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Tese de Doutorado

Contributions To Phase Transition Of Intermittent Skew-Product And Piecewise Monotone Dynamics On The Circle

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Tese de Doutorado apresentada ao Colegiado da Pós-Graduação em Matemática UFBA/UFAL como requisito parcial para obtenção do título de Doutor em Matemática.

Orientador: Prof. Dr. Thiago Bomfim São Luiz Nunes.

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AFONSO FERNANDES DA SILVA

Tese apresentada ao Colegiado do Curso de Pós-graduação em Matemática da Universidade Federal da Bahia, como requisito parcial para obtenção do Título de Doutor em Matemática.

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"O amor tudo sofre, tudo crê, tudo espera e tudo suporta."

(Paulo, Apóstolo (I Coríntios 13))

Resumo

Sabe-se que toda dinâmica uniformemente espansora ou hiperbólica transitiva não possui transição de fase com respeito aos potenciais Hölder contínuos. Em se tratando de dinâmicas mais gerais, ainda é uma questão em aberto classificar todas as dinâmicas que possuem transição com respeito a uma certa classe de potenciais regulares. Em dimensão 1, de acordo com Bomfim-Carneiro [BC21], todo $C^{1+\alpha}$-difeomorfismo local no círculo transitivo que não é expansor nem invertível tem uma única transição de fase temodinâmica com respeito ao potencial geométrico, em outras palavras, a função pressão topológica $\mathbb{R} \ni t \mapsto P_{top}(f, -t \log |Df|)$ é analítica exceto em um ponto $t_0 \in (0, 1]$. Eles também provaram transição de fase espectral, ou seja, o operador de transferência $\mathcal{L}_{f,-t \log |Df|}$ agindo no espaço das funções hölder contínuas tem gap espectral para todo $t < t_0$ e não apresenta gap spectral para $t \geq t_0$. Nosso objetivo é provar resultados similares para duas classes especiais de dinâmicas: endomorfismos de codimensão 1 parcialmente hiperbólicos e dinâmicas monótonas por partes no círculo transitivas. Para endomorfismos em dimensão alta provamos que os resultados de transição de fase termodinâmica e espectral implicam em análise multifractal para o espectro de Lyapunov. Em particular exibimos uma clase de endomorfismos parcialmente hiperbólicos que admitem transição de fase termodinâmica e espectral com relação ao potencial geométrico na direção central, e descrevemos análise multifractal dos expoente de Lyapunov central. Para dinâmicas monótonas por partes no círculo, provamos que o conjunto de potenciais Hölder contínuos que **não** possuem transição de fase termodinâmica e espectral é denso na topologia uniforme e o conjunto de potenciais Hölder contínuos que têm transição de fase **não** é denso na topologia uniforme. Também obtemos uma caracterização de transição em termos do operador de transferência e do tipo de convexidade da função pressão topológica. Em particular, descrevemos o comportamento da função pressão topológica e do operador de transferência associado.

Palavras-chave: Transição de fase; Formalismo Termodinâmico; Operador de Transferência; Princípios de Grandes Desvios; Análise Multifractal.

Abstract

It is well known that all transitive uniformly expanding or hyperbolic dynamics have no phase transition with respect to Hölder continuous potentials. For more general dynamics, It is still an open question to classify all the dynamics having phase transition with respect to a certain class of regular potential. In dimension one, due to the work of Bomfim-Carneiro [BC21], it was proved that for all transitive $C^{1+\alpha}-$local diffeomorphism f on the circle that is neither a uniformly expanding map nor invertible, has an unique thermodynamic phase transition with respect to the geometric potential, in other words, the topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(f, -t\log|Df|)$ is analytic except in a point $t_0 \in (0, 1]$. Furthermore, they proved spectral phase transitions, more specific, the transfer operator $\mathcal{L}_{f,-t\log|Df|}$ acting on the space of Hölder continuous functions, has the spectral gap property for all $t < t_0$ and does not have the spectral gap property for all $t \geq t_0$. We aim to prove similar results for two special cases of dynamics: a co-dimension 1 partially hyperbolic endomorphisms and transitive piecewise monotone on the circle. For the higher dimension we prove that thermodynamic and spectral phase transition lead to multifractal analysis of the Lyapunov spectrum, in particular we exhibit a class of partially hyperbolic endomorphism having phase transition with respect to the geometric potential in the central direction and describe the multifractal analysis of the central Lyapunov spectrum. For transitive piecewise monotone maps, we prove that the set of Hölder continuous potentials which do not have spectral and thermodynamic phase transition is dense in the uniform topology and the set of Hölder continuous potentials that has phase transition are not dense. Furthermore, we provide a description of phase transition based on the properties of the transfer operator and the type of convexity of the topological pressure function. In particular, we describe the behavior of the topological pressure function and the transfer operator associated.

Keywords: Phase Transition; Thermodynamic formalism; Transfer Operator; Large Deviation Principles; Multifractal Analysis.

Contents

Introduction

Thermodynamic and Spectral Phase Transition

In Physics, the term **phase transition** is mostly used to describe the different states of matter: solid, liquid and gaseous. During a phase transition, we often have a drastic change of properties, like a discontinuity, as a result of external changes such as temperature, pressure or others phenomenons. It corresponds to a qualitative change in the statistical properties of a dynamical system. In dynamical systems there is no unanimity on the meaning of phase transition, it depends on which settings or properties we are studying. For some authors in the literature, phase transition can be related, for example, to the non-uniqueness of equilibrium states, or lack of Gateaux differentiability of the pressure function. In this work, phase transition means that the topological pressure function is not analytic:

Definition 0.0.1. *We say that a map $f : M \to M$ has a (thermodynamic)* ***phase transition*** *with respect to a potential $\phi : M \to \mathbb{R}$, if*

$$\mathbb{R} \ni t \mapsto P_{top}(f, t\phi)$$

is not analytic at some point t_0. Additionally, if f is analytic for all t except t_0, we say that the phase transition is ***effective****. In that case, we call this t_0 the transition parameter.*

0.1 Historical context of Phase Transition

Due to the works of Sinai, Bowen and Ruelle [S72, Bow75, BR75] transitive hyperbolic or expanding dynamics do not admit phase transition with respect to Hölder continuous potentials. From a more general point of view, there are many examples in the literature of non-uniformly hyperbolic dynamics that admit phase transition with respect to the geometric potential[1] or regular potentials:

- Manneville-Pomeau maps and geometric potential [Lo93],

[1] $\phi := -\log |Df|$ is refered as the "geometric potential".

- a large class of interval maps with indifferent fixed point and geometric potential [PS92],
- certain quadratic maps and geometric potential [CRL13, CRL15, CRL19],
- certain non-degenerate smooth interval map and geometric potential [CRL21],
- porcupine horseshoes and geometric-type potential [DGR14],
- geodesic flow on Riemannian non-compact manifolds with variable pinched negative sectional curvature and suitable Hölder potential [IRV18],
- geodesic flow on certain M-puncture sphere and geometric-type potential [V17].

However, it is still an open question to decide exactly which dynamical systems admit phase transitions. Recently, Bomfim and Carneiro [BC21] proposed the following problem:

Problem A. *What is the mechanism responsible for the existence of phase transitions for C^1-local diffeomorphisms, with $h_{top}(f) > 0$, with respect to Hölder continuous potentials ?*

Remark 0.1.1. *Walters [W92] proved that if f is expansive and has finite topological entropy then the lack of differentiability of the topological pressure function is related to non-uniqueness of equilibrium states associated to a potential ϕ.*

Let's recall a very important concept in dynamical system, the transfer operator, for more details see e.g. [S12] or [PU10]. This operator is fundamental to studying thermodynamic quantities and indeed obtaining equilibrium states and its properties. We define the *Ruelle-Perron-Frobenius operator* or *transfer operator*, which acts on function spaces, as the following:

Definition 0.1.2. *Let $f : M \to M$ be a local homeomorphism on a compact and connected manifold. Given a complex continuous function $\phi : M \to \mathbb{C}$, define the Ruelle-Perron-Frobenius operator or transfer operator $\mathcal{L}_{f,\phi}$ acting on functions $g : M \to \mathbb{C}$ as following:*

$$\mathcal{L}_{f,\phi}(g)(x) := \sum_{f(y)=x} e^{\phi(y)} g(y).$$

Generally, one studies this operator acting on a Banach space E which is dense in $C^0(X, \mathbb{C})$. Classical thermodynamic results for sufficiently *chaotic* dynamics derive from good *spectral* properties from this operator. The transfer operator for dynamics such as *expanding* maps are shown to have *spectral gap* for a large set of potentials (e.g. all Hölder functions), a concept which we recall now:

Definition 0.1.3. *Given E a complex Banach space and $\mathcal{L} : E \to E$ a bounded linear operator, we say that $\mathcal{L}$ has the (strong)* ***spectral gap*** *property if there exists a decomposition of its spectrum $sp(\mathcal{L}) \subset \mathbb{C}$ as follows: $sp(\mathcal{L}) = \{\lambda_1\} \cup \Sigma_1$ where $\lambda_1 > 0$ is a leading eigenvalue for $\mathcal{L}$ with one-dimensional associated eigenspace and there exists $0 < \lambda_0 < \lambda_1$ such that $\Sigma_1 \subset \{z \in \mathbb{C} : |z| < \lambda_0\}$.*

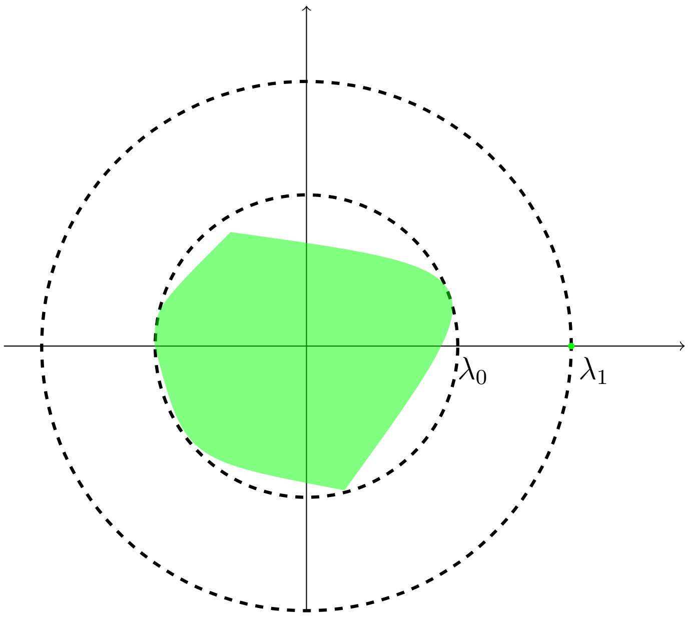

Figure 1: The figure illustrate the spectrum of a linear operator satisfying spectral gap property

From the spectral gap property, we obtain candidates for equilibrium states as follows: if ϕ is a real continuous function, via **Mazur's Separation Theorem**, every transfer operator has its spectral radius $\rho(\mathcal{L}_{f,\phi}|_{C^0})$ as an eigenvalue for its dual operator, that is, there exists a probability ν_ϕ with $(\mathcal{L}_{f,\phi}|_{C^0})^*\nu_\phi = \rho(\mathcal{L}_{f,\phi}|_{C^0})\nu_\phi$. If additionally $E \subset C(M, \mathbb{C})$ is a Banach space continuously immersed in $C(M, \mathbb{C})$ and $\mathcal{L}_{f,\phi}|_E$ has the spectral gap property, then $\rho(\mathcal{L}_{f,\phi}|_{C^0}) = \rho(\mathcal{L}_{f,\phi}|_E)$ and $\mathcal{L}_{f,\phi}|_E$ admits an eigenfunction $h_\phi \in E$ with respect to $\rho(\mathcal{L}_{f,\phi}|_E)$ which is the leading eigenvalue. We can assume, up to rescaling, that $\int h_\phi d\nu_\phi = 1$, then the probability measure $\mu_\phi = h_\phi \cdot \nu_\phi$ is proved to be $f-$invariant and is a candidate to be the equilibrium state of f with respect to ϕ.

When f is a mixing expanding or hyperbolic dynamic and ϕ is a suitable potential, the thermodynamic properties can be recovered through the transfer operator $\mathcal{L}_{f,\phi}$. That is possible by the fact that the transfer operator $\mathcal{L}_\phi$ has the spectral gap property acting on a suitable Banach space, and it can be shown that f does not have phase transition

with respect to such suitable potentials (see e.g. [PU10]). It's well known that from a dynamical system having spectral gap property, we can deduce many important statistical properties of thermodynamic quantities such as: equilibrium states, mixing properties, large deviations and limit theorems, stability of the topological pressure and equilibrium states, or even differentiability results for thermodynamic quantities. (see e.g. [Ba00, GL06, BCV16, BC19]).
Bomfim and Carneiro [BC21] conjectured the following:

Conjecture A. *Let $f : M \to M$ be a C^2-local diffeomorphism on a compact Riemannian manifold M. If f is not a uniformly expanding map or uniformly hyperbolic diffeomorphism, then there exists a suitable potential ϕ such that $\mathcal{L}_{f,\phi}$ does not have the spectral gap property acting on a suitable Banach space. (Hölder continuous or smooth functions).*

Remark 0.1.4. *Given $r \geq 1$ and a integer $\alpha \in (0, 1]$ we denote by $C^r(\mathbb{S}^1, \mathbb{C})$ and $C^\alpha(\mathbb{S}^1, \mathbb{C})$ the Banach space of C^r functions and α-Hölder continuous complex functions whose domain is $\mathbb{S}^1$, respectively.*

As a first step in this direction, we have an answer from [BC21] for Problem A and Conjecture A when $M = \mathbb{S}^1$. They proved the following:

Theorem 0.1.5 ([BC21])**.** *Let $E = C^\alpha(\mathbb{S}^1, \mathbb{C})$ or $C^r(\mathbb{S}^1, \mathbb{C})$ and let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a transitive not invertible C^1-local Diffeomorphism with Df Hölder continuous (respectively C^r). If f is not expanding, then there exist $t_0 \in (0, 1]$ such that the transfer operator $\mathcal{L}_{f,-t\log|Df|}$ has the spectral gap property on E for all $t < t_0$ and doesn't have spectral gap property for $t \geq t_0$.*

As a consequence of the previous theorem, one has effective thermodynamic phase transition:

Corollary 0.1.5.1 ([BC21])**.** *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ a transitive non-invertible C^1-local diffeomorphism with Df Hölder continuous. If f is not a expanding map, then the topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(f, -t\log|Df|)$ is analytic, strictly decreasing and strictly convex in $(-\infty, t_0)$, and constant equal to zero in $[t_0, +\infty)$. In particular, f has a unique thermodynamic phase transition with respect to the geometric potential $-\log|Df|$.*

Our first two results are to obtain similar results as in Bomfim-Carneiro [BC21] for a certain class of dynamical systems in higher dimension. The second part of our results, in the one-dimensional case, are to study existence of phase transition and its consequence for thermodynamic and spectral concepts.

0.2 Context: definition of our classes of dynamics

Now, we present the two classes of dynamics that we study. Here we set up the kind of maps we consider in our study.

0.2.1 Codimension one partially hyperbolic endomorphism

We now describe what we mean by saying that $F : M \to M$ is a codimension one partially hyperbolic endomorphism on a manifold M. Later on, we will define the class of skew-product that is a particular case and the object of our study satisfying such properties.

Definition 0.2.1. *F is a codimension one partially hyperbolic endomorphism if F is a C^1-local diffeomorphism and there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ and positive constants $\lambda > 1$, $\sigma < 1$ and $c > 0$ such that*

- *E^s, E^c, E^u are DF-invariant and $Dim(E^c) = 1$;*
- *$||DF^n|_{e^s}|| \leq e^{-\sigma n + c}$, $||DF^n|_{E^u}|| \geq e^{\lambda n + c}$;*
- *$||DF^n|_{E^c}|| \geq e^{\sigma n - c}||DF^n|_{E^s}||$, $||DF^n|_{E^c}|| \leq e^{-\lambda n + c}||DF^n|_{e^u}||$*

The subbundles E^s, E^c and E^u are called stable, central and unstable, respectively. For more details on partially hyperbolic endomorphism, the reader can check the work of Varandas-Cruz [VC18] and Álvarez and Cantarino [AC22]. The partially hyperbolic endomorphism are a natural generalization of the partially hyperbolic diffeomorphism for the non-invertible context. When $M = \mathbb{T}^2$ is the two-dimensional torus and $E^s = 0$, we know that there exists an open subset of robust transitivity partially hyperbolic endomorphism that are not expanding, see [LP13]; furthermore, a generic partially hyperbolic endomorphism admits finitely many ergodic physical measures whose union of the basin of attraction has total Lebesgue measure [T05]. Since that E^c is $DF-$invariant, then the central Lyapunov exponent $\lambda_c(x) = \lim \log \frac{1}{n} \log ||DF|_{E^c_x}(x)||$ is well defined on a total probability subset (see e.g. the Oseledets Theorem in [V14]).

Remark 0.2.2. *One cannot expect all local diffeomorphism in higher dimensions (not uniformly expanding or hyperbolic) to have thermodynamic and spectral phase transition, Bomfim and Carneiro [BC21] presented the following example:*

Example 0.2.3 ([BC21])**.** *Take the following maps on the circle*

$$f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1 \qquad\qquad R_\alpha : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$
$$x \mapsto 2x \ mod \ 1 \qquad\qquad y \mapsto y + \alpha$$

which are an expanding map and an irrational rotation ($\alpha \in \mathbb{Q}'$), respectively. Now define on the torus the direct product

$$\begin{aligned} F : \mathbb{S}^1 \times \mathbb{S}^1 &\longrightarrow \mathbb{S}^1 \times \mathbb{S}^1 \\ (x, y) &\mapsto (f(x), R_\alpha(y)) \,. \end{aligned}$$

Note that R_α is uniquely ergodic with $m :=$ Leb being its unique invariant probability. Thus given any Hölder potential $\phi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$, we can define the Hölder potential

$$\begin{aligned} \tilde{\phi} : \mathbb{S}^1 &\longrightarrow \mathbb{R} \\ x &\mapsto \int \phi(x, y) dm(y) \end{aligned}$$

And the variational principle for F becomes the variational principle for f:

$$P_{top}(F, \phi) = \sup_{\mu \in \mathcal{M}_1(f)} \{ h_\mu(f) + h_m(R_\alpha) + \int \phi(x, y) dm(y) d\mu(x) \} = P_{top}(f, \tilde{\phi}).$$

Since f is an expanding map then $C^\alpha(\mathbb{S}^1, \mathbb{R}) \ni g \mapsto P_{top}(f, g)$ is analytic. We conclude that $C^\alpha(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{R}) \ni \phi \mapsto P_{top}(F, \phi)$ is analytic, even though F is not an expanding nor hyperbolic dynamics.

Motivating by the work of Bomfim and Carneiro [BC21] in one dimensional dynamics, we are concern to solve the following problem:

Problem B. *What are the consequences of understanding the topological pressure function and the associated transfer operator ?*

Our response to problem B is to show consequences of phase transition for multifractal analysis. More precisely, we exhibit a class of **intermittent skew-product** that admit thermodynamical and spectral phase transitions with respect to the geometric-type potential and describe the multifractal analysis of its central Lyapunov spectrum. In multifractal analysis, we study invariant sets and measures with a multifractal structure. We are essentially measuring the size of those sets, in the sense of Hausdorff dimension or topological entropy, for instance. Given $\phi : M \to \mathbb{R}$ a continuous functions and $I \subset \mathbb{R}$ an interval, define

$$X(I) := \left\{ x \in M; \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x)) \in I \right\}$$

The multifractal Analysis of that sequence is basically understanding those kinds of sets metrically, geometrically (Hausdorff dimension), topologically (topological entropy, topological pressure), thermodynamically (variational principles), etc.

