Applying proposition 5.5 again, we have

$$\lim_{n\to\infty} d_{r'} \left(\varrho^n(\tau_{-x} A), \frac{1}{\varphi_x'(x)} \varphi_x(A \cap K) \right) = 0,$$

for some $r' > 0$.

□

Corollary 5.3. *Let c be a Misiurewicz point of the family $f_c(z) = z^2 + c$ and let l, p be the minimal integers such that*

$$f_c^p(f_c^l(c)) = f_c^l(c).$$

Define

$$\alpha := f_c^l(c) \quad \text{and} \quad \varrho := \left(f_c^p\right)'(\alpha).$$

Then, the Julia set J_c is asymptotically ϱ-self-similar about c and is asymptotically ϱ-self-similar about α, with the same limit model as c up to multiplication by $(f_c^l)'(c)$.

Proof. The Julia set is closed by definition and, from 2.1, is completely invariant. Also, from proposition 3.1, the eventual cycle of 0 is repelling. Then, the Julia set J_c is asymptotically self-similar about α, with limit model

$$\varphi(J_c \cap \overline{V}),$$

where φ is the König's linearization map and V is a neighborhood of α. Since 0 is the unique critical point of f_c and 0 is strictly preperiodic at a Misiurewicz point, we have $(f_c^l)'(c) \neq 0$. Then,

$$\lim_{n\to\infty} d_r \left(\varrho^n(\tau_{-c} J_c), \frac{1}{(f_c^l)'(c)} \varphi(J_c \cap \overline{V}) \right).$$

□

Remark. The proof of theorem 5.2 also gives us the existence of a neighborhood $\overline{U}$ of c that satisfies

$$\lim_{n\to\infty} d_{r'} \left(\varrho^n(\tau_{-c} J_c), \frac{1}{(f_c^l)'(c)} \varphi \circ f_c^l(J_c \cap \overline{U}) \right),$$

for some $r' > 0$.

Example 5.5. An implication of theorem 5.2 is that, given $c \in \mathbb{C}$ such that the function f_c has a repelling cycle, the Julia set J_c is asymptotically ϱ-self-similar about a point in

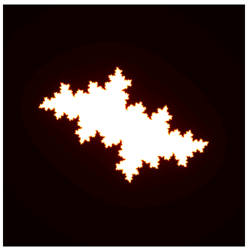

Figure 5.2: K_c at $c = -0.5 + 0.5i$

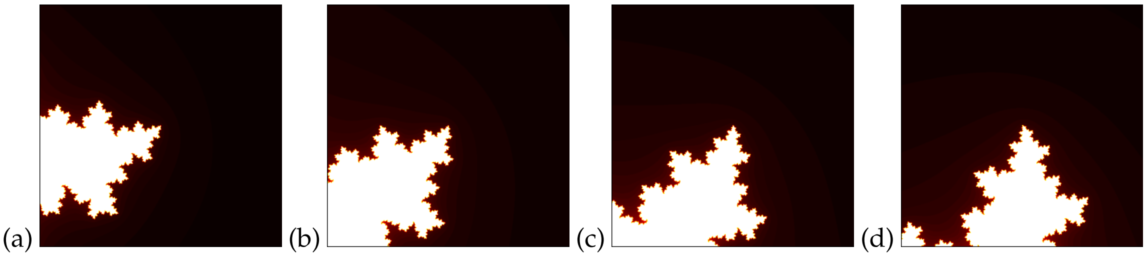

Figure 5.3: K_c centered at p. (a) Magnification factor: $|\varrho^2|$. (b) Magnification factor: $|\varrho^4|$. (c) Magnification factor: $|\varrho^6|$. (d) Magnification factor: $|\varrho^8|$.

that cycle. Take $c = -0.5 + 0.5i$ for example. The function f_c has a repelling fixed point at $p \approx 1.409 - 0.275i$. The multiplier of p is

$$\varrho := f_c'(p) = 2p.$$

Then, from theorem 5.2, the Julia set J_c is asymptotically ϱ-self-similar about p.

Above we can see magnifications of the filled-in Julia set K_c, centered at p. Note that the set on the figures is roughly the same up to rotation. This is due to the fact that to generate the images, we used $|\varrho|$ as the factor instead of ϱ. Another limitation of the method used to generate these images is the squared frame, that reveals more or less information of the magnified set depending on the position of it. Still we can observe the self-similarity.

Example 5.6. An example of a Misiurewicz point is $c = i$. Indeed, the orbit of 0 under f_i is given by

$$0 \mapsto i \mapsto i - 1 \mapsto -i \mapsto i - 1.$$

Using the notation from corollary 5.3, we have $l = 1$, $p = 2$ and $\alpha = i - 1$. Also

$$\varrho := (f_c^p)'(\alpha) = (f_i^2)'(i-1) = 4\sqrt{2} \cdot e^{\frac{\pi i}{4}} \quad \text{and} \quad (f_c^l)'(c) = (f_i)'(i) = 2i.$$

Then J_i is asymptotically ϱ-self-similar about i.

We have again a rotation of the set represented in the image. Note that the rotation angle from one magnification to another is 45^o, which is exactly the argument of ϱ.

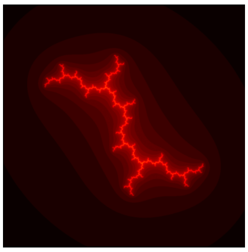

Figure 5.4: J_c at $c = i$.

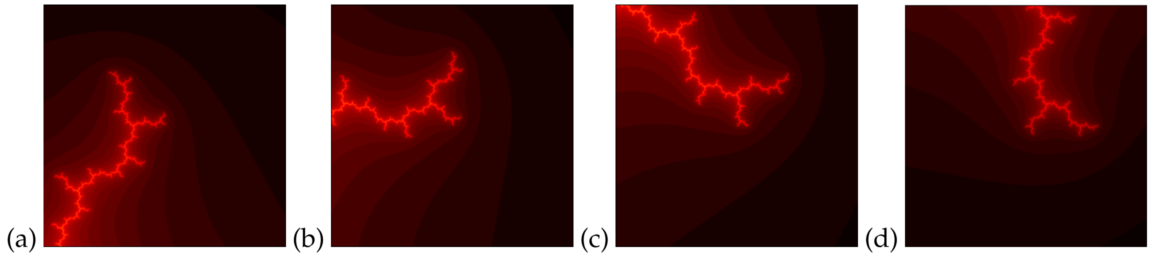

Figure 5.5: J_i centered at i. (a) Magnification factor: $|\varrho|$. (b) Magnification factor: $|\varrho^2|$. (c) Magnification factor: $|\varrho^3|$. (d) Magnification factor: $|\varrho^4|$.