In dynamical systems, given $F : M \to M$ a continuous transformation on a compact metric space we are particularly interested in, for instance,

1. $\frac{1}{n}\sum_{i=0}^{n-1} \phi \circ F^i$ (Birkhoff's averages)
2. $\frac{1}{n}\log ||DF^n||$ and $\frac{1}{n}\log ||DF^n||^{-1}$ (extreme Lyapunov exponents)
3. $\frac{1}{n}\log \mu(B(.,n,\epsilon))$, where $B(x,n,\epsilon)$ is the Bowen dynamical ball of radius ϵ and length n. (local entropy)

We are interested in the multifractal analysis of central **Lyapunov exponent** $\lambda_c(F)$, that is, of the sequence $\phi_n = \log ||DF^n|_{E^c}||$. Since that E^c is one dimensional and DF-invariant, then $\phi_n := \sum_{i=0}^{n-1} \phi \circ F^i$, where $\phi = \log ||DF|_{E^c}||$, what connects the central Lyapunov exponent with the Birkhoff's ergodic theorem. The study of the topological pressure or Hausdorff dimension of the subsets where the Birkhoff average diverges or converges to a fixed interval can be traced to Besicovitch and this topic has had contributions by many authors in recent years (see [DK01, C10, GR09, JR11, PW97, PW99, T08, TV99, Th10, ZC13, BV17, IJT17, JR21]). In our settings, we study multifractal analysis created by the central Lyapunov exponent: define the following sets

$$L^c_{a,b} = X_\phi([a,b]) = \{x \in \mathbb{T}^d \times \mathbb{S}^1; \lambda^c(x) \in [a,b]\}$$
$$\hat{L}^c := \{x \in \mathbb{T}^d \times \mathbb{S}^1; \nexists \lim \frac{1}{n}\log ||DF^n|_{E^c_x}||\}.$$

in particular, when $a = b$ we denote $\hat{L}^c_a = \hat{L}^c_{a,a}$ as the level set of points which central Lyapunov exponent is a. We notice that the previous sets decompose the space of orbits. From the ergodic point of view, given μ an ergodic $F-$invariant probability, we have that $\mu(L^c_{a,b})$ is 1 or 0 by the Birkhoff ergodic theorem, depending wether $\int \phi d\mu$ is in $[a,b]$ or not, respectively. Moreover $\mu(\hat{L}^c) = 0$. So these sets may not be that interesting from the measure theoretical point of view, but we have to be careful, these sets can be topologically large! By Thompsom [T08] and Lima-Varandas [LV21], if F has topological properties as the specification property or gluing orbit property and $\hat{L}^c \neq \emptyset$, then $\hat{L}^c$ is a Baire residual subset, has full metric mean dimension (m.m.d) and $P_{\hat{L}^c}(F,\phi) = P_{top}(F,\phi)$, where $P_Z(F,\phi)$ is the topological pressure for the continuous potential ϕ on the set Z (as defined by Pesin [P98]). For that reason, we also consider the sets

$$E^c_{a,b} := \hat{L}^c \cap \Big\{x \in \mathbb{T}^d \times \mathbb{S}^1; \liminf_{n\to\infty} \frac{1}{n}\log ||DF^n|_{E^c_x}|| \in [a,b] \quad or \quad \limsup_{n\to\infty} \frac{1}{n}\log ||DF^n|_{E^c_x}|| \in [a,b]\Big\}.$$

Then we want to answer the following question:

Question 1. *Given $N([a,b]) = L^c_{a,b}$ or $E^c_{a,b}$, describe the following functions:*

$$[a,b] \mapsto h_{N([a,b])}(F)$$
$$[a,b] \mapsto HD(N([a,b]))$$

where $h_Z(F)$ denotes the topological entropy restricted to the set Z and $HD(Z)$ denotes the Hausdorff dimension of the set Z(see Pesin [P98] for both formal definition). Do these functions vary continuously? Smoothly ? Analytically ?

The class of codimension 1 partially hyperbolic dynamics we focus on here are local diffeomorphism, neither expanding nor invertible skew-product satisfying the following properties:

Definition 0.2.4. *Let $\mathcal{D}^r$ be the space of $\mathcal{C}^r$-local diffeomorphism $F : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{T}^d \times \mathbb{S}^1$, given by $F(x,y) := (g(x), f_x(y)), \forall (x,y) \in \mathbb{T}^d \times \mathbb{S}^1$, where:*

1. *g is an expanding linear endomorphism;*
2. *$\deg(F) > \deg(g)$, where $\deg$ is the topological degree of a local homeomorphism;*
3. *There exist a measure η on $\mathbb{S}^1$ such that $\eta \in \mathcal{M}_1(f_x)$ for all $x \in \mathbb{T}^d$, with zero Lyapunov exponent, that is, $\int \log|Df_x| d\nu = 0$ for all $x \in \mathbb{T}^d$.*
4. *F is topologically conjugated to an expanding dynamical system.*

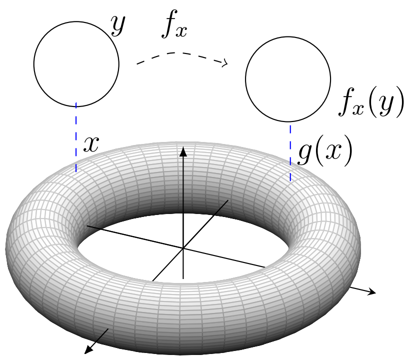

Figure 2: Points in $\mathbb{T}^d$ determine which dynamic will be used to iterate the second coordinate on the circle $\mathbb{S}^1$

Remark 0.2.5. *Given $F \in \mathcal{D}^r$, we fix the potential*

$$\phi^c(x,y) = -\log ||DF_y(x,y)|| = -\log |Df_x(y)|.$$

We denote the central Lyapunov exponent by

$$\lambda^c(z) = -\lim \frac{1}{n} \sum_{i=0}^{n-1} \phi^c(F^i(z)).$$

In the next remark we collect some immediate properties of our class of maps as in definition 0.2.4.

Remark 0.2.6. *The fourth item in definition 0.2.4 implies that F is expansive and satisfies the* ***periodic specification property****, since that F is an open map and $\mathbb{T}^d \times \mathbb{S}^1$ is a connected manifold. Note that by item (2) in definition 0.2.4, the fact that F is a local diffeomorphism and $\mathbb{T}^d \times \mathbb{S}^1$ is connected, we have that $deg(f_x) = \beta$ constant. Despite our toy model F being in a topological class of a expanding map, it admits zero Lyapunov exponent. As F necessarily has an invariant measure with zero central Lyapunov exponent (by definition), F is not an expanding map. Furthermore, by [MP77] $h_{top}(F) = \log deg(F)$.*

0.2.2 Transitive piecewise monotonous maps of the circle

Suppose that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a transitive, non invertible neither expanding, C^1-local diffeomorphism with Df Holder continuous. Following Bomfim and Carneiro [BC21], there exist a unique thermodinamical and spectral phase transition with respect to the geometric potential $-\log|Df|$ and, as a consequence, a good understanding of multifractal analysis. Motivated by that, we formulating a problem for classes of transitive piecewise monotonous maps $f : \mathbb{S}^1 \to \mathbb{S}^1$ with respect to a regular classes of potentials:

Problem C. *For* ***regular*** *potentials, can we describe the topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi)$? Does f has phase transition with respect to ϕ ? How much is common to have phase transition ? How much is common $\mathcal{L}_{f,\phi|C^\alpha}$ to have the spectral gap property ?*

Let's now define a special class of transitive piecewise monotonous maps of the circle and the class of regular potentials we are going to consider:

Definition 0.2.7. *$f : \mathbb{S}^1 \to \mathbb{S}^1$ is a continuous transitive local-diffeomorphism with break points, in other words, f is a continuous, transitive and there exist closed arcs $I_1, ..., I_k \subset \mathbb{S}^1$ such that:*

1. *$\mathbb{S}^1 = \bigcup_{i=1}^{k} I_i$ and th arcs I_i have disjoint interiors;*
2. *$f|_{I_i} : I_i \mapsto \mathbb{S}^1$ is a C^1 diffeomorphism, $i = 1, ..., k$;*
3. *the derivative of f is well defined at its fixed points.*

Later on, studying the properties of the transfer operator associated to our dynamics, we will consider the Banach space $E = C^\alpha(\mathbb{S}^1, \mathbb{R})$ of all Hölder continuous potentials, but we will also denote E as the Banach space $C^r(\mathbb{S}^1, \mathbb{R})$ of C^r potentials, in which case we suppose additionally that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a C^r local diffeomorphism.

Remark. To avoid confusion, everytime we mension E, we are consider both the Hölder and smooth case. When necessary, we will explicitly specify the regularity of our potentials.

We have an answer of Problem C for piecewise monotonous maps like in definition 0.2.7 with respect to the class of regular potential (Hölder continuous). In particular, we show that the set of potentials **not** admitting phase transition is a dense subset in the uniform topology.

The continuous potentials case

It is worth mentioning the situation when the potential is only continuous. In the literature, we have some results related to thermodynamic phase transition for continuous potentials. The following gives us an idea that if we consider only continuous potential, then everything might be possible when we talk about phase transition.
Let $d \in \mathbb{N}$ and $\mathcal{A} = \{0, 1, 2, ..., d-1\}$ be a finite alphabet with d symbols. The (two sided) shift space X on the alphabet $\mathcal{A}$ is the set of all bi-infinity sequences $x = (x_n)_{n\in\mathbb{Z}}$ with $x_n \in \mathcal{A}$. Endow X with the Tychonoff's product topology which makes X a compact metric space with the distance $d(x, y) = 2^{-\inf\{|n|; x_n \neq y_n\}}$.

The shift map $T : X \to X$ given by $T(x)_n = x_{n+1}$ is a homeomorphism. The next results by Kucherenko, Quas and Wolf showed that, in the context of shifts, there is always a continuous potential ϕ such that the topological pressure function associated to T and ϕ has multiple phase transitions.

Theorem 0.2.8. *[KQW21] Let $T : X \to X$ the two sided full shift, $\alpha > 0$ and (β_n) be a strictly increasing (finite of infinite) sequence in $[\alpha, +\infty)$. Then there exist a continuous potential $\phi : X \to \mathbb{R}$ such that the following holds:*

1. *When $\beta \geq \alpha$ the potential ϕ has a fase transition at β if and only if $\beta = \beta_n$ for some $n \in \mathbb{N}$;*

2. *If $\lim_{n\to\infty} \beta_n = \beta_\infty < \infty$, then the family of equilibrium states $\beta\phi$ is constant for all $\beta \geq \beta_\infty$.*

In Kucherenko and Quas [KQ22] the authors presented a method to explicitly construct a continuous potential whose pressure function coincides with any prescribed convex Lipschitz asymptotically linear function starting at a given positive value of the parameter:

Theorem 0.2.9. *[KQ22] Let $\alpha > 0$ and $F(t_1, t_2, ..., t_m)$ be a convex Lipschitz function on $(\alpha, \infty)^m$ such that all the supporting hyperplanes to the graph of F intersect the vertical* *axis in a closed interval* $[b, c] \subset [0, \infty)$. *Then there exists a full shift on a finite alphabet and continuous potentials* $\phi_1, ..., \phi_m$ *such that* $P_{top}(t_1\phi_1 + ... + t_m\phi_m) = F(t_1, ..., t_m)$ *for all* $(t_1, ..., t_m) \in (\alpha, \infty)^m$.

We show that the behaviour presented by the previous theorems cannot occur for maps like in definition 0.2.7 and regular potential.

The Hölder-continuous potentials case

The next result by Kloeckner [Kl20] shows that for any map on the circle which is expanding outside any arbitrary flat neutral fixed point, the set of Hölder potentials exhibiting a spectral gap is dense in the uniform topology:

Theorem 0.2.10. *(Density Of Spectral Gap Potential)[[Kl20]] Let* T *be a degree* K *self-covering of the circle with a neutral fixed point* 0*, uniformly expanding outside each neighbourhood of* 0*. For any* $\alpha \in [0, 1)$*, let* V *be the linear space of* $\mathcal{C}^\alpha$ *potentials which are constant near the neutral point. Then for all* $\phi \in V$ *the transfer operator* $\mathcal{L}_{\mathcal{T},\phi}$ *acting on* $\mathcal{C}^\alpha(\mathbb{T})$ *has spectral gap. Furthermore, for all* $\gamma \in (0, \alpha)$*,* V *is dense in* $\mathcal{C}^\alpha(\mathbb{T})$ *for the* γ*-Hölder norm.*

With the previous discussion in mind, we propose the following question:

Question 2. *How to characterize, in terms of existence of phase transition, the pressure function* $t \to P_{top}(f, t\phi)$ *with respect to a regular potential* ϕ*? Is there a generic (dense) set of potentials which admit phase transitions, or does not admit phase transitions and/or Spectral gap?*

0.3 Statement of the main results

Codimension one partially hyperbolic endomorphism

Our main objective with Theorem A below is to show the existence and uniqueness of a phase transition and give a description of the behaviour of the topological pressure function with respect to the "geometric potential" in the central direction ϕ^c. More specifically, we prove that for $F \in \mathcal{D}^r$ there exists a unique parameter $t_0 \in \mathbb{R}$ where the topological pressure function and the associated transfer operator have distinct behaviour before and after that parameter.

Theorem A. *Let $F \in \mathcal{D}^r$, with $r \geq 2$. There exists a unique $t_0 \in (0,1]$ such that:*

1. *The topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi^c)$ is analytic, strictly decreasing and strictly convex in $(-\infty, t_0)$ and constant equal to $h_{top}(g)$ in $[t_0, +\infty)$;*
2. *The transfer operator $\mathcal{L}_{f,t\phi^c}$, acting on the space $C^{r-1}(\mathbb{T}^d \times \mathbb{S}^1, \mathbb{C})$, has spectral gap for all $t < t_0$ and does not have spectral gap for all $t \geq t_0$.*

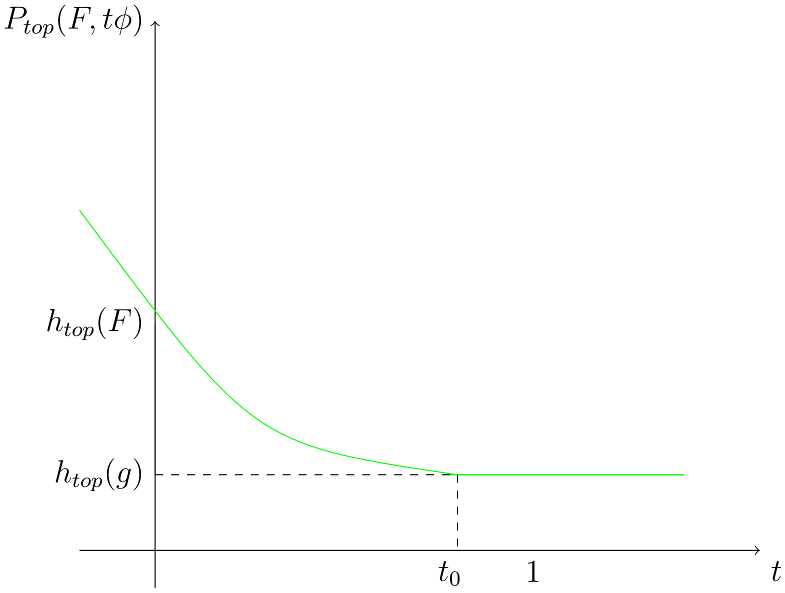

Figure 3: $\mathbb{R} \ni t \mapsto P_{top}(F, t\phi)$ as stated in theorem A.

Since each $F \in \mathcal{D}^r$ is conjugated to an expanding map, F satisfies the **periodic specification property**. Then, following Thompson [Th09], given a continuous potential $\phi : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{R}$, its **Birkhoff spectrum** is given by

$$S_\phi(F) := \Big\{ \alpha \in \mathbb{R}; \exists x \in \mathbb{T}^d \times \mathbb{S}^1 \quad \text{satisfying} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x)) = \alpha \Big\},$$

which is a non-empty bounded interval.

We are interested in studying the fractal sets created from the central Lyapunov exponents, that is, taking the observable $\phi = \log ||\frac{\partial F}{\partial y}||$. In that case, the Birkhoff spectrum of ϕ is the set of central **Lyapunov exponents**, the central Lyapunov spectrum that we denote by $L^c(F)$. So it makes sense to consider only intervals contained within the Birkhoff spectrum. Thus, we define the set:

$$\Delta := \{(a, b) \in \mathbb{R}^2; a \leq b \quad and \quad a, b \in L^c(F)\}$$

Let t_0 be the phase transition parameter provided by Theorem A. As for $t < t_0$ the transfer operator associated $\mathcal{L}_\phi$ has spectral gap property, by Theorem A, denote

$$\lambda^c_{\min} := \inf_{t<t_0} \lambda^c_{\mu_t}(F) \text{ and } \lambda^c_{\max} := \sup_{t<t_0} \lambda^c_{\mu_t}(F).$$

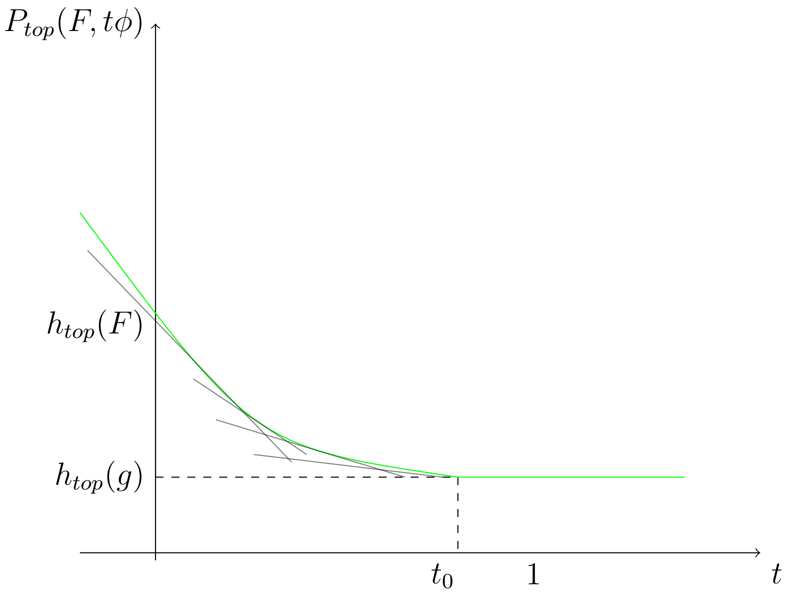

Figure 4: The absolute value of the slopes of the tangent lines to the graph of $\mathbb{R} \ni t \mapsto P_{top}(F, t\phi)$ give the central Lyapunov exponents.

as the infimum and supremum of the central Lyapunov exponents of the unique equilibrium states obtained from *spectral gap property*. We also denote $\lambda^c_{\mu_0}(F)$ as the central Lyapunov exponent of the measure of maximum entropy μ_0 for F.

We prove that the existence of phase transition as stated in Theorem A implies a good understanding of multifractal analysis of the central Lyapunov exponent:

Theorem B. *Let $F \in \mathcal{D}^r$, with $r \geq 2$. The entropy function $\Delta \ni (a, b) \mapsto h_{top}(L^c_{a,b})$ is a concave C^1 function satisfying:*

- *$h_{top}(L^c_{a,b}) = h_{top}(E^c_{a,b})$, for all $(a, b) \in \Delta$;*
- *It is constant and equal to its maximum value $h_{top}(F)$ for (a, b) in the rectangle $[0, \lambda^c_{\mu_0}] \times [\lambda^c_{\mu_0}, \lambda^c_{\max}]$;*
- *It is strictly concave and analytic everywhere, except for $b \leq \lambda^c_{\min}$, in case of the exponent $\lambda^c_{\min} > 0$ when the function is linear.*

With Theorem B, the graph of the entropy function must be as described in the following figure:

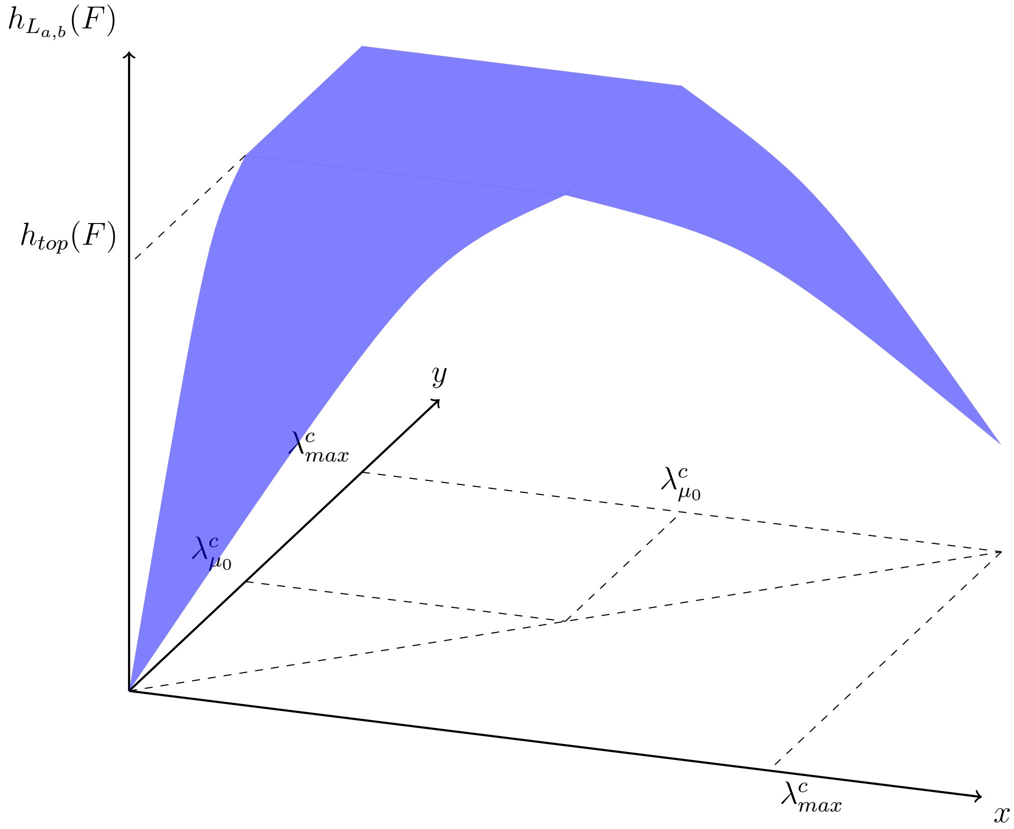

Figure 5: Ilustration of $(a, b) \mapsto h_{L^c_{a,b}}(F)$

Transitive Piecewise Monotone Maps on the Circle

For a transitive piecewise monotone map f as in definition 0.2.7, we give a description of the behaviour of the topological pressure function. In particular, we prove that the set of regular potential that does not admit phase transition is dense in the uniform topology and the set of regular potential that admit it is not dense with respect to the uniform topology. We want to understand the set of regular (Hölder continuous, C^r) potentials ϕ for which the topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi)$ has or not thermodynamic phase transition or the transfer operator has or not spectral gap property. We make clear now that for our next main results the context is transitive piecewise maps as in definition 0.2.7. Our first result in that direction are related to denseness. We prove the following:

Theorem C. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ like in definition 0.2.7. There exists an open and dense subset $\mathcal{H} \subset C^0(\mathbb{S}^1, \mathbb{R})$ in the uniform topology, such that if $\phi \in \mathcal{H}$ is Hölder continuous, then ϕ has **no** thermodynamic phase transition and $\mathbb{R} \ni t \mapsto P_{top}(f, \phi)$ is strictly convex.*

As a direct consequence of Theorem C we have:

Corollary C.
1. *The set of smooth potentials such that $t \mapsto P_{top}(f, t\phi)$ is strictly convex and has no thermodynamic phase transition is dense, in the uniform topology;*
2. *The set of Hölder continuous potential having thermodynamic phase transition is **not** dense, in the uniform topology.*

For the next results we give a characterization of the topological pressure function with respect to regular (at least Hölder continuous) potentials having or not thermodynamic phase transition. In particular, we provide information on how the graph of the topological pressure function should behave with respect to each phenomenon.

Remark 0.3.1. *We say that a continuous potential $\phi : \mathbb{S}^1 \to \mathbb{R}$ is cohomologous to a constant C, if there exists a continuous function $u : \mathbb{S}^1 \to \mathbb{R}$ such that:*

$$\phi = C + u \circ f - u$$

It follows from Thompson [Th09] that for maps as in definition 0.2.7, the family of potentials cohomologous to a constant is a closed set with empty interior is $C^0(\mathbb{S}^1, \mathbb{R})$.

Theorem D. *Let ϕ be a Hölder continuous potential. The following items are equivalent:*

1. *ϕ does not have thermodynamic phase transition;*
2. *ϕ does not have spectral phase transition, i.e., $\mathcal{L}_{f,t\phi}|_E$ has spectral gap for all $t \in \mathbb{R}$.*

If, in addition, ϕ is not cohomologous to a constant, then the previous items are equivalent to:

3. *the topological pressure function $t \mapsto P_{top}(t\phi)$ is strictly convex.*

Furthermore, if any of the previous items holds, then $t\phi$ has an unique equilibrium state, for all $t \in \mathbb{R}$.

Remark. It follows from Theorems C and D that the set

$$\{\phi \in E : \phi \text{ has an unique equilibrium state}\}$$

is dense, in the uniform topology. Furthermore,

$$\{\phi \in E : \mathcal{L}_{f,t\phi} \text{ has spectral gap property}\}$$

is dense in the uniform topology. It extends an analogous result obtained in [Kl20], when the map has a unique indifferent fixed point and the suitable Banach space is $E = C^{\alpha}(\mathbb{S}^1, \mathbb{C})$.

Remark 0.3.2. *By Leplaideur [L15], there is an example of a continuous potential defined on a mixing subshift of finite type such that the pressure function is analytic, but the uniqueness of the equilibrium state fails.*

Now, in the next theorem we describe the behaviour of the topological pressure function for potentials with thermodynamic phase transition. Furthermore, we prove that there exists at most two phase transitions for Hölder continuous potentials.

Theorem E. *Let $\phi \in E = C^{\alpha}(\mathbb{S}^1, \mathbb{R})$ be a potential having phase transition. Then, there exist $t_2 < 0 < t_1$, with at least $t_1 \in \mathbb{R}$ or $t_2 \in \mathbb{R}$, such that:*

1. *for $t \geq t_1$ or $t \leq t_2$, then the topological pressure function is an affine map and for $t_2 < t < t_1$ the topological pressure function $t \to P_{top}(t\phi)$ is analytic and strictly convex;*
2. *for $t_2 < t < t_1$ the associated transfer operator $\mathcal{L}_{f,t\phi}|_E$ has the spectral gap property, and for $t \geq t_1$ or $t \leq t_2$ the associated transfer operator $\mathcal{L}_{f,t\phi}|_E$ does not have the spectral gap property.*

Remark 0.3.3. *As a consequence of the previous theorem, there exist at most two thermodynamic phase transition for Hölder continuous potentials.*

As a particular case of the previous theorem we show that for Maneville-Pomeau-like maps there exist at most one phase transition.

Definition 0.3.4. *We say that a C^1-map $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a Maneville-Poumeau-like map if f is expanding except at a unique fixed point. More formally, $f(0) = 0$, $f'(0) = 1$ and $|f'(x)| > 1$ for all $x \neq 0$ or 1, when the derivative is well defined.*

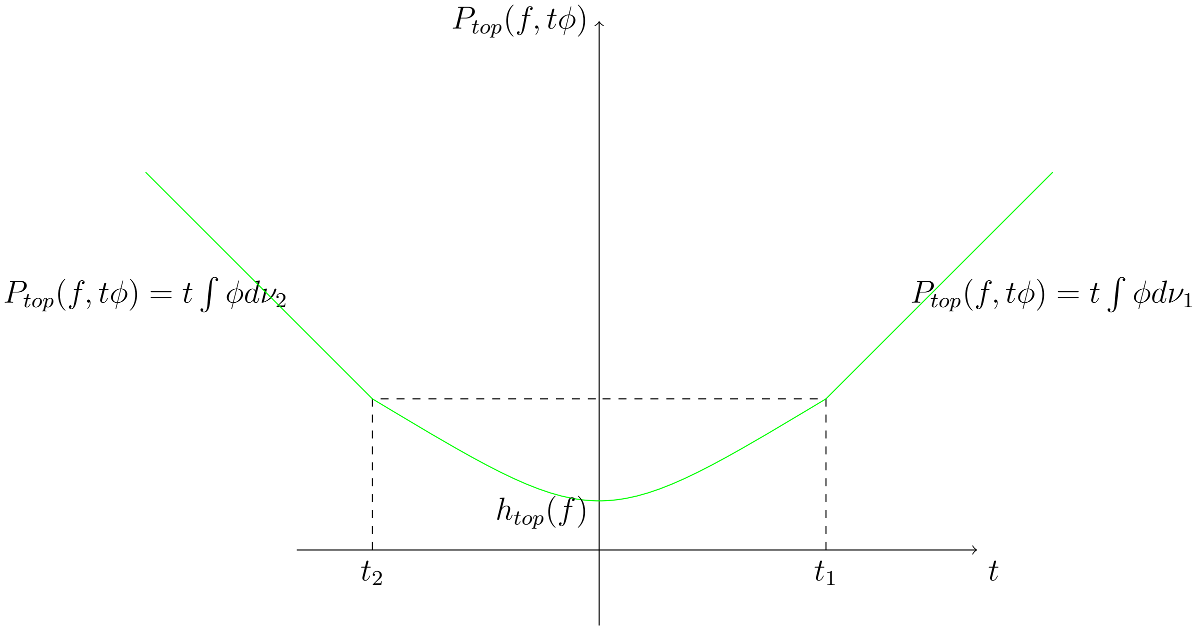

Figure 6: Illustration of a possible shape for the graph of the topological pressure function $\mathbb{R} \ni t \mapsto P(t)$ as stated in Theorem E

Corollary E. *If f is a Maneville-Pomeau-like map, then there exists at most one phase transition for Hölder continuous potentials.*

Organization of the thesis

Chapter 1 is mostly dedicated to the proof of the Theorem A, which will be divided in three steps: the existence of phase transition, absence of spectral gap after the transition and existence of spectral gap before the transition. We start the chapter by presenting a brief comment about the proof of Theorem A, then we dedicate one section to present a class of examples, another one for preliminaries and in the last one we will present the actual proof of the main result.

Chapter 2 is mostly dedicated for proving some consequences of the results of thermodynamic and spectral phase transition presented in A. In other words, we prove the Theorem B. In the beginning of the chapter we present the concept of multifractal analysis and the large deviations principle, the main concept of our study of the chapter. There are two sections where the first one we present some preliminaries results, and we end the chapter by presenting the proof of the Theorem B.

Chapter 3 is dedicated to study the existence (respectively absence) of phase transition and the consequences for the behavior of the topological pressure function for transitive piecewise monotone dynamics on the circle $\mathbb{S}^1$. We prove Theorem C, Theorem D and Theorem E and their respectively corollaries. We divided the chapter into two sections: the first one is preliminary results and the second one is dedicated to the proof of the main results of the chapter.

Chapter 4 is dedicated to some comments and further questions.

Chapter 1

Effective Spectral and Thermodynamic phase transition

As we mentioned in the introduction, thermodynamic phase transition happens when the topological pressure function associated to a potential $t\phi$ is not analytic at some point $t_0 \in \mathbb{R}$. We are going to show that in the context of our model, maps $F \in \mathcal{D}^r$, the topological pressure function has a unique phase transition parameter and that the topological pressure function has distinct behaviour before and after that parameter. We recall that we fixed the potential

$$\phi^c := -\log \left|\left|\frac{\partial F}{\partial y}\right|\right|$$

the geometric-type potential in the central direction. In this chapter, we present the proof of Theorem A, which guarantees the existence of thermodynamic phase transition for maps $F \in \mathcal{D}^r$.

Theorem A. *Let $F \in \mathcal{D}^r$, with $r \geq 2$. There exist a unique $t_0 \in (0, 1]$ such that:*

1. *The topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi^c)$ is analytic, strictly decreasing and strictly convex in $(-\infty, t_0)$ and constant equal to $h_{top}(g)$ in $[t_0, +\infty)$;*
2. *The transfer operator $\mathcal{L}_{f,t\phi^c}$, acting on the space $C^{r-1}(\mathbb{T}^d \times \mathbb{S}^1, \mathbb{C})$, has spectral gap for all $t < t_0$ and does not have spectral gap for all $t \geq t_0$.*

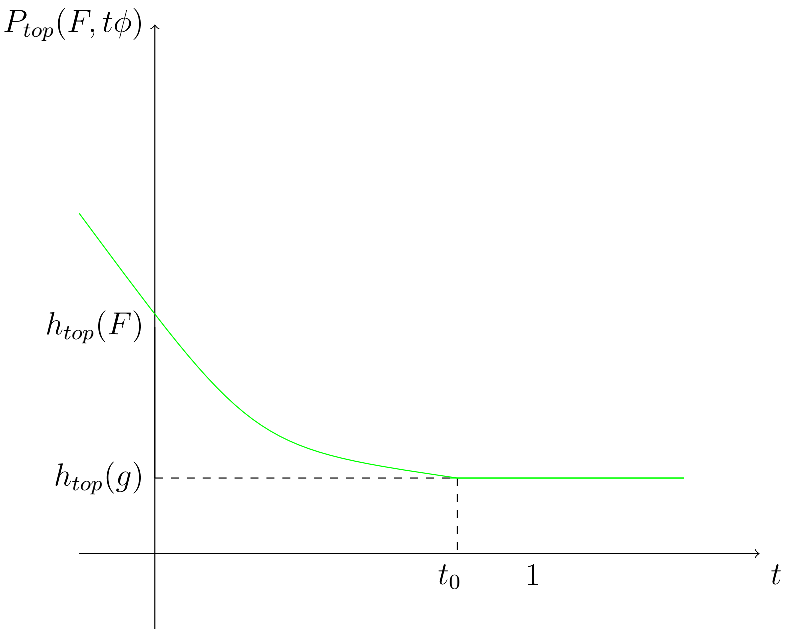

Figure 1.1: Graphical illustration of topological pressure function $\mathbb{R} \ni t \mapsto P_{top}(F, t\phi)$ as stated in theorem A.

Before the proof, we present an overview on how we organized and how the arguments work to prove theorem A.

Brief Comments About The Proof Of the Theorem A

In the proof of Theorem A we mostly follow closely the arguments in [BC21]. So, we will concentrate here on the main differences of the proof in our context. Firstly, a noticeable difference in Theorem A from the proof in [BC21] is that the topological pressure function of F is at least $h_{top}(g) > 0$, where g is the expanding linear endomorphism in the definition of $\mathcal{D}^r$. Using the fact that $F \in \mathcal{D}^r$ is conjugated to an expanding map and $\mathbb{T}^d \times \mathbb{S}^1$ is connected, which implies that F is expansive, the central Lyapunov exponent with respect to any invariant measure cannot be negative. As a consequence of that, the pressure function is non-increasing.

The proof of the Theorem A will be divided into three steps:

In the first step, following [BC21], using the fact that the topological pressure function is non-increasing, we show that for all $t \geq 1$ the pressure function is constant equal to $h_{top}(g)$. On the other side, we show that the topological pressure function is not constant for $t \leq 1$, and conclude that there exists $t_0 \in (0, 1]$ such that the topological pressure function is not analytic at t_0. With respect to the phase transition parameter t_0, we show that the transfer operator $\mathcal{L}_{F,t_0\phi^c}$ should not have the spectral gap property acting on $C^{r-1}(\mathbb{T}^d \times \mathbb{S}^1, \mathbb{C})$, using the fact that the topological pressure function is not analytic in

t_0.

In the second step, for $t > t_0$, we show that the transfer operator $\mathcal{L}_{F,t\phi^c}$ does not have the spectral gap property acting on $C^{r-1}(\mathbb{T}^d \times \mathbb{S}^1, \mathbb{C})$, as we establish **relation between the strictly convexity of the geometric pressure function and the existence of spectral gap**. For the convexity strict, we make use of Nagaev's method and the Central Limit Theorem. It associates the second derivative of the pressure function with the variance of the C.L.T. which, in turn, has its sign related to the potential being or not cohomologous to a constant. In particular, the constant behaviour of the geometric pressure function after the transition parameter implies that spectral gap does not occur for these parameters.

In the third step, we show that for $t < t_0$ the transfer operator $\mathcal{L}_{F,t\phi^c}$ has the spectral gap property acting on $C^{r-1}(\mathbb{T}^d \times \mathbb{S}^1)$. Following similar arguments as in [BC21], for $F \in \mathcal{D}^r$, **quasi-compactness is sufficient for the transfer operator to have spectral gap**. Using estimates from [CL97] to prove that the **essential spectral radius is bounded from above by a translation of the pressure function** and that the spectral radius is bounded by the pressure function, we prove the spectral gap property for $t = 0$, and therefore to t sufficiently close to zero, by the openness of spectral gap property. Finally, we extend this property to all $t < t_0$, first for $t < 0$ and similarly for $0 < t < t_0$, using the monotonicity of the pressure function and the fact that it is related to the spectral radius, not only when the gap holds but also on the boundary of that region.

We will show in the next section that an interesting variety of examples satisfy our assumptions.

1.1 Examples of Skew-Product

The following example shows us that our context extend the one dimensional case. We take a transitive local diffeomrphism on the circle $\mathbb{S}^1$ in the context of Bomfim-Carneiro [BC21] as the only map in the fiber for our skew-product, and the result will gives us the first example:

Example 1.1.1. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a transitive non-invertible C^r-local diffeomorphism. It follows from [CM86] that f is topologically conjugated to a uniformly expanding dynamic. Suppose that f is not uniformly expanding, then it follows from Theorem 2 of [CLR04] that f admits an f-invariant and ergodic probability ν such that $\chi(f) = \int \log |Df| d\nu = 0$. Therefore, the skew-product $F : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ given by*

$$F(x, y) = (2x \mod 1, f(y))$$

will belong to $\mathcal{D}^r$.

The next class of examples will show that our class of skew-product contains an interesting class of local diffeomorphims in higher dimension. Such examples can be seen as differentiable versions of a **random choice of intermittent dynamics**.

Example 1.1.2. *Let $x_1, x_2, ..., x_k = x_1$ be points in $\mathbb{S}^1$. Define $V(\{x_j\}_{j=1}^k)$ as the set of functions $f : \mathbb{S}^1 \to \mathbb{S}^1$ such that*

- *f is transitive C^r-local diffeomorphism;*
- *$f|_{[x_j, x_{j+1}]}$ is one-to-one, $f(x_j) = x_1$ and $f([x_j, x_{j+1}]) = \mathbb{S}^1$;*
- *$Df(x_j) = 1$ and $|Df(x)| > 1$, for all $x \neq x_j$, $j = 1, ..., k$.*

Remark 1.1.3. *Observe that if a local diffeomorphism $F : \mathbb{T}^n \to \mathbb{T}^n$ is (positively) expansive, then there exists a metric on $\mathbb{T}^n$ compatible with the topology such that the mapping F is uniformly expanding with respect to this metric. In particular F will be topologically conjugated to a uniformly expanding dynamics (for more details, see [PU10]).*

Proposition 1.1.4. *Fix $x_1, x_2, ..., x_k$ points in the circle $\mathbb{S}^1$. Let $F : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{T}^d \times \mathbb{S}^1$, be a C^r-local diffeomorphism given by $F(x, y) = (g(x), f_x(y))$, where g is an expanding linear endomorphism and $f_x \in V(\{x_j\}_{j=1}^k)$. Then, $F \in \mathcal{D}^r$.*

Proof. If we take $\nu = \delta_{x_1}$, then ν satisfies item (3) of the definition of $\mathcal{D}^r$, so it's enough for us to check item (4). If we know that F is (positively) expansive, we could use the results in [PU10], and that will be enough for us, by the previous remark. By definition of $V(\{x_j\}_{j=1}^k)$ and continuity $x \mapsto f_x$, there exists $\epsilon > 0$ such that: given $0 < a < \epsilon$ there is $\lambda_a > 1$ with $diam(f_x(I)) \geq \lambda_a diam(I)$, for all $a \leq diam(I) \leq \epsilon$ and $x \in \mathbb{T}^d$, where $diam$ means the diameter of a subsetin $\mathbb{S}^1$. Arguing by contradiction, suppose that F is not expansive. As g is expansive, then there exists $x \in \mathbb{T}^d$ and $y_1 \neq y_2$ with

$$|f_{g^n(x)} \circ \cdots \circ f_x(y_1) - f_{g^n(x)} \circ \cdots \circ f_x(y_2)| < \epsilon$$

for all $n \geq 0$, meanwhile:

$$diam(f_{g^n(x)} \circ \cdots \circ f_x([y_1, y_2]) \geq \lambda_a^n |y_1 - y_2|,$$

for all $n \geq 0$. This implies that exist $n \geq 0$ such that

$$|f_{g^n(x)} \circ \cdots \circ f_x(y_1) - f_{g^n(x)} \circ \cdots \circ f_x(y_2)| > \epsilon$$

That is a contradiction. We conclude that F is expansive.