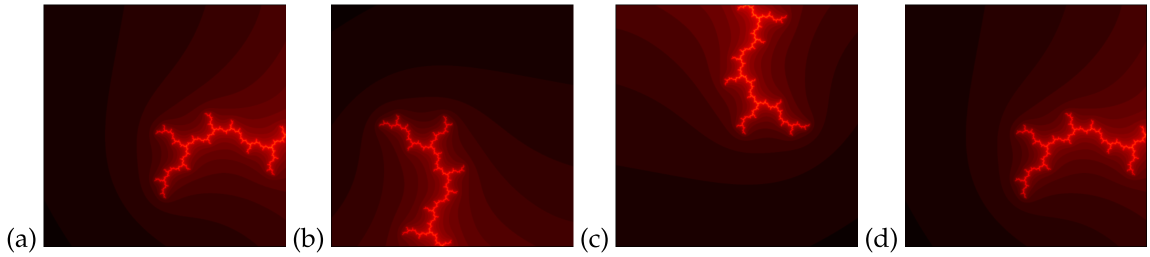

Figure 5.6: (a) Center i, magnification factor: $|\varrho^6|$. (b) Center i, magnification factor: $|\varrho^8|$. (c) Center $i-1$, magnification factor: $\frac{1}{2}|\varrho^6|$. (d) Center $i-1$, magnification factor: $\frac{1}{2}|\varrho^8|$.

Corollary 5.3 also gives us that J_i is asymptotically ϱ-self-similar about $i-1$ and if the limit model for i is X and for $i-1$ is Y, then $X = \frac{1}{2i}Y$.

Still in the example $c = i$, from theorem 5.2, the set J_i is also asymptotically ϱ-self-similar about 0. Of course, the limit model for 0 isn't the same as we've seen for i and $i-1$, as we can appreciate in the figure below. This happens because in this case

$$(f_c^l)'(0) = (f_i^2)'(0) = 0.$$

These examples can show us the beauty of this result. Still they also show that we didn't need to work exclusively with Misiurewicz points, as the Julia set is asymptotically self-similar about any pre-image of a repelling cycle according to the theorem 5.2. Since the Julia set is the closure of the repelling cycles of a given function, this property can be observed at most points of any Julia set of the quadratic family.

The next result though is even more remarkable. It enable us to compare sets on the space of parameters, in which we define the Mandelbrot set, with sets on the dynamic space, in which lies a given Julia set.

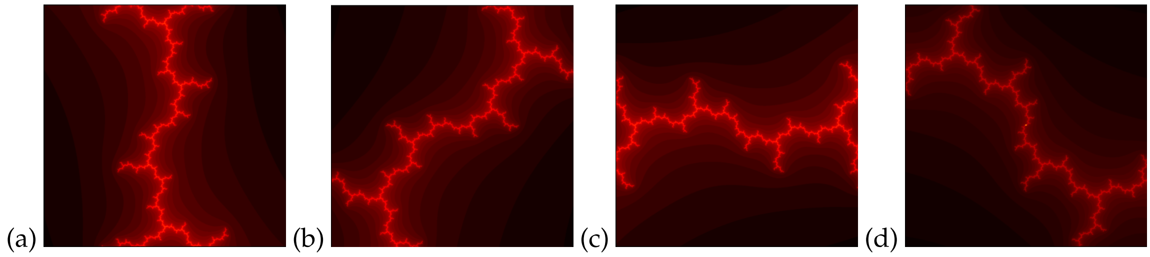

Figure 5.7: J_i centered at 0. (a) Magnification factor: $|\varrho^5|$. (b) Magnification factor: $|\varrho^6|$. (c) Magnification factor: $|\varrho^7|$. (d) Magnification factor: $|\varrho^8|$.

5.3 Similarity between $\mathcal{M}$ and J_c

Let's do an experiment and stick with the example $c = i$, but now we are going to magnify the Mandelbrot set using the same factor $|\varrho|$. In the next example, we can see that a similar phenomenon occurs around a neighborhood of c at the Mandelbrot set. By magnifying the Mandelbrot set around c, we can see that the set is converging to a limit model as well.

We aim to prove the asymptotic similarity between the Mandelbrot set and the Julia set J_c about a Misiurewicz point c. This theorem is far from trivial and to prove this we will need a proposition that gives us certain conditions for the asymptotic self-similarity of a neighborhood of a Misiurewicz point intersected with $\mathcal{M}$.

Let Λ be an open neighborhood of $c_0 \in \mathbb{C}$ and X be a subset of $\Lambda \times \mathbb{C}$. For each $c \in \Lambda$, define

$$X(c) = \{z \in \mathbb{C} \mid (c, z) \in X\}.$$

Suppose $A \subset X(c_0)$ is dense in $X(c_0)$ and, for every $z \in A$, there exists a neighborhood $U_z \subset \Lambda$ of c_0 and a continuous mapping $h_z : U_z \to \mathbb{C}$, satisfying $h_z(c_0) = z$ and $h_z(c) \in X(c)$ for every $c \in U_z$. Then we say that there exists a *dense set of continuous sections at* $(c_0, 0)$.

Proposition 5.7. *Let $X \subset \Lambda \times \mathbb{C}$, where Λ is an open neighborhood of c_0. Assume X is closed and that there exists a dense set of continuous sections at $(c_0, 0)$. Suppose there exists a holomorphic map $\varrho : \Lambda \to \mathbb{C}$ such that $X(c)$ is $\varrho(c)$-self-similar for a fixed $r' > 0$. Let $u : \Lambda \to \mathbb{C}$ be a holomorphic function, with $u(c_0) = 0$ and $u'(c_0) \neq 0$. Let M be the set*

$$M := \{c \in \Lambda \mid u(c) \in X(c)\}.$$

Then M is asymptotically $\varrho(c_0)$-self-similar about c_0 and the limit model is $X(c_0)/u'(c_0)$.

There is a certain stability related to the repelling cycle associated with a Misiurewicz point c_0. Later, we will define the function ϱ as the multiplier associated with the repelling periodic point $\alpha(c)$ of the map f_c^p, where c is near c_0. We will define the map u and the sets $X(c)$ in such a way that M will be a neighborhood of c_0 intersected with $\mathcal{M}$. Finally the limit model, for the asymptotic self-similarity of M, will be a linearization of the Julia set J_{c_0} intersected with a neighborhood of c_0, up to a rotation. All of this will be explained with more details at Theorem 5.4.

In order to prove the proposition 5.7, we will show a more general result for higher complex dimensions. We need to define similarity and self-similarity in this context. Let $\mathbb{E} = \mathbb{C}^k$ and $\mathbb{D}_r$ be the ball of $\mathbb{E}$ centered at 0 with radius r. We can define the Hausdorff-Chabauty distance the same way as in the one dimensional case. We now need to define our magnification scale.

Let $T : \mathbb{E} \to \mathbb{E}$ be a linear mapping. We say that T is *contracting* if there is $0 < \lambda < 1$ and $C > 0$ such that for all $n \in \mathbb{N}$

$$\|T^n\| \leqslant C\lambda^n.$$

And T is *expanding* if T is invertible and T^{-1} is contracting. Now, the expanding linear mappings will play the role of the scale for the definition of self-similarity.