□

Remark. Our results applied to the previous class of examples guarantee the existence and uniqueness of phase transition (Theorem A) and a good understanding of multifractal analysis of the spectrum of the central Lyapunov exponent (Theorem B)

As a byproduct of the previous example, we present a class of intermittent maps of class C^2:

Example 1.1.5. *For each $\alpha \in [0,1]$ define the constants*

$$b = b(\alpha) = \left(\left(\frac{1}{2}\right)^{3+\alpha} - \frac{4+\alpha}{4+2\alpha}\left(\frac{1}{2}\right)^{2+\alpha}\right) \text{ and } a = a(\alpha) = -\frac{-b(4+\alpha)}{4+2\alpha}$$

and the polynomial $g_\alpha : [0,1/2] \to [0,1]$, given by $g_\alpha(y) = y + ay^{3+\alpha} + by^{4+\alpha}$. Then we define the family of intermittent maps on the circle

$$f_\alpha(y) = \begin{cases} g_\alpha(y), \; if \quad 0 \leq y \leq 1/2 \\ 1 - g_\alpha(1-y), \; if \quad 1/2 \leq y \leq 1. \end{cases}$$

This family of dynamics can be seen as a C^2 version of the Manneville-Pomeau maps. Observe that $y = 0$ is an indifferent fixed point and $Df_\alpha(y) > 1$ for all $y \neq 0$. In fact, we have:

$$Df_\alpha(y) = 1 + (3+\alpha)\alpha y^{2+\alpha} + (4+\alpha)by^{3+\alpha}, \forall\, 0 \leq y \leq 1/2$$
$$Df_\alpha(y) = -1 + (3+\alpha)\alpha(1-y)^{2+\alpha} + (4+\alpha)b(1-y)^{3+\alpha}, \forall\, 1/2 \leq y \leq 1$$

$f_\alpha \in V(0,1/2,1)$ for all $\alpha \in [0,1]$. Therefore, as a byproduct of the previous construction the skew-product, if g is an expanding linear endomorphism then

$$F : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{T}^d \times \mathbb{S}^1$$

given by

$$(x,y) \mapsto (g(x), f_x(y))$$

belongs to $\mathcal{D}^r$.

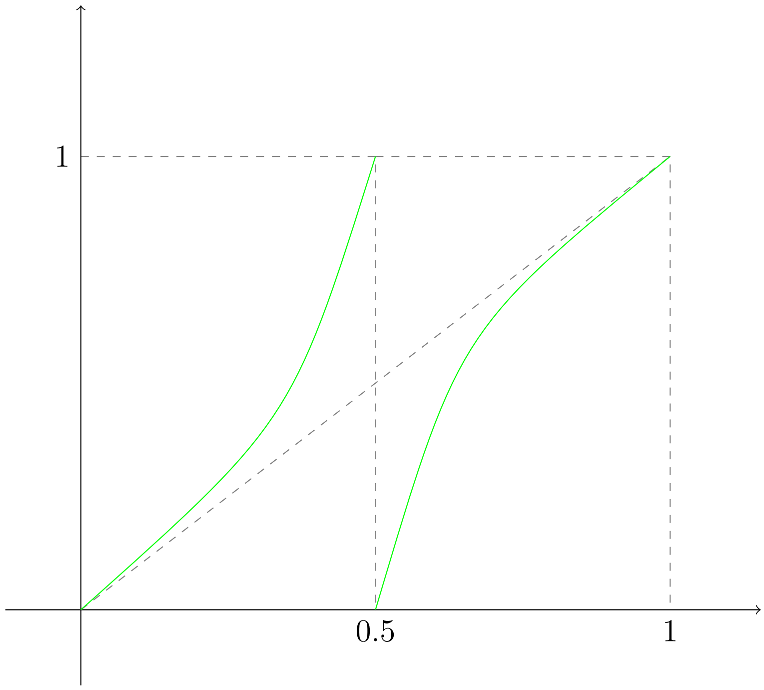

Figure 1.2: Intermittent map of class C^2

1.2 Preliminaries

Before starting the proof of Theorem A we need some classical results like the celebrated **Birkhoff Ergodic Theorem** which associates time and space averages of a given potential $\phi : X \to \mathbb{R}$. and **Oseledets Multiplicative Ergodic Theorem** on the existence of Lyapunov exponents.

Theorem 1.2.1 (Birkhoff). *Let $f : X \to X$ be a measurable transformation and μ be an f-invariant probability. Given any integrable function $\phi : X \to \mathbb{R}$, the limit:*

$$\bar{\phi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(f^j(x))$$

exists for μ-almost every $x \in X$. Furthermore, the function $\bar{\phi}$ defined this way is integrable and satisfies

$$\int \bar{\phi}(x) d\mu(x) = \int \phi(x) d\mu(x).$$

Additionally, if μ is ergodic, then $\bar{\phi} \equiv \int \phi d\mu$ for $\mu-a.e.$

We recall a very important tool introduced by Bowen [Bow71]:

Definition 1.2.2. *We say that $F : M \to M$ satisfies the specification property if given $\epsilon > 0$, there exist $N(\epsilon) \geq 1$, depending only on ϵ, such that para for any points $x_1, x_2, ..., x_k \in M$ and shadowing times $n_1, n_2, ..., n_{k-1} \geq 0$ and for all $p_1, p_2, ..., p_{k-1} \geq N(\epsilon)$ there exist a point $x \in M$ such that:*

$$d(F^j(x), F^j(x_1)) \leq \epsilon, 0 \leq j \leq n_1$$
$$d(F^{j+n_1+p_1+\ldots+n_{i-1}+p_{i-1}}(x), F^j(x_i)) \leq \epsilon,\ 2 \leq i \leq k-1 \quad e \quad 0 \leq j \leq n_i$$

If for all $p_k \geq N(\epsilon)$ we have x as describe above and $F^{+n_1+p_1+\ldots+n_{k-1}+p_{k-1}+p_k}(x) = x$, we say that F satisfies the periodic specification orbit property.

Lyapunov Exponents

Important quantities that sometimes can be linked with time averages are the *Lyapunov exponents*. That is the case of the central Lyapunov exponent for maps in $\mathcal{D}^r$ due to the codimension one of our toy model. These exponents exhibit the asymptotic rates of expansion and contraction of a smooth dynamic system. In a broader context, these are defined via the Oseledets Multiplicative Ergodic Theorem:

We say that λ is a **Lyapunov exponent** for a C^1 map f, if there exists a point x and a vector $v \in T_xM$ such that

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\|,$$

We let $L(f)$ denote the set of all Lyapunov exponents for f. The **Oseledets Multiplicative Ergodic Theorem** states that for each ergodic measure $\mu \in M_{erg}(f)$ there exists a number $0 < k \leq d$, constants $\lambda_1 > \cdots > \lambda_k$, and a filtration $T_xM = E_x^1 \supset \cdots \supset E_x^{k+1} = \{0\}$ of the tangent bundle over Λ, such that for all non-zero vector $v \in E_x^i \backslash E_x^{i+1}$, $i = 1, ..., k$

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df_x^n(v)\| = \lambda_i$$

for μ-almost every x. For non-ergodic measures the number k, the constants λ_j, and the tangent bundle decomposition which depends on x may depend on the ergodic component. The constants λ_j are called the Lyapunov exponents associated to the measure μ.

Now for an ergodic measure μ, let $m_j := m_x^j = \dim E_x^j - \dim E_x^{j-1}$ be the multiplicity for $j = 1, \ldots, k$ and μ almost every x. We define the sum of positive Lyapunov exponents, with multiplicity by

$$\lambda_+(\mu) = \sum_{\lambda_j>0} m_j \lambda_j.$$

We will need the following well known relation between entropy and positive Lyapunov exponents:

Theorem 1.2.3 (Margulis-Ruelle inequality, [Rue78]). *Let $f : M \to M$ be a C^1-local diffeomorphism that preserves an $f-$invariant and ergodic probability μ. Then*

$$h_\mu(f) \leq \lambda_+(\mu)$$

Translating the Margules-Ruelle inequality for the dynamics $F \in \mathcal{D}^r$: given an $F-$invariant and ergodic probability measure μ we denote its Lyapunov exponents as $\lambda^c_\mu, \lambda^1_\mu, \lambda^2_\mu, ..., \lambda^d_\mu$, where $\lambda_i, i = 1, ..., d$ are the Lyapunov exponents of the expanding linear endomorphism g in the definition of $\mathcal{D}^r$ and

$$\lambda^c_\mu(x,y) := \lim_{n\to\infty} \frac{1}{n} \log ||DF^n_y(x,y)|| = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(F^i(x,y)).$$

for μ-a.e. (x, y) and $\phi = \log ||DF_y||$. For an ergodic measure μ, we define its central Lyapunov exponent as

$$\lambda^c_\mu(F) := \int \log ||DF_y|| d\mu$$

The Margules-Ruelle inequality can be rewritten as

$$h_\mu(F) \le \sum_{i=1}^{d} \lambda_i + \max\{0, \lambda^c\} \tag{1.1}$$

Recall that a partition is said to be generating if its pre-images generate the Borelian σ-algebra. If its domain is a metric space, then a partition such that the diameter of the elements of $\bigvee_{i=m}^{+\infty} f^{-i}(\mathcal{P})$ gets arbitrarily small as m grows is a generating partition.

Next, we recall the definition of *Jacobian*, which together with generating partition gives the Rokhlin formula. Let f be a locally invertible map and a given probability ν (not necessarily invariant), we define the **Jacobian** of f with respect to ν as the measurable function $J_\nu(f)$, which is essentially unique, satisfying:

$$\nu(f(A)) = \int_A J_\nu(f) d\nu$$

for any invertibility domain A. Up to restricting f to a full measure subset, f always admits a Jacobian with respect to an invariant probability, in that case, we call it weak Jacobian.

The existence of generating partition gives a fundamental tool for calculating the metric entropy:

Theorem 1.2.4 (Rokhlin formula). *Let $f : M \to M$ be a locally invertible transformation and ν be an f-invariant probability measure. Assume that there is some generating partition $\mathcal{P}$ up to measure zero such that every $P \in \mathcal{P}$ is an invertibility domain of f. Then*

$$h_\mu(f) = \int \log J_\mu(f) d\mu.$$

This equality will be fundamental later on to relate topological pressure function and spectral radius. Another property regarding the metric entropy is the *upper semi-continuity* of the function $\mathcal{M}_1(f) \ni \mu \mapsto h_\mu(f) \in \mathbb{R}^+$ which guarantees the existence of equilibrium states. We have that expansiveness of the system implies semi-continuity, which is the case for our maps, by remark 0.2.6.

We want to obtain analyticity of thermodynamic quantities, such as the pressure function. We derive it from the analyticity of spectral objects such as the spectral radius.

Let $\mathcal{L} : E \to E$ be a bounded linear operator on a complex Banach space. We denote the spectral radius of $\mathcal{L}$ by $\rho(\mathcal{L})$.

Definition 1.2.5. *A complex number λ is said to be in **resolvent set** of $\mathcal{L}$ if $\lambda I - \mathcal{L}$ is a bijection with bounded inverse. If λ is not in the resolvent set, then λ is said to be in the **spectrum** of $\mathcal{L}$, which we denote by $sp(\mathcal{L})$.*

Suppose that $\mathcal{L}$ has the spectral gap property as we defined in 0.1.3. By [R55] this is an open property, that is, there exists $\delta > 0$ such that if $\tilde{\mathcal{L}} : E \to E$ is a bounded linear operator with $||\mathcal{L} - \tilde{\mathcal{L}}|| < \delta$ then $\tilde{\mathcal{L}}$ has the spectral gap property. Moreover, $B(\mathcal{L}, \delta) \ni \tilde{\mathcal{L}} \mapsto \left(\rho(\tilde{\mathcal{L}}), \mathtt{P}_{\rho(\tilde{\mathcal{L}})}\right)$ is analytic, where $\mathtt{P}_{\rho(\tilde{\mathcal{L}})}$ is the spectral projection of $\tilde{\mathcal{L}}$ with respect to the leading eigenvalue $\rho(\tilde{\mathcal{L}})$.

From there, following the same proof of [BCV16, Proposition 4.1] we have

Proposition 1.2.6. *Let $f : M \to M$ be a local homeomorphism on a compact and connected manifold and E be a Banach algebra of functions such that $\mathcal{L}_{f,\phi}$ as defined in 0.1.2 is a bounded linear operator of E on E, for all $\phi \in E$. Then*

$$E \ni \phi \to \mathcal{L}_{f,\phi}$$

is analytic, where we endow on the codomain the topology generated by the operator norm. In particular, $\mathbb{R} \ni t \mapsto \mathcal{L}_{f,t\phi}$ is real analytic for all ϕ in E.

Note that if $T = \mathcal{L}_{f,\phi}|_E$ has the spectral gap property, we have the following equality for the projection $\mathtt{P}_{\rho(\tilde{T})}(g) = \int g d\nu_\phi \cdot h_\phi$, since the eigenspace associated to the leading eigenvalue is unidimensional. Define $SG(E) := \{\phi \in E : \mathcal{L}_{f,\phi}|_E \text{ has the spectral gap property}\}$, we then conclude:

Corollary 1.2.7. *$SG(E) \subset E$ is an open subset and the following map is analytic:*

$$SG(E) \ni \phi \mapsto (\rho(\mathcal{L}_{f,\phi}|_E), h_\phi, \nu_\phi)\,.$$

Most of the time, it is hard to verify that a certain linear operator has the spectral gap property. Sometimes it's convenient to consider a weaker spectral property, for instance, **quasi-compactness**:

Definition 1.2.8. *Given E a complex Banach space and $\mathcal{L} : E \to E$ a bounded linear operator, we say that $\mathcal{L}$ is quasi-compact if there exists $0 < \sigma < \rho(\mathcal{L})$ and a decomposition of $E = F \oplus H$ as follows: F and H are closed and $\mathcal{L}$-invariant, $\dim F < \infty$, $\rho(\mathcal{L}|_F) > \sigma$ and $\rho(\mathcal{L}|_H) \leq \sigma$.*

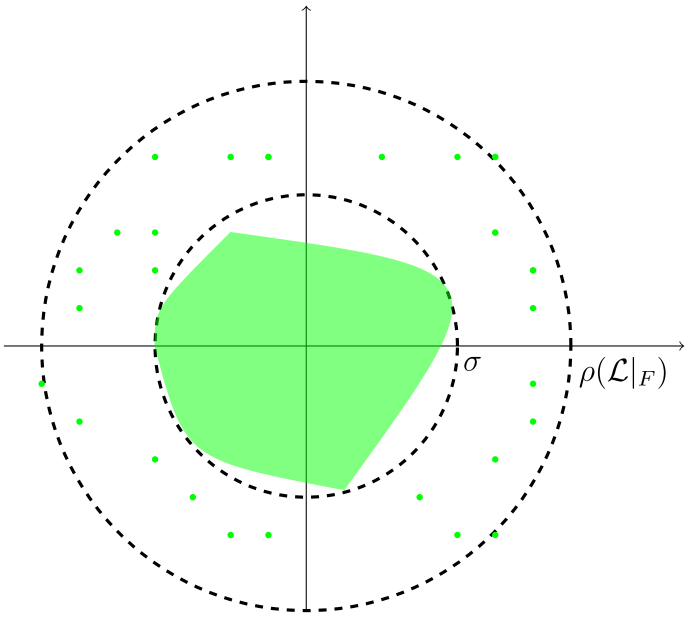

Figure 1.3: The figure illustrate the spectrum of a linear operator satisfying quasi-compactness

In general, spectral gap implies quasi-compactness. Later we will prove that, in the context of Theorem A, quasi-compactness is a sufficient condition for spectral gap to hold. In order to obtain quasi-compactness, we use an alternative equivalent definition for it via the **essential spectral radius**:

Definition 1.2.9. *Given E a complex Banach space and $\mathcal{L} : E \to E$ a bounded linear operator, define the essential spectral radius:*

$$\rho_{ess}(\mathcal{L}) := \inf\{r > 0;\ sp(L) \setminus \overline{B(0,r)} \text{ contains only eigenvalues of finite multiplicity}\}$$

Thus quasi-compactness is equivalent to having $\rho_{ess}(\mathcal{L}) < \rho(\mathcal{L})$, and so estimates on the essential spectral radius and the spectral radius will be crucial.

The proof of the following two lemmas can be found in [BC21], where their proof is in the context of local diffeomorphism on the circle, but their argument can be extended for more general local diffeomorphism on a compact and connected manifold with dense pre-images.

Lemma 1.2.10 (Lemma 5.5 of [BC21])**.** *Let $f : M \to M$ be a C^r-local diffeomorphism on a compact and connected manifold M, such that $\{f^{-n}(x) : n \geq 0\}$ is dense in M for* *all $x \in M$, and let $\phi \in C^r(M,\mathbb{R})$. If $\mathcal{L}_{f,\phi}\varphi = \lambda\varphi$ with $|\lambda| = \rho(\mathcal{L}_{f,\phi}|_{C^r})$ and $\varphi \in E \setminus \{0\}$, then $\mathcal{L}_{f,\phi}|\varphi| = \rho(\mathcal{L}_{f,\phi}|_{C^r})|\varphi|$. Furthermore, $\rho(\mathcal{L}_{f,\phi}|_{C^0}) = \rho(\mathcal{L}_{f,\phi}|_{C^r})$, φ is bounded away from zero and* $\dim\ker(\mathcal{L}_{f,\phi}|_{C^r} - \lambda I) = 1$.

Basically, by the previous lemma, if the transfer operator has a peripheral eigenvalue associated to a non-zero potential φ, the spectral radius is an eigenvalue associated to the absolute value of φ and it is a simple eigenvalue. As a corollary we have:

Corollary 1.2.10.1 (Corollary 5.6 in [BC21]). *Let $f : M \to M$ be a C^r-local diffeomorphism on a compact and connected manifold M, such that $\{f^{-n}(x) : n \geq 0\}$ is dense in M for all $x \in M$, and let $\phi \in C^r(M,\mathbb{R})$. If $\mathcal{L}_{f,\phi}|_{C^r}$ has the spectral gap property then there exists a unique probability ν_ϕ on M such that $(\mathcal{L}_{f,\phi}|_{C^r})^*\nu_\phi = \rho(\mathcal{L}_{f,\phi}|_{C^r})\nu_\phi$. Moreover, $supp(\nu_\phi) = M$.*

Remark 1.2.11. *Given $\mathcal{L}_{f,\phi}|_{C^r}$ with the spectral gap property, by Lemma 1.2.10 there exists a unique $h_\phi \in C^r(M,\mathbb{C})$ such that $\int h_\phi d\nu_\phi = 1$. Moreover, denote the $f-$invariant probability $h_\phi d\nu_\phi$ by μ_ϕ. Note that $supp(\nu_\phi) = M$ and $h_\phi > 0$, also by this lemma, thus $supp(\mu_\phi) = M$.*

The next lemma establishes a connection between the pressure function and spectral gap. Here, as in [BC21], we use the Rohklin's formula for the metric entropy as $F \in \mathcal{D}^r$ admits a generating partition such that each element is a domain of invertibility.

Lemma 1.2.12. *Let $f : M \to M$ be a C^r-local diffeomorphism be on a compact and connected manifold M, such that $\{f^{-n}(x) : n \geq 0\}$ is dense in M for all $x \in M$ and f admits generating partition by domains of injectivity. Let $\phi \in C^r(M,\mathbb{R})$. If $\mathcal{L}_{f,s\phi}|_{C^r}$ has spectral gap for a given $s \in \mathbb{R}$, then:*

1. *$P_{top}(f, s\phi) = \log\rho(\mathcal{L}_{f,s\phi}|_{C^r})$ and $\mu_{s\phi}$ is an equilibrium state of f with respect to $s\phi$;*
2. *$\mathbb{R} \ni t \mapsto P(f, t\phi)$ is analytic on s.*

Proof. The proof follows closely the same proof as in [BC21]. □

1.3 Proof of Theorem A

The main goal in this section is to prove Theorem A. Roughly speaking, we want to understand the regularity of the topological pressure function, which by the **variational principle** is given by:

$$P(t) = P_{top}(F, t\phi^c) = \sup\{h_\mu(F) - t\lambda^c_\mu(F),\ \mu \in \mathcal{M}_1(F)\},\ t \in \mathbb{R}.$$

where $\mathcal{M}_1(F)$ is the space of all invariant probability measure by F, $h_\mu(F)$ is the metric entropy and $\lambda^c_\mu(F)$ is the central Lyapunov exponent associated to the invariant probability measure μ. It is known, via the Variational Principle (see e.g. [OV16]), that this suprememum can be taken over the space of ergodic probabilities measures, $\mathcal{M}_{erg}(F)$.

1.3.1 Step 1: Existence of phase transition

We first show that, for $F \in \mathcal{D}^r$, negative central Lyapunov exponent cannot occur and, as a consequence, the topological pressure function is monotone non-increasing.

Lemma 1.3.1. *Let $F \in \mathcal{D}^r$. Then $\lambda^c_\mu(F) \geq 0$, for every $\mu \in \mathcal{M}_1(F)$.*

Proof. This lemma is a consequence of F being expansive and satisfying the periodic specification property, by remark 0.2.6. Suppose $\lambda^c_\mu(F) < 0$. As F satisfies the periodic specification property, by Sigmund [S74] the periodic points are dense, then there exist $0 < \lambda < 1$ and $(x, y) \in \mathbb{T}^d \times \mathbb{S}^1$ such that $F^n(x, y) = (x, y)$ and $||Df^{kn}(x, y)|| < \lambda^k, \forall k \geq 1$. In particular $f(y) := f_{g^{n-1}(x)} \circ f_{g^{n-2}(x)} \circ ... \circ f_x(y)$ satisfies $f(y) = y$ and $|Df(y)| \leq \lambda$. By continuity, triangular inequality and mean value theorem there exist $\delta > 0$ such that $|f(z_1) - f(z_2)| \leq 2\lambda|z_1 - z_2|$, for all $z_1, z_2 \in B(y, \delta)$. As F is uniformly continuous, given $\epsilon > 0$ we can find a $\tilde{\delta} \leq \delta$ such that

$$|f_{g^{m-1}(x)} \circ f_{g^{m-2}(x)} \circ ... \circ f_x(z_1) - f_{g^{m-1}(x)} \circ f_{g^{m-2}(x)} \circ ... \circ f_x(z_2)| < \epsilon$$

for all $m \geq 0$ and $z_1, z_2 \in B(y, \tilde{\delta})$. As ϵ is arbitrary this implies that F is not expansive, which is a contradiction. □

The fact that F does not have negative Lyapunov exponent is very useful especially for properties related to the transfer operator. With the previous lemma, the main properties of our toy model and the Margules-Ruelle's inequality, 1.1, we are going to show that should exist a point t_0 where the topological pressure function changes behaviour, and so it cannot be analytic at t_0.

Lemma 1.3.2. *Let $F \in \mathcal{D}^r$. Then, there exist $t_0 \in (0, 1]$ such that the pressure function $\mathbb{R} \ni t \mapsto P_{top}(F, t\phi^c)$ is not analytic in t_0.*

Proof. By Lemma 1.3.1, F cannot have negative central Lyapunov exponent. So, if we denote $P(t) := P_{top}(F, t\phi^c)$, the pressure function $\mathbb{R} \ni t \mapsto P(t)$ is monotone non-increasing. As a consequence of that fact and the definition of $F \in \mathcal{D}^r$,

- $P(0) = h_{top}(F) > h_{top}(g) > 0$

- If $\eta \in \mathcal{M}_1(f_x)$ for all $x \in \mathbb{T}^d$ is the measure in item (3) in the definition of $F \in \mathcal{D}^r$ together with item (1) in definition of $\mathcal{D}^r$, then $Leb \times \eta \in \mathcal{M}_1(F)$ and for all $t \in \mathbb{R}$:

$$h_{Leb\times\eta}(F) - t\lambda^c_{Leb\times\eta}(F) = h_{Leb\times\eta}(F) = h_{top}(g)$$

Then we have $P(t) \geq h_{Leb\times\eta}(F) \geq h_{Leb}(g) = h_{top}(g)$. We remind now that by the **Margules-Ruelle** inequality 1.1, we have that for all $F-$invariant and ergodic μ,

$$h_\mu(F) - \lambda^c_\mu(F) \leq \sum_{\lambda_j(F)>0} m_j\lambda_j(F) \leq \sum_{i=1}^{d} \lambda_i(g, Leb).$$

This implies that

$$P(1) \leq \sum_{i=1}^{d} \lambda_i(g, Leb) = h_{Leb}(g) = h_{top}(g) \leq P(t)$$

and so, by the fact that the $P(t) \geq h_{top}(g)$,

$$P(t) = h_{top}(g), \forall t \geq 1$$

As F is differentiable and by remark 0.2.6 together with item (2) in the definition of $\mathcal{D}^r$ 0.2.4, $h_{top}(F) = \log(deg(F)) > \log(deg(g)) = h_{top}(g)$. We conclude that there exists $t_0 \in (0, 1]$ such that the function $\mathbb{R} \ni t \mapsto P(t)$ has phase transition in t_0, with $t_0 = \inf\{t \in (0, 1]; P(t) = h_{top}(g)\}$. □

From the previous lemma, the topological pressure function is constant and equal to $h_{top}(g)$ for all $t \geq t_0$, and it is convex for all $t < t_0$. Further ahead, we will show that $P(t)$ is actually strictly convex, using the properties of the transfer operator associated to F and ϕ^c.

1.3.2 Step 2: Absence of spectral gap after transition

Now we can show the absence of spectral gap on the parameter t_0. We already know, by Lemma 1.3.2, that the topological pressure function $t \mapsto P(t)$ is convex monotone non-increasing. We will show now that for each parameter s where the transfer operator has spectral gap, the topological pressure function is actually strictly convex in a neighbourhood of s. As a consequence of that, we establish a relation between the spectral gap property and strict convexity of the pressure function (and its consequence on the phase transition).

Another way to define the transfer operator is through duality. In [S12] it is defined in the following way:

Definition 1.3.3. *The transfer operator of a non-singular map* $(M, \mathcal{B}, \mu, T)$ *is the operator* $\hat{T} : L^1(\mu) \to L^1(\mu)$ *such that* $\hat{T}(f)$ *is the unique element of* $L^1(\mu)$ *such that for all test functions* $\varphi \in L^\infty$,

$$\int \varphi(\hat{T}(f))d\mu = \int (\varphi \circ T).f d\mu$$

For the next theorem, we assume the following assumptions: let $(M, \mathcal{B}, T, \mu)$ be a mixing, probability preserving map. Suppose that $\hat{T}$ has the spectral gap property in some Banack space $(\mathcal{E}, ||.||)$ of functions, subset of $L_1(\mu)$, which contains the constants, is closed under multiplication and which satisfies the inequalities:

$$||fg|| \leq ||f||||g||, \ ||.|| \geq ||.||_1$$

With the previous assumptions in mind, Recall the following central limit theorem (C.L.T.):

Theorem 1.3.4. *(Central Limit Theorem [S12]) Let* $\varphi \in \mathcal{E}$ *be bounded with* μ*-integral zero. If there does not exist* $v \in \mathcal{E}$ *such that* $\varphi = v - v \circ T$ μ*-a.e., then there exist* $\sigma > 0$ *such that*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ T^n \xrightarrow[n \to \infty]{dist} \mathcal{N}(0, \sigma^2)$$

i.e.

$$\mu\Big(x; \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \varphi \circ T^n(x) \in [a, b]\Big) \longrightarrow \frac{1}{\sqrt{2\pi\sigma^2}} \int_a^b e^{\frac{-t^2}{2\sigma^2}} dt$$

for all intervals [a.b] and $\mathcal{N}(0, \sigma)$ *denotes the Gaussian distribution with mean zero and standard deviation* σ.

The proof of the C.L.T. follows closely the well-known **Nagaev's method**, which we briefly recall (for more details, we refer to [S12]). The method consist on doing perturbations of the operator $\hat{T} = T_0$: define a new operator

$$\hat{T}_t(f) := \hat{T}(e^{it\varphi} f)$$

Defined that way, $\hat{T}_t$ are bounded linear operators acting on $\mathcal{E}$. In our context, we have $\hat{T} := \mathcal{L}_{F, s\phi^c}$, $\mathcal{E} = E$ and $\mu = \mu_{s\phi^c}$, so we can translate the perturbations as

$$t \mapsto \mathcal{L}_{s\phi^c + it\varphi} / \lambda_\phi^c$$

If we denote λ_t as the spectral radius for this perturbed operator $\hat{T}$, Sarig [S12], proved the following expansion of λ_t near zero: $\lambda_t = 1 - \frac{1}{2}\sigma^2 t^2 + O(t^3)$ as $t \to 0$, where

$$\sigma := \sqrt{\lim_{n \to \infty} \frac{1}{n} \int (\varphi_n)^2 d\mu_\phi} \geq 0$$

and $\varphi_n := \varphi + \varphi(F) + ... + \varphi(F^{n-1})$. The first two derivatives of λ_t at zero are

$$\lambda_0' = \int \varphi d\mu = 0 \text{ and } \lambda_0'' = -\sigma^2.$$

Furthermore, supposing $\sigma = 0$, we can construct a solution u to the equation $\varphi = c + u - u \circ F$ for μ-a.e., which means that φ cohomologous to a constant. In our context, again taking $\mu = \mu_\phi$, we have

$$\lambda_0'' = \frac{d^2\left(\lambda_{\phi^c+it\varphi}/\lambda_{\phi^c}\right)}{dt^2} = \sigma^2_{F,\phi^c}(\varphi) \tag{1.2}$$

By differentiating, we have

$$\frac{d\,\lambda_{\phi^c+it\varphi}}{dt} = i\frac{d\,\lambda_f}{d\,f}\bigg|_{f=\phi^c+it\varphi} \cdot \varphi$$

and differentiating one more time,

$$\frac{d^2\,\lambda_{\phi^c+it\varphi}}{dt^2}\bigg|_{t=0} = -\frac{d^2\,\lambda_f}{d\,f^2}\bigg|_{f=\phi^c} \cdot (\varphi, \varphi) = -\frac{d^2\,\lambda_{\phi^c+t\varphi}}{d\,t^2}\bigg|_{t=0}$$

Now for $s\phi^c$ with spectral gap property, we have

$$\sigma^2_{F,s\phi^c}(\varphi) = \Upsilon''(0),$$

where $\Upsilon(t) = \dfrac{\lambda_{-s\phi^c+t\varphi}}{\lambda_{-s\phi^c}}$.

For the next lemma we will use **Nagaev's Method** and the fact that in the set of parameters where the topological pressure function is analytic, the second derivative is the variance of the central limit theorem.

Proposition 1.3.5. *Let $F \in \mathcal{D}^r, r \geq 2$. If $\mathcal{L}_{F,s\phi^c}$ has the spectral gap property on C^{r-1}, for some $s \in \mathbb{R}$, then the topological pressure function $t \mapsto P(t)$ is strictly convex in a neighbourhood of s. In particular, $\mathcal{L}_{F,t\phi^c}|_{C^{r-1}}$ does not have spectral gap property for all $t \geq t_0$.*

Proof. By the corollary of Proposition 1.2.6 , spectral gap still holds for small perturbations of the transfer operator. We denote by λ_{ϕ^c} the spectral radius $\rho(\mathcal{L}_{F,\phi^c})$ and fix $\varphi \in C^{r-1}$. Then, the function Υ defined by

$$t \mapsto \Upsilon(t) = \frac{\lambda_{s\phi^c+it\varphi}}{\lambda_{s\phi^c}}$$

is well defined and analytic in a small neighbourhood of zero. In particular, we take $\varphi = \phi^c + \int \varphi d\mu_{s\phi^c}$, where $\mu_{s\phi^c}$ is the equilibrium state provided by the spectral gap property. Define

$$G(t) := P_{top}(F, s\phi^c + t\varphi) = \log \rho(\mathcal{L}_{s\phi^c+t\varphi}).$$

We can see now that $G(t) = P_{top}(t+s) + t\int \varphi d\mu_{s\phi^c}$. Then, P is strictly convex around s if, and only if G is strictly convex around zero. Furthermore, for small values of t close to zero $\mathcal{L}_{F,s\phi^c+t\varphi}$ has spectral gap, which means that $G(t) = \log(\lambda_{s\phi^c+t\varphi})$. Note that $G'(0) = \int \varphi d\mu_{s\phi^c} = 0$ and $P''(s) = G''(0)$. By **Nagaev's method**,

$$\sigma^2 := \sigma^2_{F,s\phi^c} = -\frac{d^2\Upsilon(t)}{dt^2}|_{t=0}$$

where $\sigma^2_{F,s\phi^c}$ is the variance provided by the **Central Limit theorem** with respect to $F, \mu_{s\phi^c}$ and φ. Moreover, $\sigma = 0$ if, and only if, there is a constant $c \in \mathbb{R}$ and $u \in C^{r-1}$ such that for $\mu_{-s\phi^c} - a.e \quad x \in \mathbb{T}^d \times \mathbb{S}^1$

$$\varphi(x) = C + u(x) + u(F(x)).$$

By the Chain rule,

$$\frac{d\lambda_{s\phi^c+it\varphi}}{dt} = i\frac{\partial\lambda_\phi}{d\phi}|_{\phi=s\phi^c+it\varphi}.\varphi$$

Which means that

$$\frac{d^2T(t)}{dt^2} = -\frac{\frac{d^2\lambda_\phi}{d\phi^2}|_{\phi=s\phi^c}(\varphi,\varphi)}{\lambda_{s\phi^c}} = -\frac{\frac{d^2\lambda_{s\phi^c+t\varphi}}{dt^2}}{\lambda_{s\phi^c}}|_{t=0}$$

By differentiating $G(t) = \log(\lambda_{-s\log|DF_y|+t\varphi})$ two times, we get the following:

$$\frac{d^2\lambda_{s\phi^c+t\varphi}}{dt^2} = e^{G(t)}((G'(t))^2 + G''(t))$$

Now, we have that

$$\sigma^2 := \sigma^2_{s\phi^c}(\varphi) = \frac{\frac{\partial^2\lambda_{s\phi^c+t\varphi}}{\partial t^2}|_{t=0}}{\lambda_{s\phi^c}} = ((G'(0))^2 + G''(0))$$

In conclusion, $\sigma^2 = G''(0)$.

Arguing by contradiction, suppose that $G''(0) = 0$. By Nagaev's method, there exist $a \in \mathbb{R}$ and $u \in E$ such that

$$-\phi^c(x,y) = \log|\frac{\partial F}{\partial y}(x,y)| = a + u(F(x,y)) + u(x,y), \mu_\phi - a.e.(x,y)$$

As $supp(\mu_\phi) = \mathbb{T}^d \times \mathbb{S}^1$, we get $-\phi^c = \log|\frac{\partial F}{\partial y}| = a + u\circ F - u$. As a consequence, the Birkhoff average $\frac{1}{n}\log|\frac{\partial F}{\partial y}|$ converge uniformly to a, and so a is the unique central Lyapunov exponent of F. Then, a must be zero, by the definition of $F \in \mathcal{D}^r$, and then

$$\log|Df_x(y)| = u(F(x,y)) + u(x,y).$$

In particular for all $x \in \mathbb{T}^d$ we have that the unique Lyapunov exponent of $f_x : \mathbb{S}^1 \to \mathbb{S}^1$ is zero, which means that f_x is invertible. That contradicts the fact that, in the

definition of our model 0.2.4, $deg(F) > deg(g)$, which means that for all $x \in \mathbb{T}^d$ the local diffeomorphism on the circle f_x is a non-invertible function. As a conclusion $P''(s) = G(0) > 0$. Therefore, because P is analytic, $P'' > 0$ in a small neighbourhood of the parameter s. So P is strictly convex in a neighbourhood of s.
As a consequence, we have that spectral gap property cannot occur for $t \geq t_0$, otherwise, the topological pressure function would be strictly convex, contradicting the fact that $P(t) = h_{top}(g)$ for all $t \geq t_0$ by definition of t_0. □

1.3.3 Step 3: Existence of spectral gap before transition

There is a sufficient criterion for quasi-compactness, the **Lasota-yorke inequality**, which is in general hard to verify. We will use an alternative condition estimating the essential radius via an inequality with origins in the work of Ruelle [Rue89, Rue90] and show that our toy model satisfies quasi-compactness. We start by showing that for the class of dynamics $F \in \mathcal{D}^r$, quasi-compactness is equivalent to spectral gap. The proof can be found in [BC21] in the context of local diffeomorphism in the circle, but with similar arguments, we can prove it in more general settings. For completeness, we present the proof here:

Lemma 1.3.6. *Let $f : M \to M$ be a C^r-local diffeomorphism on a compact and connected manifold M, such that $\{f^{-n}(x); n \geq 0\}$ is dense in M for all $x \in M$, and let $\phi \in C^r(M, \mathbb{R})$ be a continuous real function. If $\mathcal{L}_{f,\phi}|_{C^{r-1}}$ is quasi−compact, then it has the spectral gap property.*

Proof. As $\mathcal{L}_{f,\phi}|_{C^{r-1}}$ is quasi-compact, it admits a peripheral eigenvalue, lets say the complex number $\gamma = \xi \cdot \rho(\mathcal{L}_{f,\phi}|_{C^{r-1}})$, with $|\xi| = 1$, that is, there exist an eigenfunction $\varphi \in C^{r-1} - \{0\}$ with $\mathcal{L}_{f,\phi}|_{C^{r-1}}\varphi = \gamma\varphi$. By lemma 1.2.10 $\lambda = \rho(\mathcal{L}_{f,\phi}|_{C^{r-1}})$ is a simple eigenvalue associated to the eigenfunction $|\varphi|$. we are going to show that it is the only peripheral eigenvalue. Define

$$s(x) := \begin{cases} \varphi(x)/|\varphi(x)|, \text{if } \varphi(x) \neq 0 \\ 1, \text{otherwise,} \end{cases}$$

Claim. *For all $n \in \mathbb{Z}$, $\mathcal{L}_{f,\phi}|_{C^{r-1}}(s^n|\varphi|) = \rho(\mathcal{L}_{f,\phi}|_{C^{r-1}})\xi^n s^n|\varphi|$*

Proof. Note that $s|\varphi| = \varphi$, thus

$$\mathcal{L}_{f,\phi}|_{C^{r-1}}(s|\varphi|) = \mathcal{L}_{f,\phi}|_{C^{r-1}}(\varphi) = \gamma\varphi = \gamma s|\varphi|.$$

This implies that

$$\mathcal{L}_{f,\phi}|_{C^{r-1}}\left(\frac{s}{\gamma(s\circ f)}\cdot|\varphi|\right)(x)=\sum_{f(y)=x}e^{\phi(y)}\frac{s(y)}{\gamma s(x)}|\varphi|(y)$$
$$=\frac{1}{\gamma s(x)}\mathcal{L}_{f,\phi}|_{C^{r-1}}(s|\varphi|)(x)=\frac{1}{\gamma s(x)}\gamma s(x)|\varphi|(x)$$

Thus

$$\mathcal{L}_{f,\phi}|_{C^{r-1}}\left(\frac{s}{\gamma(s\circ f)}|\varphi|\right)=|\varphi|.$$

By corollary 1.2.10.1 there exists a unique eigenmeasure ν such that $(\mathcal{L}_{f,\phi}|_{C^{r-1}})^*\nu=\rho(\mathcal{L}_{f,\phi}|_{C^{r-1}})\nu$ with $supp(\nu)=M$, integrate both sides and using duality:

$$\int\frac{s}{\rho(\mathcal{L}_{\phi}|_{C^{r-1}})\xi\cdot s\circ f}|\varphi|d\nu\cdot\rho(\mathcal{L}_{\phi}|_{C^{0}})=\int|\varphi|d\nu.$$