Definition 5.6. A closed subset B of $\mathbb{E}$ is *T-self-similar* about 0 if $T : \mathbb{E} \to \mathbb{E}$ is expanding and there exists $r > 0$ such that

$$(TB)_r = B_r$$

Definition 5.7. A closed subset A of $\mathbb{E}$ is *asymptotically T-self-similar* about $x \in A$ if $T : \mathbb{E} \to \mathbb{E}$ is expanding and there exist $r > 0$ and a closed subset B of $\mathbb{E}$ such that

$$\lim_{n\to\infty} (T^n \tau_{-x} A)_r = B_r,$$

for the Hausdorff distance. The set B is called the limit model of A at x.

It is possible to show that if B is a limit model for a asymptotically T-self-similar set A, then B is T-self-similar about 0. The proof is analogue to proposition 5.1, since T preserves intersection and union of sets by virtue of being invertible.

For the next results, analogously to the one dimensional case, let Λ be a neighborhood of $\lambda_0 \in \mathbb{E}$ and X be a subset of $\Lambda \times \mathbb{E}$. For each $\lambda \in \Lambda$, define

$$X(\lambda) := \{x \in \mathbb{E} \mid (\lambda, x) \in X\}.$$

Lemma 5.3. *The set X is closed if and only if for each $\zeta \in \Lambda$, the set $X(\zeta)$ is closed in $\mathbb{E}$ and for every $r > 0$*

$$\lim_{\lambda\to\zeta} \delta((X(\lambda))_r, (X(\zeta))_r) = 0.$$

Proof. Suppose that X is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of points in $X(\zeta)$, with $x_n \to x$. Then, the sequence $(\zeta, x_n)_{n\in\mathbb{N}} \subset X$ is convergent. Since X is closed, the limit of the sequence (ζ, x) lies in X, which implies that $x \in X(\zeta)$. So $X(\zeta)$ is closed.

Now, fix $r > 0$ and assume by contradiction that there exists $\varepsilon > 0$ such that for every $n > 0$, there exist λ_n and $x_n \in X(\lambda_n) \cap \overline{\mathbb{D}}_r$ such that

$$|\lambda_n - \zeta| < \frac{1}{n} \quad \text{and} \quad d(x_n, X(\zeta) \cap \overline{\mathbb{D}}_r) \geqslant \varepsilon. \tag{5.14}$$

Consider the sequence $(\lambda_n, x_n) \in X$. It is a sequence contained in a closed set X and it is limited, since λ_n converges to ζ and $x_n \in \overline{\mathbb{D}}_r$. Then, there is a subsequence (λ_{n_k}, x_{n_k}) converging to some point $(\zeta, x) \in X$. It follows that $x \in X(\zeta) \cap \overline{\mathbb{D}}_r$. Well, take $N \in \mathbb{N}$ such that for every $n_k \geqslant N$ we have

$$|x_{n_k} - x| < \varepsilon.$$

This yields an absurd, since the sequence $(x_n)_{n\in\mathbb{N}}$ was constructed as in (5.14). Then, for every $\varepsilon > 0$, there exists $\eta > 0$ such that if $|\lambda - \zeta| < \eta$, then

$$d(x, X(\zeta) \cap \overline{\mathbb{D}}_r) < \varepsilon,$$

for every $x \in X(\lambda) \cap \overline{\mathbb{D}}_r$.

On the other hand, suppose that for each $\zeta \in \Lambda$, the set $X(\zeta)$ is closed and

$$\lim_{\lambda\to\zeta} \delta((X(\lambda))_r, (X(\zeta))_r) = 0,$$

for all $r > 0$. Let $(\lambda_n, x_n) \to (\zeta, x)$ be a convergent sequence in X. Since the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent, it is limited. So we can take $r > 0$ big enough such that $x_n \in \mathbb{D}_r$, for every $n \in \mathbb{N}$. Then, since $\lambda_n \to \zeta$, for every $\varepsilon > 0$, there exists $n_0 > 0$ such that if $n > n_0$, then

$$d(x_n, (X(\zeta))_r) < \frac{\varepsilon}{3}.$$

By taking a larger $r > 0$ we can assume

$$d(x_n, X(\zeta) \cap \overline{\mathbb{D}}_r) < \frac{\varepsilon}{3}. \tag{5.15}$$

Since $X(\zeta) \cap \overline{\mathbb{D}}_r$ is a compact set, it follows that for every $n > n_0$ the distance in (5.15) is realized by some $x'_n \in X(\zeta) \cap \overline{\mathbb{D}}_r$. This creates a sequence $(x'_n)_{n>n_0}$ on a compact set. It follows that there exists a subsequence x'_{n_k} converging to some point $x' \in X(\zeta) \cap \overline{\mathbb{D}}_r$.

Now, let $N > 0$ be large enough such that (5.15) is satisfied and

$$|x_n - x| < \frac{\varepsilon}{3} \quad \text{and} \quad |x'_{n_k} - x'| < \frac{\varepsilon}{3},$$

for all $n \geqslant N$. Hence,

$$|x - x'| \leqslant |x - x_{n_k}| + |x_{n_k} - x'_{n_k}| + |x'_{n_k} - x'| < \varepsilon.$$

Thus, $x = x'$ and $(\zeta, x) \in X$. We conclude that X is closed.

□

Proposition 5.8. *Suppose that X is closed in $\Lambda \times \mathbb{E}$ and assume there exists a dense set A of continuous sections at $(\lambda_0, 0)$. Then there is $r > 0$ such that*

$$\lim_{\lambda \to \lambda_0} d_r(X(\lambda), X(\lambda_0)) = 0.$$

Proof. We already know from lemma 5.3 that if X is closed, then $X(\lambda)$ is closed for every $\lambda \in \Lambda$ and

$$\lim_{\lambda \to \lambda_0} \delta((X(\lambda))_r, (X(\lambda_0))_r) = 0.$$

We just need to prove that

$$\lim_{\lambda \to \lambda_0} \delta((X(\lambda_0))_r, (X(\lambda))_r) = 0.$$

Since A is dense in $X(\lambda_0)$, for every $\varepsilon > 0$ the set

$$\bigcup_{x \in A} D(x, \varepsilon/2)$$

is an open cover of $X(\lambda_0)$, where $D(x, \varepsilon/2)$ is the disk centered at x, with radius $\varepsilon/2$. But the set $X(\lambda_0) \cap \overline{\mathbb{D}}_r$ is compact, so there exist finitely many $x_1, ..., x_k$ in A such that

$$X(\lambda_0) \cap \overline{\mathbb{D}}_r \subset \bigcup_{i=1}^{k} D(x_i, \varepsilon/2).$$

For each x_i, there is a continuous mapping h_{x_i} such that $h_{x_i}(\lambda_0) = x_i$. Then, there is $\delta_i > 0$ such that if $\|\lambda - \lambda_0\| < \delta_i$, then

$$d(h_{x_i}(\lambda), x_i) < \frac{\varepsilon}{2}.$$

Hence, since $h_{x_i}(\lambda) \in X(\lambda)$,

$$d(x_i, X(\lambda)) \leqslant \frac{\varepsilon}{2}.$$

Thus, for every $\varepsilon > 0$, there exists $\delta = \min\{\delta_i\}$ such that if $\|\lambda - \lambda_0\| < \delta$, then for all $x \in X(\lambda_0) \cap \overline{\mathbb{D}}_r$ we have

$$d(x, X(\lambda)) \leqslant \|x - x_i\| + d(x_i, X(\lambda)) < \varepsilon,$$

for some $i \in \{1, ..., k\}$. We already know that

$$\delta((X(\lambda_0)_r, (X(\lambda))_r) \leqslant \sup_{x \in X(\lambda_0) \cap \overline{\mathbb{D}}_r} d(x, X(\lambda)).$$

Since $X(\lambda_0) \cap \overline{\mathbb{D}}_r$ is compact and $X(\lambda)$ is closed, the supreme is realized and we have the desired result.

□

To prove the next proposition, we are going to need a lemma from algebraic topology.

Lemma 5.4. *Let $\Lambda' \subset \mathbb{E}$ be a neighborhood of* 0 *and $u : \Lambda' \to \mathbb{E}$ be continuous such that $u(0) = 0$. Suppose the derivative $Du(0)$ exists and is non-singular. Then there exists $\eta > 0$ such that for any continuous mapping $w : \Lambda' \to \mathbb{E}$, with $\|w(x)\| < \eta$, the mapping $u + w$ has at least one zero in Λ'.*

Proof. Since $Du(0)$ is non-singular there exists $\overline{\mathbb{D}}_r \subset \Lambda'$ such that u restricted to $\overline{\mathbb{D}}_r$ is a diffeomorphism. We can take r small enough so that 0 is the only zero of u. It follows that the induced homomorphism on the $(k-1)$-th homology group

$$u^* : H_{k-1}(S_r) \to H_{k-1}(E - \{0\}) \cong \mathbb{Z}$$

is an isomorphism, where S_r is the sphere centered at 0, with radius r. Set

$$\eta := \inf_{x \in S_r} \|u(x)\|.$$

Then, for every continuous mapping $w : \Lambda' \to \mathbb{E}$ that satisfies $\|w(x)\| < \eta$, for every $x \in \Lambda'$, the mapping $u + w : S_r \to \mathbb{E} - \{0\}$ is well defined. The induced homomorphism $(u + w)^*$ is also an isomorphism, since u is homotopic to $u + w$, by the homotopy $u + tw$. Suppose there are no zeroes in $\overline{\mathbb{D}}_r$. Then,

$$(u + w)^* : H_{k-1}(\overline{B}_r) \to H_{k-1}(\mathbb{E} - \{0\})$$

would be an isomorphism. This yields a contradiction, because $\overline{\mathbb{D}}_r$ is convex, so its homology group is trivial, while $H_{k-1}(\mathbb{E} - \{0\}) \cong \mathbb{Z}$.

□

Proposition 5.9. *Assume X is closed and that there exists a dense set A of continuous sections at $(\lambda_0, 0)$. Let $u : \Lambda \to \mathbb{E}$ be a continuous mapping, with $u(\lambda_0) = 0$. Suppose the derivative $S := Du(\lambda_0)$ exists and is non-singular, satisfying*

$$\|u(\lambda) - S(\lambda - \lambda_0)\| = \mathcal{O}(\|\lambda - \lambda_0\|^2).$$

Set

$$M := \{\lambda \in \Lambda \mid u(\lambda) \in X(\lambda)\}.$$

Suppose that, for all $\lambda \in \Lambda$, the set $X(\lambda)$ is $T(\lambda)$-self-similar about 0, for a fixed $r' > 0$, where $\lambda \mapsto T(\lambda)$ is a continuous mapping that maps a point $\lambda \in \Lambda$ into a expanding linear transformation $T(\lambda)$, satisfying

$$\|T(\lambda) - T(\lambda_0)\| = \mathcal{O}(\|\lambda - \lambda_0\|).$$

Also, assume there exist $\mu_1, \mu_2 \in \mathbb{R}$, with $\mu_2 < 1$, such that for all $\lambda \in \Lambda$,

$$\|T(\lambda)\| \leqslant \mu_1 \quad \textit{and} \quad \|T(\lambda)^{-1}\| \leqslant \mu_2$$

with

$$\mu := \mu_1 \cdot \mu_2^2 < 1.$$

Then, there exists $r > 0$ such that the set $S\tau_{-\lambda_0} M$ is asymptotically $T(\lambda_0)$-self-similar about 0, with limit model $X(\lambda_0)$. That is

$$\lim_{n\to\infty} (T(\lambda_0)^n \cdot S\tau_{-\lambda_0} M)_r = (X(\lambda_0))_r.$$

Furthermore, if the linear transformations S and $T(\lambda_0)$ commute, then M is asymptotically $T(\lambda_0)$-self-similar about λ_0 and the limit model is $S^{-1}X(\lambda_0)$. In other words, there exists $s > 0$ such that

$$\lim_{n\to\infty} (T(\lambda_0)^n \tau_{-\lambda_0} M)_s = (S^{-1}X(\lambda_0))_s.$$

Proof. Let $0 < r < r'$ and set

$$M(n) := T(\lambda_0)^n \cdot S\tau_{-\lambda_0} M.$$

We are going to prove that

$$\delta((M(n))_r, (X(\lambda_0))_r) \to 0 \quad \text{when} \quad n \to \infty.$$

It's sufficient to prove that for every point

$$y = T(\lambda_0)^n \cdot S\tau_{-\lambda_0} \lambda \in M(n) \cap \overline{\mathbb{D}}_r, \tag{5.16}$$

we have

$$\lim_{n\to\infty} d(T(\lambda_0)^n \cdot S\tau_{-\lambda_0} \lambda, (X(\lambda_0))_r) = 0. \tag{5.17}$$

We begin by estimating $\|\lambda - \lambda_0\|$. From (5.16) and since $\|T(\lambda_0)\| < \mu_2$ and $y \in \overline{\mathbb{D}}_r$, we have

$$\|\lambda - \lambda_0\| \leqslant \|S^{-1}\| \cdot \|T(\lambda_0)\|^n \cdot \|y\| \leqslant \|S^{-1}\| \cdot \mu_2^n \cdot r = c\mu_2^n, \tag{5.18}$$

where c is a non-zero constant.