Hence

$$\frac{\rho(\mathcal{L}_{f,\phi}|_{C^{0}})}{\rho(\mathcal{L}_{f,\phi}|_{C^{r-1}})}\int\frac{s}{\xi\cdot s\circ f}|\varphi|d\nu=\int|\varphi|d\nu.$$

Since $|s/\xi{\cdot}s{\circ}f|\equiv 1$ and $|\varphi|\geq 0$, we must have $\rho(\mathcal{L}_{\phi}|_{C^{0}})=\rho(\mathcal{L}_{\phi}|_{C^{r-1}})$ and $s(x)=\xi{\cdot}s{\circ}f(x)$, for $|\varphi|$dν-a.e. $s(x)=\xi\cdot s\circ f(x)$, for $|\varphi|$dν-ae. By taking exponents on both sides, we have $s^n=\xi^n\cdot s^n\circ f$, φdν-a.e. which means $s^n|\varphi|=\xi^n\cdot s^n\circ f|\varphi|$, ν-ae. Furthermore, as supp $\nu=M$, the functions coincide a.e. Now applying the transfer operator we have:

$$\mathcal{L}_{f,\phi}(s^n|\varphi|)=\mathcal{L}(s^{n-1}s|\varphi|)=\mathcal{L}_{f,\phi}(\xi^{n-1}s^{n-1}\circ f\cdot s|\varphi|)=\xi^{n-1}s^{n-1}\mathcal{L}_{f,\phi}(s|\varphi|)$$
$$=\xi^{n-1}s^{n-1}\rho(\mathcal{L}_{f,\phi})\xi\cdot s|\varphi|=\xi^{n}\rho(\mathcal{L}_{f,\phi})s^{n}|\varphi|.$$

□

By Lemma 1.2.10, φ is bounded away from zero, thus s and $|\varphi|\in C^{r-1}$. Thereby $s^n|\varphi|\in C^{r-1}$, and, by previous claim, $\xi^n{\cdot}r(\mathcal{L}_{f,\phi}|_{C^{r-1}})\in\text{sp}(\mathcal{L}_{f,\phi})$ for all $n\in\mathbb{Z}$. Then the set $\{\xi^n:n\in\mathbb{Z}\}$ forms a subgroup of the circle, which is either dense or periodic. However by quasi-compactness all eigenvalues are isolated and it can't be a dense subgroup, so there is a $k>0$ such that $\xi^k=1$ and

$$\xi^k r(\mathcal{L}_{f,\phi}|_{C^{r-1}})=r(\mathcal{L}_{f,\phi}|_{C^{r-1}}).$$

We already know $|\varphi|$ is an eigenvector for the spectral radius by Lemma 1.2.10, thus we have both

$$\mathcal{L}^k_{f,\phi}(|\varphi|)=\rho(\mathcal{L}_{f,\phi})^k|\varphi|$$
$$\mathcal{L}^k_{f,\phi}(\varphi)=\rho(\mathcal{L}_{f,\phi})^k\varphi$$

and then

$$\mathcal{L}^k_{f,\phi}(|\varphi| - \varphi) = \rho(\mathcal{L}_{f,\phi})^k(|\varphi| - \varphi).$$

Take x_0 such that $\varphi(x_0) > 0$, since φ is bounded away from zero, one can always replace it by $\varphi/\varphi(x_0)$ to that end. This means that $(|\varphi| - \varphi)(x_0) = 0$, and also $||\varphi| - \varphi|(x_0) = 0$. On the other hand, rewrite $\mathcal{L}^k_{f,\phi} = \mathcal{L}_{f^k, S_k\phi}$ where $S_k\phi = \phi \circ f^{n-1} + \cdots + \phi \circ f + \phi \in C^{r-1}$, and thus arguing by denseness of pre-orbits and continuity, it follows that $|\varphi| - \varphi \equiv 0$. Now applying the transfer operator:

$$\xi\lambda\varphi = \mathcal{L}_{f,\phi}|_{C^{r-1}}\varphi = \mathcal{L}_{f,\phi}|_{C^{r-1}}|\varphi| = \lambda\varphi.$$

Finally $\xi = 1$ and $\lambda = r(\mathcal{L}_{f,\phi})$ is the unique peripheral eigenvector. □

According to the previous lemma, it is enough for us to prove quasi-compactness to conclude that the spectral gap holds for the transfer operator associated to $F \in \mathcal{D}^r$ and $\phi \in C^r(\mathbb{T}^d \times \mathbb{S}^1, \mathbb{R})$ because of the property of dense pre-images holds for F. We already know that the topological pressure function is non-increasing, but the next result tells us something more: before the phase transition t_0 the pressure function is strictly decreasing.

Lemma 1.3.7. *Let $F \in \mathcal{D}^r$ be given. Then, the topological pressure function $\mathbb{R} \ni t \mapsto P(t) := P_{top}(F, t\phi^c)$ is strictly decreasing in $(-\infty, t_0)$.*

Proof. Suppose that there is an interval $(a, b) \subset (-\infty, t_0)$ such that $P(t) = C$, for all $t \in (a, b)$. Fix $t \in (a, b)$ and take $w < 0$ small such that $t + w \in (a, b)$. As F is expansive there exists at least an invariant measure $\mu_{t\phi^c}$ which is an equilibrium state with respect to F and $t\phi^c$. Then:

$$h_{\mu_{t\phi^c}}(F) - (t + w)\lambda^c_{\mu_{t\phi}}(F) \le P_{top}(t + w) = P(t) = h_{\mu_{t\phi}}(F) - t\lambda^c_{\mu_{t\phi}}(F)$$

which means that $w\lambda^c_{\mu_{t\phi}}(F) \ge 0$. As $\lambda^c_{\mu_{t\phi^c}}(F) \ge 0$ thus $\lambda^c_{\mu_{t\phi}}(F) = 0$, and so

$$P(t) = C = h_\nu(F)$$

for all $t \in (a, b)$, where ν is an invariant measure satisfying $h_\nu(F) = \max\{h_\mu(F); \lambda^c_\mu(F) = 0\}$, and so $\lambda^c_\nu(F) = 0$. In conclusion, using Margulis-Ruelle's inequality, we have:

$$h_{top}(g) = \sum_{i=1}^{d} \lambda_i(g, Leb) \ge h_\nu(F) \ge h_{Leb\times\eta}(F) \ge h_{top}(g),$$

The first inequality is due to the fact that by Margulis-Ruelle inequality 1.2.3:

$$h_\nu(F) \le \sum_{i=1}^{d} \lambda_i(g, \nu) \le \sum_{i=1}^{d} \lambda_i(g, Leb) = h_{top}(g)$$

the second inequality holds by definition of ν, and η is the invariant measure in the definition of $\mathcal{D}^r$ with zero Lyapunov exponent. In that way, we have $a \geq t_0$, by definition of t_0. □

The previous lemma gives us an important information about the topological pressure function for $t < t_0$, it means that we are in the right direction to prove theorem A. Our next goal is to show that before the phase transition, $t < t_0$, the transfer operator $\mathcal{L}_{F,t\phi^c}$ is quasi-compact, and then conclude that it satisfies the spectral gap property, by Lemma 1.3.6, as F has dense pre-images. In order to do that, we need to estimate the essential radius $\rho_{ess}(\mathcal{L}_{F,t\phi})$ using the following theorem proved by Campbell and Latushkin [CL97].

Definition 1.3.8. *Suppose X and Y are connected smooth manifolds. A map $f : X \to Y$ is a **smooth covering map** if*

1. *X is path-connected and locally path-connected;*
2. *f is surjective and continuous;*
3. *each point $p \in Y$ has a neighbourhood U that is evenly cover by f, meaning that U is connected and each component of $f^{-1}(U)$ is mapping diffeomorphically onto U by f.*

Remark 1.3.9. *If $F \in \mathcal{D}^r$, then clearly F is a smooth covering map.*

Theorem 1.3.10 ([CL97])**.** *Assume that $f : M \to M$ is any smooth covering map and $\phi \in C^r(M, \mathbb{R})$. Then*

$$\rho_{ess}(\mathcal{L}_{f,\phi}|_{C^k}) \leq \exp\big[\sup_{\mu\in\mathcal{M}_e(f)} \{h_\mu(f) + \int \phi d\mu - k\lambda_{\min}(f,\mu)\}\big] \text{ and}$$

$$\rho(\mathcal{L}_{f,\phi}|_{C^k}) \leq \exp\big[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi d\mu\}\big],$$

for $k = 0, 1, \ldots, r.$, where $\lambda_{\min}(f,\mu)$ denotes the smallest Lyapunov-Oseledec exponent of μ.

Now we have all the ingredients to show that before the transition the transfer operator is quasi-compact. With the previous theorem and using Lemma 1.3.7 we can estimate both essential and spectral radius.

Proposition 1.3.11 (**Fundamental Lemma**). *Let $F \in \mathcal{D}^r$ be given, with $r > 1$. Then, $\rho_{ess}(\mathcal{L}_{F,t\phi^c}) < e^{P(t)}$, for all $t < t_0$.*

Proof. Firstly, consider the case where μ is an F-invariant probability such that $\lambda_{\min}(F,\mu) = \lambda^c_\mu(F)$. Then, using that the pressure function is strictly decreasing for $t < t_0$, by Lemma 1.3.7, we have

$$h_\mu(F) - t\lambda^c_\mu - k\lambda_{\min}(F,\mu) = h_\mu(F) - (t+k)\lambda^c_\mu(F) \leq P(t+k) < P(t)$$

for $k = 1, ..., r$ like in Campbell-Latushkin's Theorem 1.3.10.
Secondly, suppose that $\lambda_{\min}(F,\mu) = \lambda^1_\mu(F) \neq \lambda^c_\mu(F)$. In that case we must have $\lambda^1_\mu(F) > 0$, by defnition of $F \in \mathcal{D}^r$. Then:

$$h_\mu(F) - t\lambda^c_\mu(F) - k\lambda_{\min}(F,\mu) = h_\mu(F) - t\lambda^c_\mu(F) - k\lambda^1_\mu(F) \leq P(t) - k\lambda^1_\mu(F) < P(t).$$

The result now follows from Campbell-Latushkin's Theorem. □

Until now, almost all the results that we proved in the previous subsections has the spectral gap property as one of their assumptions. In order to show that the previous results hold, we now show that the transfer operator has spectral gap property before the transition, and we can apply the results of this chapter to our toy model.

Lemma 1.3.12. *Let $F \in \mathcal{D}^r$ be given, with $r > 1$. Then, for $t \leq 0$, $\mathcal{L}_{F,t\phi}|_{C^{r-1}}$ has the spectral gap property.*

Proof. We first show the spectral gap property for $t = 0$. By the fundamental lemma, Lemma 1.3.11 in $[0, t_0)$ and Remark 0.2.6,

$$\rho_{ess}(\mathcal{L}_{F,0}|_{c^{r-1}}) < e^{P_{top}(0)} = e^{h_{top}(F)} = deg(F).$$

Since $\mathcal{L}_{F,0}(1) = \deg(F)1$, then $\rho(\mathcal{L}_{F,0}) \geq deg(F)$. By Campbell and Latushkin's Theorem 1.3.10

$$\rho(\mathcal{L}_{F,0}) \leq e^{h_{top}(F)} = deg(F),$$

which means that $deg(F) = \rho(\mathcal{L}_{F,0}|_{C^{r-1}})$. Hence, $\rho_{ess}(\mathcal{L}_{F,0}) < \rho(\mathcal{L}_{F,0})$ and $\mathcal{L}_{F,0}|_{C^{r-1}}$ is quasi compact. By Lemma 1.3.6, $\mathcal{L}_{F,0}|_{C^{r-1}}$ has the spectral gap property. Next, take

$$t_1 := \inf\{t < 0; \mathcal{L}_{F,t\phi} \quad \text{has spectral gap property}\}$$

Arguing by contradiction, suppose that $t_1 > -\infty$. By the fact that spectral gap is an open property, see corollary 1.2.7, $\mathcal{L}_{F,t_1\phi^c}|_{C^{r-1}}$ cannot have spectral gap property.

Claim 1. $\rho(\mathcal{L}_{F,t_1\phi^c}|_{C^{r-1}}) = e^{P(t_1)}$

Proof. In fact, suppose $t_1 > \infty$. Due to the spectral gap property of $\mathcal{L}_{F,t\phi^c}$ and Lemma 1.2.12, for $t \in (t_1, 0]$ we have that $\rho(\mathcal{L}_{F,t\phi^c}) = e^{P(t)}$. By Lemma 1.3.7, $(t_1, 0] \ni t \mapsto \rho(\mathcal{L}_{F,t\phi^c}|_{C^{r-1}})$ is strictly decreasing. On one hand, take $t_n \searrow t_1$ and $\rho(\mathcal{L}_{F,t_1\phi^c}) < \rho(\mathcal{L}_{F,t_n\phi^c})$. By the semi-continuity of the spectral components (see e.g. [K95]):

$$\rho(\mathcal{L}_{F,t_1\phi}|_{C^{r-1}}) \geq \limsup_{n\to\infty} \rho(\mathcal{L}_{F,t_n\phi^c}) = \limsup_{n\to\infty} e^{P(t_n)} = e^{P(t_1)}.$$

On the other hand, $\rho(\mathcal{L}_{F,t_1\phi^c}) \leq e^{P(t_1)}$, by theorem 1.3.10, and we have proved Claim 2. □

As a consequence, $\rho(\mathcal{L}_{F,t_1\phi^c}|_{C^{r-1}}) = e^{P(t_1)} > \rho_{ess}(\mathcal{L}_{F,t_1\phi^c}|_{C^{r-1}})$ and it means that $\mathcal{L}_{F,t_1\phi^c}|_{C^{r-1}}$ is quasi-compact, and so it satisfies the spectral gap property, contradicting the definition of t_1. Hence, $t_1 = -\infty$. □

Given $F \in \mathcal{D}^r$, with $r \geq 2$, define analogously:

$$\tau_2 := \sup\{t > 0; \mathcal{L}_{F,t\phi^c}|_{C^{r-1}} \textit{has the spectral gap property}\}$$

Note that τ_2 exists and is at most t_0 by the definition of t_0. Analogously to Lemma 1.3.12, it holds $\rho(\mathcal{L}_{F,\tau_2\phi^c}|_{C^{r-1}}) = e^{P(\tau_2)}$. Since the spectral gap property is open, $\mathcal{L}_{F,\tau_2\phi^c}|_{C^{r-1}}$ has not the spectral gap property and thus $\rho(\mathcal{L}_{F,\tau_2\phi^c}|_{C^{r-1}}) = \rho_{ess}(\mathcal{L}_{F,\tau_2\phi^c}|_{C^{r-1}})$ by Lemma 1.2.12.

Claim 2. $\tau_2 = t_0$

Proof. Arguing by contradiction, suppose that $\tau_2 < t_0$. By Lemma 1.3.11, we get:

$$e^{P(\tau_2)} = \rho(\mathcal{L}_{\tau_2\phi^c}|_{C^{r-1}}) = \rho_{ess}(\mathcal{L}_{\tau_2\phi^c}|_{C^{r-1}}) < e^{P(\tau_2)}.$$

Which is absurd. Therefore, $\tau_2 \geq t_0$. Since $\tau_2 \leq t_0$, we get the equality. □

From definition of τ_2 we get: If $F \in \mathcal{D}^r$, with $r \geq 2$, then $\mathcal{L}_{F,t\phi^c}|_{C^{r-1}}$ has spectral gap for all $t < t_0$. With the previous lemma, together with the Proposition 1.3.2, Lemma 1.2.12 and Proposition 1.3.5, we have completed the proof of the Theorem A. □

Chapter 2

Multifractal Analysis for Skew-Product

In this chapter, we apply our results about the spectral gap in the proof of Theorem A to obtain consequences for multifractal analysis. Generally, in multifractal analysis, we want to understand the behaviour and differentiability of functions like $I \mapsto h_{top}(L_I)$ and $I \mapsto \dim_H(L_I)$ where L_I is the set of points which have Lyapunov exponents lying on the given interval I. Our main objective is to prove Theorem B. One of the tools we'll use in this chapter is the theory of **Large Deviations Principle** that basically provides exponential bounds of rare events, i.e., we characterize "rare" events in terms of a **rate function**. Here we first give the classical definition of large deviations principle:

Definition 2.0.1. *(Rate function) Let M be a compact metric space. A rate function $\mathcal{I} : M \to [0, \infty]$ is a lower semi-continuous map (such that for all $\alpha \in [0, \infty)$, the level set $\Psi_{\mathcal{I}}(\alpha) := \{x; \mathcal{I}(x) \leq \alpha\}$ is a closed subset of M). A good rate function is a rate function for which all the level sets $\Psi_{\mathcal{I}}(\alpha)$ are compact subsets of M. Then, given a observable $\psi : M \to \mathbb{R}$, a probability $\mu \in \mathcal{M}_1(M)$ satisfies the **Large deviations principle** with rate function $\mathcal{I}$ if:*

$$- \inf_{x \in int(A)} \mathcal{I}(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log \mu(A_{n,\psi}) \leq \limsup_{n \to \infty} \frac{1}{n} \log \mu(\overline{A}_{n,\psi}) \leq - \inf_{x \in \overline{A}} \mathcal{I}(x)$$

for any measurable set A, where

$$A_{n,\psi} := \Big\{ x \in M; \frac{1}{n} \sum_{i=0}^{n-1} F^i(\psi(x)) \in int(A) \Big\}$$

and

$$\overline{A}_{n,\psi} := \Big\{ x \in M; \frac{1}{n} \sum_{i=0}^{n-1} F^i(\psi(x)) \in \overline{A} \Big\}$$

For more details on large deviations, see [DZ98, RY08] and [You90].

Given $F \in \mathcal{D}^r$, as F is topologically conjugated to an transitive uniformly expanding dynamics, then F has the periodic specification property, see definition 1.2.2. In particular, given a continuous observable $\psi : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{R}$ the Birkhoff's spectrum

$$S_\psi := \{\alpha \in \mathbb{R} : \exists x \in \mathbb{T}^d \times \mathbb{S}^1 \text{ with } \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \psi(F^i(x)) = \alpha\}$$

is a non-empty bounded interval; furthermore

$$S_\psi = \Big\{ \int \psi d\mu : \mu \text{ is a } F - \text{invariant probability} \Big\}$$

(for more details and proof, see e.g. [Th09]).