Now, we turn our attention to the limit in (5.17). We have

$$d(T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda, (X(\lambda_0))_r) \leqslant \|T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda - T(\lambda)^n u(\lambda)\| + d(T(\lambda)^n u(\lambda), (X(\lambda_0))_r).$$

In order to prove that the limit is 0, we must show that

$$\lim_{n\to\infty} \|T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda - T(\lambda)^n u(\lambda)\| = 0 \tag{5.19}$$

and

$$\lim_{n\to\infty} d(T(\lambda)^n u(\lambda), (X(\lambda_0))_r) = 0 \tag{5.20}$$

For (5.19), note that

$$\begin{aligned} \|T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda - T(\lambda)^n u(\lambda)\| \leqslant & \|T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda - T(\lambda_0)^n u(\lambda)\| \\ & + \|T(\lambda_0)^n u(\lambda) - T(\lambda)^n u(\lambda)\|. \end{aligned} \tag{5.21}$$

We have

$$\|T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda - T(\lambda_0)^n u(\lambda)\| \leqslant \|T(\lambda_0)\|^n \cdot \|S(\lambda - \lambda_0) - u(\lambda)\|$$

Using the hypotheses and the estimate in (5.18),

$$\begin{aligned} \|T(\lambda_0)^n \cdot S\tau_{-\lambda_0}\lambda - T(\lambda_0)^n u(\lambda)\| &\leqslant c_1\mu_1^n \cdot \|\lambda - \lambda_0\|^2 \\ &\leqslant c_1 c \mu_1^n \cdot \mu_2^{2n} = c_2\mu^n. \end{aligned} \tag{5.22}$$

On the other hand by summing and subtracting the terms

$$T(\lambda_0)^{n-i} \cdot T(\lambda)^{i+1} u(\lambda),$$

for $i = 1, ..., n-1$, we get

$$\begin{aligned} \left\|T(\lambda_0)^n u(\lambda) - T(\lambda)^n u(\lambda)\right\| &= \left\|\sum_{i=0}^{n-1} (T(\lambda_0)^{n-i} T(\lambda)^i - T(\lambda_0)^{n-i-1} T(\lambda)^{i+1}) u(\lambda)\right\| \\ &\leqslant \sum_{i=0}^{n-1} \left\|(T(\lambda_0)^{n-i-1} T(\lambda)^i (T(\lambda_0) - T(\lambda)) u(\lambda)\right\| \\ &\leqslant \sum_{i=0}^{n-1} \left\|(T(\lambda_0)\right\|^{n-i-1} \left\|T(\lambda)^i\right\| \left\|T(\lambda_0) - T(\lambda)\right\| \left\|u(\lambda)\right\|. \end{aligned}$$

Since $\|T(\lambda)\| < \mu_1$ for every $\lambda \in \Lambda$, we have

$$\left\|T(\lambda_0)^n u(\lambda) - T(\lambda)^n u(\lambda)\right\| \leqslant n\mu_1^{n-1} \left\|T(\lambda_0) - T(\lambda)\right\| \left\|u(\lambda)\right\|.$$

From the hypotheses, we have that

$$\|T(\lambda) - T(\lambda_0)\| = \mathcal{O}(\|\lambda - \lambda_0\|).$$

Also, from the mean value inequality,

$$\|u(\lambda)\| \leqslant \sup_{\zeta \in [\lambda, \lambda_0]} \|Du(\zeta)\| \cdot \|\lambda - \lambda_0\|.$$

Then, by using the estimate at (5.18),

$$\begin{aligned} \left\| T(\lambda_0)^n u(\lambda) - T(\lambda)^n u(\lambda) \right\| &\leqslant c_3 n \mu_1^{n-1} \|\lambda - \lambda_0\|^2 \\ &\leqslant c_4 n \mu_1^{n-1} \cdot \mu_2^{2n} \\ &= c_4 n \mu_1^{-1} \mu^n \\ &= c_5 n \mu^n. \end{aligned} \tag{5.23}$$

Since

$$\lim_{n \to \infty} \mu^n = \lim_{n \to \infty} n \mu^n = 0,$$

from (5.23), (5.22) and (5.21), we get (5.19).

Now, we are going to use (5.19) to prove (5.20). For every $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that if $n \geqslant n_0$, then

$$\|T(\lambda_0)^n \cdot S\tau_{-\lambda_0} \lambda - T(\lambda)^n u(\lambda)\| < \varepsilon.$$

Then, by triangular inequality and using the fact that $T(\lambda_0)^n \cdot S\tau_{-\lambda_0} \lambda \in \overline{\mathbb{D}}_r$, we get

$$\|T(\lambda)^n u(\lambda)\| < r + \varepsilon. \tag{5.24}$$

We can make ε small enough such that $r + \varepsilon < r'$. Since $\lambda \in M$, it follows that $u(\lambda) \in X(\lambda)$. But $X(\lambda)$ is $T(\lambda)$-self-similar about 0. Hence,

$$T(\lambda)^n u(\lambda) \in X(\lambda). \tag{5.25}$$

The equation (5.24) guarantees

$$d(T(\lambda)^n u(\lambda), \partial \mathbb{D}_r) < \varepsilon \quad \text{or} \quad T(\lambda)^n u(\lambda) \in X(\lambda) \cap \overline{\mathbb{D}}_r.$$

If the first is true, the limit in (5.20) follows directly. If not, then $T(\lambda)^n u(\lambda) \in (X(\lambda))_r$. From proposition 5.8 we have

$$\lim_{\lambda \to \lambda_0} d((X(\lambda))_r, (X(\lambda_0))_r) = 0.$$

From (5.18), if $n \to \infty$ then $\lambda \to \lambda_0$. Thus, we get (5.20) and we've proven the limit for one of the semi-distances:

$$\lim_{n \to \infty} \delta((M(n))_r, (X(\lambda_0))_r) = 0.$$

Now, we aim to prove the limit for the other semi-distance. That is

$$\delta((X(\lambda_0))_r, (M(n))_r) \to 0 \quad \text{when} \quad n \to \infty.$$

Firstly, we may prove that for each $x \in A \cap \overline{\mathbb{D}}_r$ we have

$$d(x, (M(n))_r) \to 0 \quad \text{when} \quad n \to \infty.$$

Remember A is a dense set in $X(\lambda_0)$ for which, given $x \in A$, there exists a continuous map $h_x : \Lambda \to \mathbb{E}$ satisfying $h_x(\lambda_0) = x$ and $h_x(\lambda) \in X(\lambda)$, for every $\lambda \in \Lambda$.

We can use lemma 5.4 to show that, for any open neighborhood $\Lambda' \subset \Lambda$ of λ_0, satisfying $\overline{\Lambda'} \subseteq \Lambda$, there exists $n_0 > 0$ such that if $n \geqslant n_0$, then the following equation has at least one solution:

$$u(\lambda_0 + \lambda) - T(\lambda_0 + \lambda)^{-n} h_x(\lambda_0 + \lambda) = 0. \tag{5.26}$$

Indeed, the mapping $\tilde{u}(\lambda) = u(\lambda_0 + \lambda)$ satisfies $\tilde{u}(0) = 0$. Then, using the notation from the lemma, set $w(\lambda) = -T(\lambda_0 + \lambda)^{-n} h_x(\lambda_0 + \lambda)$. Since h_x is continuous,

$$\|h_x(\lambda_0 + \lambda)\| \leqslant \|\sup_{\lambda \in \Lambda'} \|h_x(\lambda)\| = C.$$

Which gives us

$$\|w(\lambda)\| = \|T(\lambda_0 + \lambda)^{-n} h_x(\lambda_0 + \lambda)\| < C\mu_2^n. \tag{5.27}$$

Hence, for any given η from lemma 5.4, we can make n big enough so that the hypotheses of the lemma are satisfied.

Let $\lambda_n \in \Lambda'$ be the solution in the form $\lambda_n := \lambda_0 + \lambda$ of (5.26) for $n \geqslant n_0$. Then

$$h_x(\lambda_n) = T(\lambda_n)^n u(\lambda_n).$$

Take $\varepsilon > 0$ such that $r + \varepsilon < r'$. We can make Λ' small enough such that

$$\|h_x(\lambda_n)\| < \|x\| + \varepsilon < r + \varepsilon < r'.$$

Then $h_x(\lambda_n) \in X(\lambda_n) \cap \overline{\mathbb{D}}_{r'}$. Combining with the fact that $X(\lambda_n)$ is $T(\lambda_n)$-self-similar, we have $\lambda_n \in M$. Hence,

$$T(\lambda_0) S \tau_{-\lambda_0} \lambda_n \in M(n).$$

Then,

$$\begin{aligned} d(x, (M(n))_r) &\leqslant d(x, M(n)) \\ &\leqslant \|x - T(\lambda_0) S \tau_{-\lambda_0} \lambda_n\| \\ &\leqslant \|x - h_x(\lambda_n)\| + \|T(\lambda_n)^n u(\lambda_n) - T(\lambda_0) S \tau_{-\lambda_0} \lambda_n\|. \end{aligned}$$

We must show the following limits to get the desired result:

$$\lim_{n \to \infty} \|x - h_x(\lambda_n)\| \tag{5.28}$$

and

$$\lim_{n\to\infty} \|T(\lambda_n)^n u(\lambda_n) - T(\lambda_0) S\tau_{-\lambda_0} \lambda_n\|. \tag{5.29}$$

Note that (5.29) is the same as (5.19) with the only difference being that we don't necessarily have $T(\lambda_0)S\tau_{-\lambda_0}\lambda_n \in \overline{\mathbb{D}}_r$. This property permitted us to make the estimate in (5.18), which was crucial to get (5.19). We may surpass this obstacle by getting to (5.18) by another path. From (5.26) and (5.27), we already have

$$\|u(\lambda_n)\| \leqslant C\mu_2^n. \tag{5.30}$$

Also, we have

$$\begin{aligned} \frac{\|u(\lambda)\|}{\|\lambda - \lambda_0\|} &\geqslant \left\| S\left(\frac{\lambda - \lambda_0}{\|\lambda - \lambda_0\|}\right)\right\| - \frac{\|u(\lambda) - S(\lambda - \lambda_0)\|}{\|\lambda - \lambda_0\|} \\ &\geqslant \inf_{\|v\|=1} \|Sv\| - \frac{\|u(\lambda) - S(\lambda - \lambda_0)\|}{\|\lambda - \lambda_0\|}. \end{aligned} \tag{5.31}$$

But $S = Du(\lambda_0)$ is non-singular. Then,

$$\inf_{\|v\|=1} \|Sv\| = C_1 > 0 \quad \text{and} \quad \lim_{\lambda\to\lambda_0} \frac{\|u(\lambda) - S(\lambda - \lambda_0)\|}{\|\lambda - \lambda_0\|} = 0.$$

Then, we can make Λ' small enough such that

$$\frac{\|u(\lambda) - S(\lambda - \lambda_0)\|}{\|\lambda - \lambda_0\|} < \beta < C_1,$$

for some $\beta > 0$. Hence, from (5.31),

$$\frac{\|u(\lambda)\|}{\|\lambda - \lambda_0\|} \geqslant C_1 - \beta := C_2 > 0.$$

Using (5.30), we have

$$\|\lambda_n - \lambda_0\| \leqslant \frac{1}{C_2}\|u(\lambda_n)\| \leqslant \frac{C}{C_2}\mu_2^n.$$

Thus, from (5.21), (5.22) and (5.23), we get (5.29). And, since h_x is continuous, we also get (5.28). So, for $x \in A \cap \overline{\mathbb{D}}_r$, we can conclude that

$$d(x, (M(n))_r) \to 0 \quad \text{when} \quad n \to \infty.$$

For the general case, we use the same idea as in proposition 5.8. Given $\varepsilon > 0$, there exist $x_1, ..., x_k$ in A such that

$$X(\lambda_0) \cap \overline{\mathbb{D}}_r \subset \bigcup_{i=1}^{k} D(x_i, \varepsilon/2).$$

Then, for every $x \in X(\lambda_0) \cap \overline{\mathbb{D}}_r$, for n large enough, we have

$$d(x, (M(n))_r) \leqslant \|x - x_i\| + d(x_i, (M(n))_r) < \varepsilon,$$

for some $i \in \{1, ..., k\}$. Of course, the case $x \in \partial\mathbb{D}_r$ is trivial. Thus, we finally have

$$\delta((X(\lambda_0))_r, (M(n))_r) \to 0 \quad \text{when} \quad n \to \infty$$

and consequently,

$$\lim_{n\to\infty} d_r(T(\lambda_0)^n \cdot S\tau_{-\lambda_0} M, X(\lambda_0)) = 0.$$

To prove the second claim of the proposition, assume $\lambda_0 = 0$ and let $s < r/\|S\|$. Then,

$$S(\overline{\mathbb{D}}_s) \subseteq \overline{\mathbb{D}}_r.$$

Since S and $T(0)$ commute, we have

$$\lim_{n\to\infty} d_r(S \cdot T(\lambda_0)^n M, X(\lambda_0)) = 0.$$

Hence,

$$\begin{aligned}
\lim_{n\to\infty} (T(0)^n M)_s &= \lim_{n\to\infty} (S^{-1} \cdot S \cdot T(0)^n M)_s \\
&= S^{-1} \left[\lim_{n\to\infty} \left(S \cdot T(0)^n M \cap S(\overline{\mathbb{D}}_s) \right) \cup S(\partial\mathbb{D}_s) \right] \\
&= S^{-1} \left[\lim_{n\to\infty} \left[\left(\left(S \cdot T(0)^n M \cap \overline{\mathbb{D}}_r \right) \cup \partial\mathbb{D}_r \right) \cap S(\overline{\mathbb{D}}_s) \right] \cup S(\partial\mathbb{D}_s) \right] \\
&= S^{-1} \left[\left(X(0) \cap S(\overline{\mathbb{D}}_s) \right) \cup S(\partial\mathbb{D}_s) \right] \\
&= \left(S^{-1} X(0) \cap \overline{\mathbb{D}}_s \right) \cup \partial\mathbb{D}_s.
\end{aligned}$$