Here, for $F \in \mathcal{D}^r$, we study multifractal analysis of the central Lyapunov exponent for maps $F \in \mathcal{D}^r$: let $[a, b]$ a closed interval, we define

$$L^c_{a,b} = X_\phi([a, b]) = \{x \in \mathbb{T}^d \times \mathbb{S}^1; \lambda^c(x) \in [a, b]\}$$
$$\hat{L}^c := \{x \in \mathbb{T}^d \times \mathbb{S}^1; \nexists \lim \frac{1}{n} \log ||DF^n|_{E^c_x}||\}.$$

We also consider the sets

$$E^c_{a,b} := \hat{L}^c \cap \Big\{x \in \mathbb{T}^d \times \mathbb{S}^1; \liminf_{n\to\infty} \frac{1}{n} \log ||DF^n|_{E^c_x}|| \in [a, b] \quad or \quad \limsup_{n\to\infty} \frac{1}{n} \log ||DF^n|_{E^c_x}|| \in [a, b]\Big\}$$

We recall a result by Bomfim and Varandas [BV17], in which we have an upper bound for the topological pressure of these fractal sets. We define for any interval I:

$$\underline{X}_I := \{x \in M; \liminf_{n\to\infty} \frac{1}{n} S_n(\psi(x)) \in I\} \text{ and } \overline{X}_I := \{x \in M; \limsup_{n\to\infty} \frac{1}{n} S_n(\psi(x)) \in I\}$$

If I^δ is δ-neighbourhood of the interval I, we define:

$$L_{I^\delta,\nu} := -\limsup_{n\to\infty} \frac{1}{n} \log \nu \Big\{x \in \mathbb{T}^d \times \mathbb{S}^1; \frac{1}{n} S_n(\psi(x)) \in I^\delta\Big\}$$

Theorem 2.0.2. *Let M be a compact metric space, $F : M \to M$ a continuous map, $\phi : M \to \mathbb{R}$ a continuous potential and ν (a not necessarily invariant) Gibbs measure on M and $\mu_\phi \ll \nu$ be the unique equilibrium state of F with respect to ϕ. For any continuous observable $\psi : M \to \mathbb{R}$, any closed interval $I \subset \mathbb{R}$ and any small $\delta > 0$,*

$$P_{\underline{X}_I}(F, \phi) \le P_{\overline{X}_I}(F, \phi) \le P_{top}(F, \phi) - L_{I^\delta,\nu} \le P_{top}(F, \phi)$$

Proof. See [BV17]. □

In our context, $\lim \frac{1}{n}\log|\frac{\partial F^n}{\partial y}(w)| = \lambda^c_{\mu_0}(w)$ for μ_0-almost every w, where μ_0 is the measure of maximum entropy of F. Thus, we are interested in understanding asymptotically

$$\frac{1}{n}\log \mu_0\Big(\{w \in \mathbb{T}^d \times \mathbb{S}^1 : \frac{1}{n}\log\Big|\frac{\partial F^n}{\partial y}(w)\Big| \in [a,b]\}\Big),$$

with $[a,b]$ intersecting the central Lyapunov spectra

$$L^c(F) := \Big\{\alpha \in \mathbb{R} : \exists w \in \mathbb{S}^1 \times T^d \text{ with } \frac{1}{n}\log\Big|\frac{\partial F^n}{\partial y}(w)\Big| = \alpha\Big\}$$

Note that since F has specification property and

$$\frac{1}{n}\log\Big|\frac{\partial F^n}{\partial y}(w)\Big| = \frac{1}{n}\sum_{j=1}^{n-1}\log\Big|\frac{\partial F}{\partial y}(F^j(w))\Big|$$

then the central Lyapunov spectra satisfies

$$L^c(F) = \Big\{\int \log\Big|\frac{\partial F}{\partial y}\Big|d\mu : \mu \in \mathcal{M}_1(F)\Big\} = \overline{\Big\{\int \log\Big|\frac{\partial F}{\partial y}\Big|d\nu ; \nu \in \mathcal{M}_{erg}(F)\Big\}}$$

and it is a non-empty compact interval (see [Th09]). In fact, $L^c(F)$ will have non-empty interior and by Lemma 1.3.1 and the definition of $F \in \mathcal{D}^r$, we have $\inf L^c(F) = 0$.

Definition 2.0.3. *We set $\lambda^c_{\min} := \inf_{t<t_0}\lambda^c_{\mu_{t\phi^c}}$ and $\lambda^c_{\max} := \sup_{t<t_0}\lambda^c_{\mu_{t\phi^c}}$. where $\mu_{t\phi^c}$ is the equilibrium state of F with respect to $t\phi^c$ obtained in Lemma 1.2.12.*

Moreover, denote

$$\Delta := \{(a,b) \in L^c(F) \times L^c(F) : a \leq b\}$$

and write $\lambda^c_{\mu_0}$ the central Lyapunov exponent of the measure of maximum entropy of F. Recalling what we want to prove:

Theorem B. *The entropy function $\Delta \ni (a,b) \mapsto h_{top}(L^c_{a,b})$ is a concave C^1 function satisfying:*

- *$h_{top}(L^c_{a,b}) = h_{top}(E^c_{a,b})$, for all $(a,b) \in \Delta$;*
- *It is constant and equal to its maximum value $h_{top}(F)$ for (a,b) in the rectangle $[0,\lambda^c_{\mu_0}] \times [\lambda^c_{\mu_0}, \lambda^c_{\max}]$;*
- *It is strictly concave and analytic everywhere, except for $b \leq \lambda^c_{\min}$, in the case of the exponent $\lambda_{\min} > 0$ where the function is linear.*

Theorem B is a consequence of the spectral phase transition proved in Theorem A. We will use the fact that in the context of spectral gap, the **free energy** converges to a "pressure's variation". Then we will show that the **Legendre transform** of the free energy can be seen both as an analytic and thermodynamic rate function as well, in that part of Birkhoff's spectrum where the spectral gap property holds. Combining with the result by Thompson [Th09], the variational principle for level sets of Birkhoff average, and Theorem 2.0.2 by [BV17], we show that the entropy of the sets $L^c_{a,b}$ can be written as a variational principle.

2.1 Preliminaries

As in the previous chapter, we consider $F \in \mathcal{D}^r$, $r > 1$ as in definition 0.2.4, and the potential

$$\phi^c : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{R} \textit{ given by } (x, y) \mapsto -\log\left|\frac{\partial F}{\partial y}(x, y)\right|$$

and consider the associate transfer operator $\mathcal{L}_{F,t\phi^c}|_{C^{r-1}}$, which has spectral gap property for all $t < t_0$, by Theorem A. We will describe $L^c(F)$ in terms of multifractal analysis using the equilibrium states $\mu_{t\phi^c}$. It is important for us to specify the interval of definition of $L^c(F)$.

Lemma 2.1.1. *$L^c(F) = [0, \lambda^c_{\max}]$ and $\lambda^c_{\min} \in L^c(F)$.*

Proof. Since F has the specification property, applying [Th09], we have that $L^c(F) = \{\int \log\left|\frac{\partial F}{\partial y}\right| d\eta : \eta \in \mathcal{M}_1(F)\}$. In particular, $\lambda^c_{\mu_{t\phi^c}} \in L^c(F)$ for all $t < t_0$ and $\lambda^c_{\max}, \lambda^c_{\min} \in L^c(F)$. We will show that $\sup L^c(F) = \lambda^c_{\max}$. Take a probability $\eta \in \mathcal{M}_1(F)$ and $t < 0$. Then:

$$h_\eta(F) - t\int \log\left|\frac{\partial F}{\partial y}\right| d\eta \leq h_{\mu_{t\phi^c}} - t\int \log\left|\frac{\partial F}{\partial y}\right| d\mu_{t\phi^c} \text{ implies}$$
$$\frac{h_\eta(F)}{t} - \int \log\left|\frac{\partial F}{\partial y}\right| d\eta \geq \frac{h_{\mu_{t\phi^c}}(F)}{t} - \int \log\left|\frac{\partial F}{\partial y}\right| d\mu_{t\phi^c}.$$

Letting $t \mapsto -\infty$, we have that

$$\int \log\left|\frac{\partial F}{\partial y}\right| d\eta \leq \lim_{t\to-\infty} \int \log\left|\frac{\partial F}{\partial y}\right| d\mu_{t\phi^c} \leq \sup_{t<t_0} \int \log\left|\frac{\partial F}{\partial y}\right| d\mu_{t\phi^c}.$$

Thus, $\sup L^c(F) = \lambda^c_{\max}$ and conclude that $L^c(F) = [0, \lambda^c_{\max}]$. □

Following the same idea as in [DZ98], we are going to define the **free energy** and show a connection with an important thermodynamical quantity. Then, the **Legendre transform** of the free energy can often be proved to be a rate function of a large deviations principle. That is the content of **Gärtner Ellis**'s theorem.

Definition 2.1.2. *We define the **free energy** as*

$$\mathcal{E}(t) = \mathcal{E}_{F,\phi^c}(t) = \limsup_{n\to\infty} \frac{1}{n} \log \int e^{tS_n(\phi^c)} d\mu_0$$

where μ_0 is the measure of maximum entropy for F.

In our setting, due to the fact that for $t < t_0$, where t_0 is the transition parameter provided by Theorem A, the transfer operator with respect to F and $t\phi^c$ has spectral gap, we will prove that the limit above does exist for $t < t_0$ and its limit is related with the pressure function.

Lemma 2.1.3. *Let $F \in \mathcal{D}^r$ be fixed, with $r > 1$. Then, for all $t < t_0$ the following limit exists*

$$\mathcal{E}(t) = \lim_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\phi^c}\, d\mu_0 = P(t) - \log deg(F)$$

Moreover, $\mathcal{E} : (-\infty, t_0) \to \mathbb{R}$ is real analytic and strictly convex.

Proof. The proof that

$$\mathcal{E}(t) = \lim_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\phi^c}\, d\mu_0 = \log \rho(\mathcal{L}_{F,t\phi^c}) - \log \rho(\mathcal{L}_{F,0})$$

is analogous to [BCV16, Proposition 5.2], we present it here for completeness. For all $n \in \mathbb{N}$,

$$\int e^{tS_n(\phi^c)} d\mu_0 = \int \lambda_{F,0}^{-n} \mathcal{L}_{F,0}^n (h_{F,0} e^{tS_n\phi^c}) d\nu_{F,0} = \left(\frac{\lambda_{F,t\phi^c}}{\lambda_{F,0}}\right)^n \int (\lambda_{F,t\phi^c})^{-n} \mathcal{L}_{F,t\phi^c}^n (h_{F,0}) d\nu_{F,0}.$$

On the one hand, as $h_{F,0}$ is bounded away from zero and infinity, we have that:

$$(\lambda_{F,t\phi^c})^{-n} \mathcal{L}_{F,t\phi^c}^n (h_{F,0}) \xrightarrow{unif.} h_{F,t\phi^c} \int h_{F,0} d\nu_{F,t\phi^c}$$

using that $\mathcal{L}_{F,t\phi^c}$ has the spectral gap property for $t < t_0$ (see Theorem A). Observe that the limit above is again bounded away from zero and infinity. Using the dominated convergence theorem, we get

$$\lim_{n\to\infty} \frac{1}{n} \log \int e^{tS_n\phi^c} d\mu_0 = \log(\lambda_{F,t\phi^c}) - \log(\lambda_{F,0}) = P(t) - \log(\deg(F))$$

On the other hand, $\log \rho(\mathcal{L}_{F,t\phi^c}) = P(t)$ because $\mathcal{L}_{F,t\phi^c}$ has the spectral gap property, so we can apply Lemma 1.2.12. Finally, since $P(t)$ is analytic and strictly convex in $(-\infty, t_0)$ (by Theorem A) we have finished the proof of the lemma. □

Definition 2.1.4. *Since the function $(-\infty, t_0) \ni t \mapsto \mathcal{E}(t)$ is strictly convex, the local **Legendre's transform** of the free energy is well defined as follows:*

$$\mathcal{I}(s) = \sup_{t<t_0} \{ts - \mathcal{E}(t)\}.$$

As the function $\mathcal{E}$ is strictly convex and differentiable, its domain is given by $Dom(\mathcal{I}) = \{\mathcal{E}'(t); t < t_0\}$. In fact, we have

$$\frac{d}{dt}(st - \mathcal{E}(t)) = s - \mathcal{E}'(t) = 0 \Leftrightarrow s = \mathcal{E}'(t)$$

Hence, $\mathcal{I}(\mathcal{E}'(t)) = t\mathcal{E}'(t) - \mathcal{E}(t)$, for all $t < t_0$.

Remark 2.1.5. *Given $s < t_0$, by Theorem A and openness of the spectral gap property, $\mathcal{L}_{F,s\phi^c+\psi}$ has the spectral gap property on C^{r-1} for $\psi \in C^{r-1}$ and $||\psi||_{r-1} < \epsilon$. Applying Lemma 1.2.12 we have that $\{\Psi \in C^{r-1}(\mathbb{T}^d \times \mathbb{S}^1, \mathbb{R}); ||\psi||_{r-1} < \epsilon\} \ni \psi \mapsto P_{top}(F, s\phi^c + \psi)$ is analytic. By [W92], $P'(t) = \int \phi^c d\mu_{t\phi^c}$.*

The previous remark insures that the domain of $\mathcal{I}$ is equal to $\{-\int \phi^c d\mu_{t\phi^c} : t < t_0\} = (\lambda^c_{\min}, \lambda^c_{\max})$. Furthermore, by the expression of the free energy:

$$\begin{aligned} \mathcal{I}(\mathcal{E}'(t)) = t\mathcal{E}'(t) - \mathcal{E}(t) &= t\int \phi^c d\mu_{t\phi^c} - \Big(h_{\mu_{t\phi^c}}(F) + t\int \phi^c d\mu_{t\phi^c} - h_{\mu_0}(F)\Big) \\ &= -h_{\mu_{t\phi^c}}(F) + h_{\mu_0}(F), \end{aligned}$$

then $\mathcal{I} \geq 0$ and, $\mathcal{I}(s) = 0$ if, and only if $s = \lambda_{\mu_0}$. We summarize the property of the Legendre transform with the following lemma.

Lemma 2.1.6. *The Legendre transform of the free energy has the following properties:*

i) The domain of $\mathcal{I}$ is $(\lambda^c_{\min}, \lambda^c_{\max})$;

ii) $\mathcal{I}$ is a positive, strictly convex function and $\mathcal{I}(s) = 0$ if and only $s = \lambda^c(\mu_0)$;

iii) $\mathcal{I}$ is a real analytic function.

The following results hold from Gartner-Ellis theorem (see e.g. [DZ98, RY08]) as a consequence of the differentiability of the free energy function.

Proposition 2.1.7. *Given any interval $[a, b] \subset (\lambda^c_{\min}, \lambda^c_{\max})$ it holds that*

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_0 \left(w \in \mathbb{S}^1 \times \mathbb{T}^d : \frac{1}{n} \log \left| \frac{\partial F^n}{\partial y}(w) \right| \in [a, b] \right) = - \inf_{s\in[a,b]} \mathcal{I}(s).$$

Since F is expansive and has the specification property, applying [Bo74] we have that μ_0 is a Gibbs probability with respect to the dynamics of F and null potential. Thus, applying [You90], the measure of maximum entropy μ_0 satisfies a large deviations principle for all continuous observables, with rate function obtained through thermodynamical quantities. More formally:

Proposition 2.1.8. *Given any interval* $[a,b] \subset L^c(F)$ *it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_0 \left(w \in \mathbb{S}^1 \times \mathbb{T}^d : \frac{1}{n} \log \left| \frac{\partial F^n}{\partial y}(w) \right| \in [a,b] \right)$$
$$\leq -h_{top}(F) + \sup\{h_\eta(F) : \lambda^c(\eta) \in [a,b]\}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_0 \left(w \in \mathbb{S}^1 \times \mathbb{T}^d : \frac{1}{n} \log \left| \frac{\partial F^n}{\partial y}(w) \right| \in (a,b) \right)$$
$$\geq -h_{top}(F) + \sup\{h_\eta(F) : \lambda^c(\eta) \in [a,b]\}$$

Remark 2.1.9. *Since* F *is expansive, then* $\eta \mapsto h_\eta(F)$ *is upper semicontinuous (see e.g. [OV16]). By the contraction's principle, we know that there is uniqueness of the large deviation rate function (see e.g. [DZ98]). It follows from the two previous propositions that*

$$\mathcal{I}(s) = -h_{top}(F) + \sup\{h_\eta(F) : \lambda^c_\eta(F) = s\},$$

for all $s \in (\lambda^c_{\min}, \lambda^c_{\max})$.

2.2 Proof Of Theorem B

We know that the Legendre transform has some distinct behaviour whether it is positive or zero. The next lemma gives us a way to compute the topological entropy of the sets $L^c_{a,b}$ and $E^c_{a,b}$.

Lemma 2.2.1. *Given* $[a,b] \subset L^c(F)$ *we have that* $h_{L^c_{a,b}}(F) = \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}$. *Moreover, if* $a < b$ *then* $h_{L^c_{a,b}}(F) = h_{E_{a,b}}(F)$.

Proof. We start the proof by defining the large deviations rate:

$$LD_{c,d} := \limsup_{n\to\infty} \frac{1}{n} \log \mu_0 \left(w \in \mathbb{S}^1 \times \mathbb{T}^d : \frac{1}{n} \log |\frac{\partial F^n}{\partial y}(w)| \in [c,d] \right).$$

We already know that the measure of maximum entropy μ_0 satisfies the Gibbs property with respect to the dynamic F and null potential. Applying Theorem 2.0.2 we have that $h_{L^c_{a,b}}(F) \leq h_{top}(F) + LD_{a-\delta,b+\delta}$, for all $\delta > 0$. By Proposition 2.1.8

$$LD_{a-\delta,b+\delta} \leq -h_{top}(F) + \sup\{h_\nu(F) : \lambda^c(\nu) \in [a-\delta, b+\delta]\}.$$

By the upper semi-continuity of the function $\nu \mapsto h_\nu(F)$ we can let δ go to zero, resulting in

$$h_{L^c_{a,b}}(F) \leq \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}.$$

Similarly, if $a < b$ then:

$$h_{E_{a,b}}(F) \leq \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}.$$

On the other hand, following Thompson [Th09] we know that

$$h_{L^c_{d,d}}(F) = \sup\{h_\nu(F) : \lambda^c(\nu) = d\}.$$

The upper semi-continuity of $\nu \mapsto h_\nu(F)$ also guarantees that there exists $d \in [a,b]$ with $\sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\} = \sup\{h_\nu(F) : \lambda^c(\nu) = d\}$. Hence:

$$\begin{aligned}\sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\} = \sup\{h_\nu(F) : \lambda^c(\nu) = d\} = h_{L^c_{d,d}}(F) \leq h_{L^c_{a,b}}(F) \\ \leq \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}.\end{aligned}$$

We conclude that $h_{L^c_{a,b}}(F) = \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}$, finishing the first claim of the lemma.

To show the inverse inequality of the second claim, suppose that $a < b$. We can take a measure $\tilde{\nu}_1 \in \mathcal{M}_1(F)$ satisfying

$$h_{\tilde{\nu}_1}(F) = \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\} \quad and \quad \lambda^c(\tilde{\nu}_1) \in [a,b].$$

Fix now a measure $\mu \in \mathcal{M}_1(F)$ such that $\lambda^c(\mu) \in (a,b)$ and $\lambda^c(\mu) \neq \lambda^c(\tilde{\nu}_1)$. Next, we take the convex combination $\mu_t := (1-t)\tilde{\nu}_1 + t\mu$. We can see that if we take t small enough, we can find $\tilde{\nu}_2 \in \mathcal{M}_1(F)$ such that $|h_{\tilde{\nu}_1}(F) - h_{\tilde{\nu}_2}(F)| < \gamma$, $\lambda^c(\tilde{\nu}_2) \in [a,b]$ and $\lambda^c(\tilde{\nu}_2) \neq \lambda^c(\tilde{\nu}_1)$.

Since F has the specification property, then F is *entropy dense* by [EKW94], that is, for any F-invariant probability measure μ, there exists a sequence of F-invariant ergodic probability measures μ_n so that $\mu_n \to \mu$ in the weak* topology and $h_{\mu_n}(f) \to h_\mu(f)$ as $n \to \infty$. Without loss of generality we may suppose that the measures are ergodic (c.f. [EKW94, Theorem B]). Therefore, we can find $F-$invariant and ergodic probabilities ν_1 and ν_2 such that:

i) $\lambda^c(\nu_i) \in [a,b]$, for $i = 1, 2$;

ii) $\lambda^c(\nu_1) \neq \lambda^c(\nu_2)$;

iii) $|h_{\tilde{\nu}_1}(F) - \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}| < 2\gamma$.