□

Remark. Note that the hypothesis of Proposition 5.7 satisfy all hypothesis from Proposition 5.9 in the one dimensional case. Indeed, since u is analytic, it has an expansion in power series around c_0:

$$u(c) = u'(c_0)(c - c_0) + \mathcal{O}(|c - c_0|^2).$$

Then, if $S = u'(c_0)$, it follows that

$$|u(c) - S(c - c_0)| = \mathcal{O}(|c - c_0|^2).$$

If we define $T(c) := I\varrho(c)$, for each $c \in \Lambda$, the mean value inequality gives us

$$|\varrho(c) - \varrho_0| = \mathcal{O}(|c - c_0|).$$

Moreover,

$$\|T(c)\| \cdot \|T(c)^{-1}\|^2 = |\varrho(c)| \cdot |\varrho(c)|^{-2} = |\varrho(c)|^{-1} < 1.$$

Then Proposition 5.7 can be seen as a particular case of Proposition 5.9.

We now go back to the quadratic family $f_c(z) = z^2 + c$ to prove our main result. A quick remark, before stating the theorem, recall that the Mandelbrot set is defined as

$$\mathcal{M} := \{c \in \mathbb{C} \mid 0 \in K_c\}.$$

It is equivalent to say

$$\mathcal{M} = \{c \in \mathbb{C} \mid c \in K_c\}.$$

Theorem 5.4. *Suppose c_0 is a Misiurewicz point. Then the Mandelbrot set $\mathcal{M}$ is asymptotically similar to J_{c_0} about c_0 up to multiplication by a constant $\lambda \in \mathbb{C} - \{0\}$. In other words, there exists $s > 0$ such that*

$$\lim_{t \in \mathbb{C},\, t \to \infty} d_s(t\tau_{-c_0}\mathcal{M}, t\lambda\tau_{-c_0}J_c) = 0.$$

Proof. Let c_0 be a Misiurewicz point and let l and p be the minimal integers such that

$$f_{c_0}^p(f_{c_0}^l(c_0)) = f_{c_0}^l(c_0).$$

Set

$$\alpha_0 = f_{c_0}^l(c_0) \quad \text{and} \quad (f_{c_0}^p)'(\alpha_0) = \varrho_0.$$

Define $g : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$g(c, z) = f_c^p(z) - z.$$

We have

$$g(c_0, \alpha_0) = 0 \quad \text{and} \quad \frac{\partial g}{\partial z}(c_0, \alpha_0) = \varrho_0 - 1 \neq 0.$$

Then, by the implicit mapping theorem, there exists a neighborhood W of c_0 and an analytic function $\alpha : W \to \mathbb{C}$, with $c \mapsto \alpha(c)$ such that

$$f_c^p(\alpha(c)) = \alpha(c),$$

for every $c \in W$. Consequently, we can define an analytic function $\varrho : W \to \mathbb{C}$ given by

$$\varrho(c) = (f_c^p)'(\alpha(c)).$$

Since ϱ is continuous, we can restrict W in such a way that $|\varrho(c)| > 1$, for every $c \in W$.

We aim to use proposition 5.7. We will determinate an explicit form for $X(c)$ at each $c \in W$, so we can define the set X and the function u that will satisfy the hypotheses of the proposition. For every $c \in W$, it is defined the König's linearization map φ_c about $\alpha(c)$ and the map φ_c varies analytically with respect to c. In other words, there exists a analytic map $\phi : U \subseteq W \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ defined by

$$\phi(c, z) = (c, \varphi_c(z)).$$

Since the function $(c, z) \mapsto (c, f_c^l(z))$ is analytic, the composition

$$\Phi(c, z) = (c, \varphi_c \circ f_c^l(z))$$

is also analytic.

The set $U = W' \times V$ for some $W' \subseteq W$ and an open set V. Let $r > 0$ such that $\overline{D(\alpha_0, r)} \subseteq V$. Restrict W', if necessary, so if $c \in W'$ then $\alpha(c) \in D(\alpha_0, r/2)$. By the mean value inequality,

$$|\alpha(c) - \varphi_c^{-1}(x)| \leqslant M \cdot |x|,$$

for every $x \in \varphi_c(\overline{D(\alpha_0, r)})$. Note that we can restrict V so we get an uniform constant M for all $c \in W'$. Then,

$$\frac{r}{2} \leqslant \inf_{x \in \varphi_c(\partial D(\alpha_0, r))} |\alpha(c) - \varphi_c^{-1}(x)| \leqslant \inf_{x \in \varphi_c(\partial D(\alpha_0, r))} M \cdot |x|.$$

Hence, given $r' < r/(2 \cdot M)$, we get

$$\overline{\mathbb{D}}_{r'} \subseteq \varphi_c(V),$$

for all $c \in W'$.

Now, assume W' is closed and restrict it, if necessary, so that we have

$$\Phi(c, c) \in W' \times \overline{\mathbb{D}}_{r'}, \tag{5.32}$$

for every $c \in W'$. Set

$$\Omega = \Phi^{-1}(W' \times \overline{\mathbb{D}}_{r'}) \quad \text{and} \quad \Omega_c = \{z \in \mathbb{C} \mid (c, z) \in \Omega\}$$

Since $W' \times \overline{\mathbb{D}}_{r'}$ is closed and Φ continuous, the set Ω is closed. From lemma 5.3, the set Ω_c is closed, for each $c \in W'$. Also, note that (5.32) guarantees that $c \in \Omega_c$, for each $c \in W'$.

Now we can express an explicit form for $X(c)$ and X. Define

$$\mathcal{K} = \{(c, z) \mid z \in K_c\}$$

$$X(c) = \varphi_c \circ f_c^l(K_c \cap \Omega_c)$$

and

$$X = \Phi(\mathcal{K} \cap \Omega).$$

Using the notation from proposition 5.7, we can put $\Lambda = W'$. Then,

$$X(c) = \{z \in \mathbb{C} \mid (c, z) \in X\}.$$

Indeed, let $z \in \mathbb{C}$ such that $(c, z) \in X$. By definition, there exists $x \in \mathbb{C}$ such that $(c, x) \in \mathcal{K} \cap \Omega$ and $\varphi_c \circ f_c^l(x) = z$. But $(c, x) \in \mathcal{K} \cap \Omega$ if and only if $x \in K_c \cap \Omega_c$. Hence, $z \in \varphi_c \circ f_c^l(K_c \cap \Omega_c)$.

The set X is closed, since $\mathcal{K}$ is closed, according to proposition 4.2, and $X(c)$ is $\varrho(c)$-self-similar about 0. Indeed, this can be proven using the same ideas from theorem 5.2 and setting $U = \varphi_c^{-1}(\mathbb{D}_{r'/|\varrho(c)|})$.

As in proposition 5.7, define $u : W' \to \mathbb{C}$ by

$$u(c) = \varphi_c(f_c^l(c)).$$

Then, $u(c_0) = 0$ and

$$M = \{c \in W' \mid u(c) \in X(c)\} = \left\{c \in W' \mid \varphi_c \circ f_c^l(c) \in \varphi_c \circ f_c^l(K_c \cap \Omega_c)\right\}.$$

Which means

$$M = \{c \in W' \mid c \in K_c \cap \Omega_c\}.$$

But $c \in \Omega_c$ for each $c \in W'$. Hence

$$M = \{c \in W' \mid c \in K_c\} = \mathcal{M} \cap W'.$$

So M is actually a neighborhood of c_0 intersected with the Mandelbrot set. The fact that $u'(c_0) \neq 0$ is proven by lemma 5.5. We will see that it has an explicit form, which is quite useful to create examples.

We have all but one of the hypotheses satisfied from proposition 5.7. Let $R_{c_0} \subseteq K_{c_0} \cap \Omega_{c_0}$ be the set of repelling periodic points in $K_{c_0} \cap \Omega_{c_0}$. We know that R_{c_0} is dense in $K_{c_0} \cap \Omega_{c_0}$. So $A := \varphi_{c_0} \circ f_{c_0}^l(R_{c_0})$ is dense in $X(c_0)$. Let $x \in R_{c_0}$ be a k-periodic point under f_{c_0}. Define the function $\psi : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by

$$\psi(c, z) = f_c^k(z) - z.$$

Then

$$\psi(c_0, x) = 0 \quad \text{and} \quad \frac{\partial \psi}{\partial z}(c_0, x) = (f_{c_0}^k)'(x) - 1 \neq 0.$$

Therefore, by the implicit mapping theorem, there exists a neighborhood U_x of c_0 and a function $\psi_x : U_x \to \mathbb{C}$ such that

$$\psi_x(c_0) = x \quad \text{and} \quad f_c^k(\psi_x(c)) = \psi_x(c),$$

for all $c \in U_x$. We can assume that $\psi_x(c)$ will be a repelling periodic point under f_c. Hence, $\psi_x(c) \in K_c$. Also, since $x \in \Omega_{c_0}$, it follows that

$$\Phi(c_0, x) \in W' \times \overline{\mathbb{D}}_{r'}.$$

We can restrict U_x in a way that $U_x \subseteq W'$ and, since Φ is continuous, again by restricting U_x, if necessary, we get $\psi_x(c) \in \Omega_c$, for all $c \in U_x$. Now, set $x' = \varphi_c \circ f_c^l(x)$ and $h_{x'}(c) = \varphi_c \circ f_c^l \circ \psi_x(c)$.We conclude that $h_{x'}(c_0) = x'$ and $h_{x'}(c) \in X(c)$.

Finally, all the hypotheses of proposition 5.7 are satisfied. We can conclude that $\mathcal{M}$ is asymptotically ϱ_0-self-similar about c_0, with limit model.

$$\frac{1}{u'(c_0)}X(c_0).$$

But the Julia set J_{c_0} is asymptotically ϱ_0-self-similar about c_0 with limit model

$$Z = \frac{1}{(f_{c_0}^l)'(c_0)}X(c_0).$$

Then, the limit model of $\mathcal{M}$ can be written as

$$\frac{(f_{c_0}^l)'(c_0)}{u'(c_0)}Z.$$

Set

$$\lambda = \frac{(f_{c_0}^l)'(c_0)}{u'(c_0)}.$$

Thus, there exists $s > 0$ such that

$$\lim_{n\infty} d_s(\varrho_0^n \tau_{-c_0}\mathcal{M}, \varrho_0^n \lambda \tau_{-c_0} J_{c_0}) = 0.$$

From proposition 5.3, we get the desired result.

□

The only missing part is the proof that $u'(c_0) \neq 0$.

Lemma 5.5. *Let $u : W' \to \mathbb{C}$ be as in theorem 5.4. Then,*

$$u'(c_0) = \frac{d}{dc}(f_c^l(c))|_{c=c_0} - \frac{d}{dc}(\alpha(c))|_{c=c_0} \neq 0.$$

Proof. Using the same notation as in the previous theorem, set $\alpha : W' \to \mathbb{C}$ and define $\beta : \mathbb{C} \to \mathbb{C}$ by

$$\beta(c) = f_c^l(c) = f_c^{l+1}(0).$$

It follows that $\beta(c_0) = \alpha_0$. Also, set $F(c,z) = \varphi_c(z)$. We have $F(c, \beta(c)) = u(c)$. Then,

$$u'(c_0) = \frac{\partial F}{\partial c}(c_0, \alpha_0) + \frac{\partial F}{\partial z}(c_0, \alpha_0) \cdot \beta'(c_0). \tag{5.33}$$

We have

$$\frac{\partial F}{\partial z}(c_0, \alpha_0) = \varphi_{c_0}'(\alpha_0) = 1 \tag{5.34}$$

and, on the other hand,

$$\frac{\partial F}{\partial c}(c_0, \alpha_0) = \lim_{c \to c_0} \frac{F(c, \alpha_0) - F(c_0, \alpha_0)}{c - c_0}.$$

Since $F(c_0, \alpha_0) = F(c, \alpha(c)) = 0$, we can substitute one by the other to get

$$\frac{\partial F}{\partial c}(c_0, \alpha_0) = \lim_{c \to c_0} \frac{F(c, \alpha_0) - F(c, \alpha(c))}{c - c_0} = -\lim_{c \to c_0} \frac{F(c, \alpha(c)) - F(c, \alpha_0)}{c - c_0}. \tag{5.35}$$

But

$$\begin{aligned}
\frac{\partial F}{\partial z}(c_0, \alpha_0) \cdot \alpha'(c_0) &= \lim_{z \to \alpha_0} \frac{F(c_0, z) - F(c_0, \alpha_0)}{z - \alpha_0} \cdot \lim_{c \to c_0} \frac{\alpha(c) - \alpha_0}{c - c_0} \\
&= \lim_{c \to c_0} \frac{F(c, \alpha(c)) - F(c, \alpha_0)}{\alpha(c) - \alpha_0} \cdot \lim_{c \to c_0} \frac{\alpha(c) - \alpha_0}{c - c_0} \\
&= \lim_{c \to c_0} \frac{F(c, \alpha(c)) - F(c, \alpha_0)}{c - c_0}.
\end{aligned}$$

Hence, from (5.33), (5.34) and (5.35), we get

$$u'(c_0) = \beta'(c_0) - \alpha'(c_0). \tag{5.36}$$

The proof that $u'(c_0) \neq 0$ is in Theorem 4.3.

$\square$

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