Now the proof is inspired by the proof of Theorem 2.6 in [Th09]. Define $\psi := \log\left|\frac{\partial F^n}{\partial y}\right|$. Consider a strictly decreasing sequence $(\delta_k)_{k\geq 1}$ of positive numbers converging to zero, a strictly increasing sequence of positive integers $(\ell_k)_{k\geq 1}$, so that the sets

$$Y_k = \Big\{w \in T^d \times \mathbb{S}^1 \colon |\frac{1}{n} S_n\psi(w) - \int \psi \, d\nu_{\rho(k)}| < \delta_k \text{ for every } n \geq \ell_k\Big\},$$

where $\rho : \mathbb{N} \to \{1,2\}$ is given by $\rho(k) = 1 + kmod(1)$, satisfy $\nu_{\rho(k)}(Y_k) > 1 - \gamma$ for every k (that last assumption is possible due to the Birkhoff ergodic theorem). We will construct a fractal $\tilde{F}$ inspired *ipsis literis* by the construction of Subsection 3.1 in [Th09] with ν_i replacing μ_i, $\sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\} = C$ and $\tilde{F} \subset E_{a,b}$. Our starting point is the following lemma proved by Thompson:

Lemma 2.2.2 ([Th09])**.** *For any sufficiently small $\epsilon > 0$, we can find a sequence $n_k \to \infty$ and a countable collection of finite sets $\mathcal{S}_k$ such that each $\mathcal{S}_k$ is $(n_k, 4\epsilon)$ separated set for Y_k and $M_k := \sum_{x\in\mathcal{S}_k} \exp\{\sum_{j=0}^{n_k-1} \psi(F^j(x))\}$ satisfies $M_k \geq \exp\{n_k(C - 4\gamma)\}$. Furthermore, the sequence n_k can be chosen so that $n_k > l_k$ and $n_k \geq 2^{m_k}$, where $m_k = m(\epsilon/2^k)$ is as in the definition 1.2.2 of the specification property.*

With regard to the construction of the fractal $\mathcal{F}$, we will give a brief discussion on how to construct it, for the details the reader can check out section 3.1 in [Th09]. We first enumerate the sets $\mathcal{S}_k := \{x_i^k ; i = 1, ..., \#\mathcal{S}_k\}$. We choose k and consider the set of words $\underline{i} := (i_1, ..., i_{N_k})$ of length N_k with entries on points of $\mathcal{S}_k$, that is, each word represents a point in $\mathcal{S}_k^{N_k}$.Using the Specification Property 1.2.2, we can construct a point $y(\underline{i})$ which satisfies

$$d_{n_k}(x_{i_j}^k, F^{a_j}(y(\underline{i}))) < \frac{\epsilon}{2^k},\ \forall j = 1, ..., k.$$

where $a_j = (j-1)(n_k + m_k)$. Then, $C_k := \{y(\underline{i}), \underline{i} \in \mathcal{S}_k^{N_k}\}$. Hence, we consider the family $\{C_k\}_{k\in\mathbb{N}}$. By Thompson [Th09], each C_k is a $(c_k, 3\epsilon)-$separated set, where $c_k = N_k n_k + (N_k - 1)m_k$. Once again, we use the specification property to construct a second family of separated sets $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$ inductively in the following way: define $\mathcal{T}_1 := C_1$, and if $x \in \mathcal{T}_k$ and $y \in C_{(k+1)}$, set $t_1 = c_1$ and $t_{k+1} = t_k + m_{k+1} + c_{k+1}$. By the specification property, there exists a point $z = z(x,y)$ satisfying:

$$d_{t_k}(x,z) < \epsilon/2^{k+1} \text{ and } d_{c_{k+1}}(y, F^{t_k+m_{k+1}}) < \epsilon/2^{k+1}.$$

Then, we define $\mathcal{T}_{k+1} := \{z(x,y); x \in \mathcal{T}_k, y \in C_{k+1}\}$. Again, as we can take the time skip in the specification property, the sets $\mathcal{T}_k$ are $(t_k, 2\epsilon)$-separated and $\#\mathcal{T}_k = \prod_{j=1}^{k} \#\mathcal{C}_j$. Finally, to construct the fractal $\mathcal{F}$, define $\mathcal{F}_k := \bigcup_{x\in\mathcal{T}_k} \overline{B}(x, \epsilon/2^{k-1})$. It's not hard to check that $\mathcal{F}_{k+1} \subset \mathcal{F}_k$, Then define $\mathcal{F} := \bigcap_k \mathcal{F}_k$ which is a nested sequence of non-empty compact sets, therefore $\mathcal{F} \neq \emptyset$. The following can be found in [Th10]:

Lemma 2.2.3 ([Th10], Lemma 3.8)**.** *If $p \in \mathcal{F}$, then $\frac{1}{t_k}\sum_{i=0}^{t_k-1} \psi(F^i(p))$ diverges.*

Therefore, there exist sub-sequences $(n_{k_i})_{k\geq 1}, i = 1,2$ so that

$$\lim_{k\to\infty} |\frac{1}{n_{k_i}} S_{n_{k_i}}\psi(w) - \int \psi \, d\nu_i| = 0 \quad \text{for every } w \in \mathcal{F}$$

and again by Thompson [Th09], $h_{\mathcal{F}}(F) \geq \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\} - 8\gamma$. In particular $\mathcal{F}$ is contained in the irregular set $\tilde{L}^c_\psi$. Furthermore, since that $\lambda^c(\nu_i) \in (a,b)$, for $i = 1,2$, we have that $\mathcal{F} \subset E_{a,b}$. Thus

$$\sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\} - 8\gamma \leq h_{\mathcal{F}}(F) \leq h_{E_{a,b}}(F) \leq \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}$$

and so $h_{E_{a,b}}(F) = \sup\{h_\nu(F) : \lambda^c(\nu) \in [a,b]\}$. □

Our next step is to use the previous results to give a precise description of the topological entropy of the sets $L^c_{a,b}$ in terms of the deviations rate function $\mathcal{I}$.

Proposition 2.2.4. *Let $F \in \mathcal{D}^r$ be given, with $r > 1$. Let $[a,b] \subset L^c(F)$, $\lambda^c_{\min}$ and $\lambda^c_{\max}$ as in Definition 2.0.3. We have:*

(i) If $[a,b] \subset [0, \lambda^c_{\min}]$, then: $h_{L^c_{a,b}}(F) = bt_0 + h_{top}(g)$;

(ii) If $[a,b] \subset [\lambda^c_{\min}, \lambda^c_{\max}]$, then: $h_{L^c_{a,b}}(F) = h_{top}(F) - \inf_{s\in[a,b]} \mathcal{I}(s) = h_{top}(F) - \mathcal{I}(d)$. where $d = \lambda^c(\mu_0)$ when $\lambda^c(\mu_0) \in [a,b]$; $d = b$ when $b \leq \lambda^c(\mu_0)$ and $d = a$ when $\lambda^c(\mu_0) \leq a$.

Proof. (i) Take $d \in [0, \lambda^c_{\min}]$ and let ν an $F-$invariant probability such that $\lambda^c(\nu) = d$. Then $h_\nu(F) - t_0\lambda^c(\nu) \leq P(t_0) = h_{top}(g)$ implies $h_\nu(F) \leq h_{top}(g) + dt_0$. On other hand, let η the probability that appears in item (3) of the definition of $F \in \mathcal{D}^r$ 0.2.4. Then $\lambda^c(Leb \times \eta) = 0$ and $h_{Leb\times\eta}(F) = h_{top}(g)$. Define $\mu_d := \frac{d}{\lambda^c_{\min}}\mu_{t_0} + \left(1 - \frac{d}{\lambda^c_{\min}}\right) Leb \times \eta$. Then, $\lambda^c(\mu_d) = d$ and using the equalities $h_{\mu_{t_0}}(F) + t_0\lambda_{\mu^c_{t_0}} = h_{top}(g)$ and $\lambda_{\mu^c_{t_0}} = \lambda^c_{\min}$ we have:

$$h_{\mu_d}(F) = \frac{d}{\lambda^c_{\min}}\left(h_{top}(g) + t_0\lambda_{\mu_{t_0}}\right) + \left(1 - \frac{d}{\lambda^c_{\min}}\right) h_{top}(g) = h_{top}(g) + dt_0.$$

Therefore, applying the Lemma 2.2.1, we conclude that $h_{L^c_{a,b}}(F) = bt_0 + h_{top}(g)$ for all $[a,b] \subset [0, \lambda^c_{\min}]$.

(ii) Suppose that $[a,b] \subset [\lambda^c_{\min}, \lambda^c_{\max}]$. Applying Lemma 2.2.1 and Remark 2.1.9 we conclude that

$$h_{L^c_{a,b}}(F) = h_{top}(F) - \inf_{s\in[a,b]} \mathcal{I}(s)$$

Thus, follows from Lemma 2.1.6, by the properties of the function $\mathcal{I}$

$$\inf_{s\in[a,b]} \mathcal{I}(s) = \mathcal{I}(d),$$

where $d = \lambda^c(\mu_0)$ when $\lambda^c(\mu_0) \in [a,b]$; $d = b$ when $b \leq \lambda^c(\mu_0)$; and $d = a$ when $\lambda^c(\mu_0) \leq a$. □

It follows from the previous result that $h_{L^c_{a,\lambda^c_{\min}}}(F) = \lambda^c_{\min}t_0 + h_{top}(g)$. On the other hand, $\lim_{s\mapsto\lambda^c_{\min}} \mathcal{I}(s) = \lim_{t\mapsto t_0}(t\mathcal{E}'(t) - \mathcal{E}(t)) = -\lambda^c_{\min}t_0 - h_{top}(g) + h_{top}(F)$. Thus, if $[a,b] \cap [\lambda^c_{\min}, \lambda^c_{\max}]$, then $h_{L^c_{a,b}}(F) = h_{L^c_{\lambda^c_{\min},b}}(F)$. Therefore, using the previous proposition and the properties of the function $\mathcal{I}$, we complete the proof of the Theorem B.

The graphics below illustrate the behaviour of the function $\mathcal{I}$:

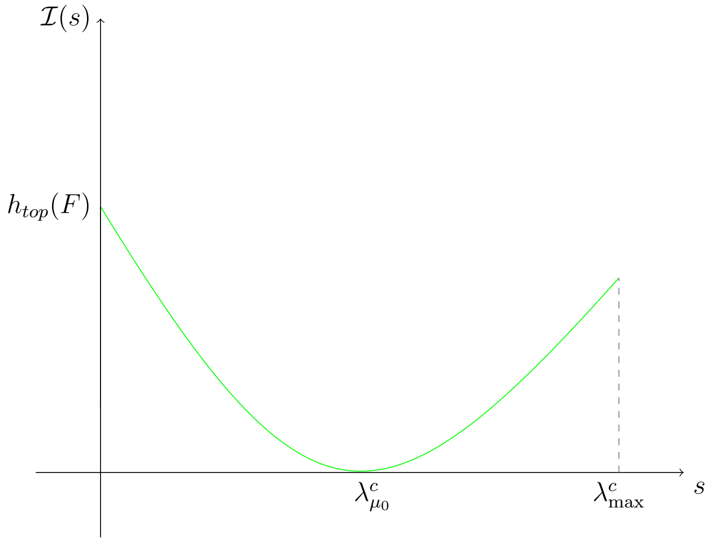

Figure 2.1: Graph of $s \mapsto \mathcal{I}(s)$ for the case $\lambda^c_{min} = 0$

and

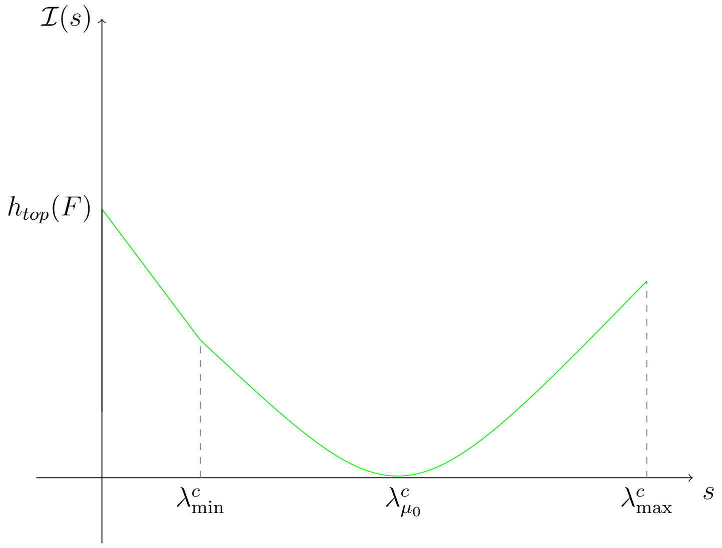

Figure 2.2: Graph of $s \mapsto \mathcal{I}(s)$ for the case $\lambda^c_{min} > 0$.

Chapter 3

Phase transitions for transitive piecewise monotonous maps of the circle

We start this chapter by recalling the class of dynamics we are considering from now on. It is a class of continuous transitive piecewise monotonous dynamics on the circle $\mathbb{S}^1$ as stated in definition 0.2.7, more precisely:

Definition 3.0.1. *$f:\mathbb{S}^1\to\mathbb{S}^1$ is a continuous transitive local-diffeomorphism with break points, in other words, f is a continuous, transitive and there exist closed arcs $I_1,...,I_k\subset\mathbb{S}^1$ such that:*

1. *$\mathbb{S}^1=\bigcup_{i=1}^{k} I_i$ and the arcs I_i have disjoint interiors;*
2. *$f|_{I_i}:I_i\mapsto\mathbb{S}^1$ is a C^1 diffeomorphism;*
3. *the derivative of f is well defined at its fixed points.*

Remark 3.0.2. *We also recall that we denote E as the class of potentials $C^{\alpha}(\mathbb{S}^1,\mathbb{R})$ of Hölder continuous potentials or $C^r(\mathbb{S}^1,\mathbb{R})$ of r-times continuous differentiable potentials, and in the last case we suppose f a C^r-local diffeomorphism.*

As a simple example, we mention the intermittent maps or Maneville-Pomeau-like maps [MP80]:

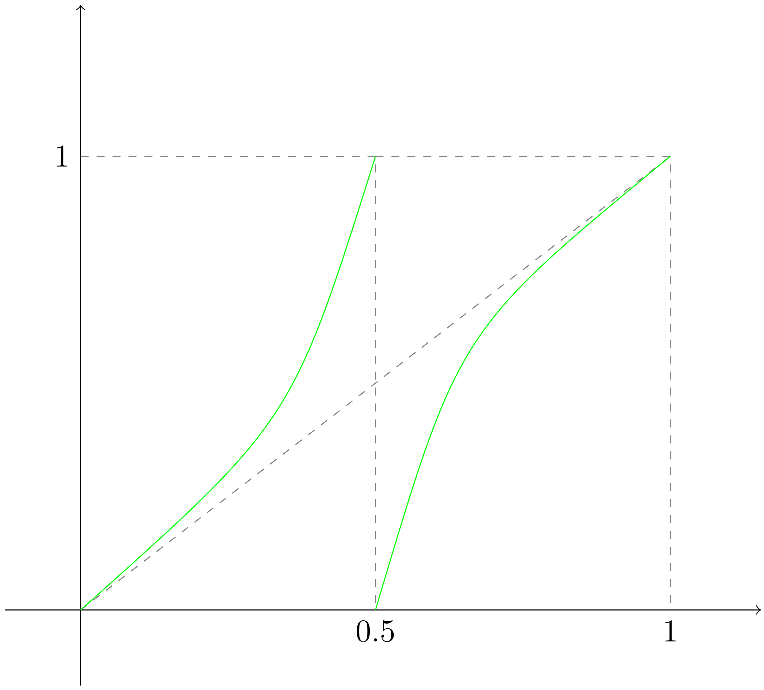

Figure 3.1: Intermittent maps are a very well-known example of transitive piecewise monotone dynamics.

A very important fact about the class of maps as in definition 0.2.7 is that they are topologically conjugated to an expanding map [CM86]:

Theorem 3.0.3 (Coven-Mulvey, 86). *Every transitive, piecewise monotone map f of the interval is topologically conjugate to a piecewise linear map whose linear pieces have slopes $\pm\beta$ (where $\log\beta$ is the topological entropy of f).*

Follows of the previous theorem that if f is a map like in definition 0.2.7 then f is expansive, has the periodic specification property and admits generating partition by domains of injectivity. In particular, we can apply Rokhlin's formula. Moreover, we have that every set of pre-orbits $\bigcup_{n\geq 0} f^{-n}\{x\}$ is uniformly dense on $\mathbb{S}^1$ for every $x \in \mathbb{S}^1$.

As a first results, we show that the Lyapunov exponent varies continuously in the space of invariant probability measures. As we are reduced to the one-dimensional case, we denote the **Lyapunov exponent** of a invariant probability μ as $\lambda(\mu)$.

Lemma 3.0.4. *The function $\mathcal{M}_1(f) \ni \mu \mapsto \int \log|Df|d\mu$ is continuous.*

Proof. By assumption, f does not have critical points and Df has at most a finite number of discontinuity points, let's say $D := \{x_1, ..., x_k\}$, we must show that for any invariant probability $\mu \in \mathcal{M}_1(f)$ we have $\mu(\{x_i\}) = 0$ for each discontinuity point x_i. Suppose $\mu(x_i) > 0$. We claim that for any $m > n \in \mathbb{N}$, $f^{-n}(x_i) \cap f^{-m}(x_i) = \emptyset$. Otherwise, let $x \in f^{-n}(x_i) \cap f^{-m}(x_i)$ be fixed, then we would have $f^{m-n}(x_i) = x_i$ which means that x_i

would be a periodic point. This contradicts the fact that x_i cannot be a periodic point, by definition of f. From that, we conclude by the invariance of μ that

$$\mu(\bigcup_{n\in\mathbb{N}} f^{-n}(x_i)) = \sum_{i=0}^{\infty}\mu(f^{-i}(x_i)) = \sum_{i=0}^{\infty}\mu(x_i) = \infty$$

which is a contradiction. So, $\mu(x_i) = 0$ for every invariant probability measure μ. Given $\delta > 0$, take the closed subspace $A_\delta = [0,1] - \bigcup_{x_i} B(x_i, \delta)$. By **Tietze's extension theorem** let $\phi_\delta : [0,1] \to \mathbb{R}$ be a continuous extension of $\log|Df||_{A_\delta}$. Take any sequence $\mathcal{M}_1(f) \ni \mu_n \xrightarrow{weak^*} \mu$, then by construction we have:

$$\int \phi_\delta d\mu_n \to \int \phi_\delta d\mu.$$

Now, for $\epsilon > 0$ let $n_0 \in \mathbb{N}$ and $\delta > 0$ be sufficiently small such that:

$$|\int \phi_\delta d\mu_n - \int \phi_\delta d\mu| < \epsilon/3,\ \forall n \geq n_0$$

and

$$\sup_{x\in[0,1]\backslash D} |\phi_\delta(x) - \log|Df(x)|| < \epsilon/3$$

which implies that (taking the integrals in $[0,1]\backslash D$)

$$\Big|\int \phi_\delta d\mu - \int \log|Df|d\mu\Big| < \epsilon/3 \text{ and } \Big|\int \phi_\delta d\mu_n - \int \log|Df|d\mu_n\Big| < \epsilon/3.$$

Thus,

$$\Big|\int \log|Df|d\mu_n - \int \log|Df|d\mu\Big| \leq \Big|\int \log|Df|d\mu_n - \int \phi_\delta d\mu_n\Big| +$$

$$\Big|\int \phi_\delta d\mu_n - \int \phi_\delta d\mu\Big| + \Big|\int \phi_\delta d\mu - \int \log|Df|d\mu\Big| < \epsilon$$

As ϵ is arbitrary, we have the result. □

As a consequence of the previous lemma, we show that f does not admit negative Lyapunov exponent.

Definition 3.0.5. *Let $p \in \mathbb{S}^1$ be a periodic point for f with period $n \in \mathbb{N}$. We define a empirical measure $\mu(p,n)$ associated to p and n as*

$$\mu(p,n) := \frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i(p)}.$$

Lemma 3.0.6. *Let $\mu \in \mathcal{M}_1(f)$. Then $\lambda(\mu) \geq 0$.*

Proof. By Sigmund [S74], as f satisfies the Periodic Specification Property 1.2.2, the set of empirical measure is dense in the weak* topology. Arguing by contradiction, suppose $\nu \in \mathcal{M}_1(f)$ is an invariant probability with $\lambda(\nu) < 0$. By the previous lemma, there exist an open set $\nu \ni \mathcal{U} \subset \mathcal{M}_1(f)$ such that for any $\eta \in \mathcal{U}$ we have $\lambda(\eta) < 0$. In particular, there exist a periodic point p and a natural number n such that $\mu(p, n) \in \mathcal{U}$, which means that p is a periodic attracting point. Since f is transitive, we have a contradiction. □

We saw in the previous chapter that knowing that the spectral gap property of the transfer operator holds with respect to a dynamic f and a potential ϕ is a very important tool and has many applications in the world of dynamical systems. For the dynamics that we study here and a given regular potential, we want to understand the obstacles for the spectral gap to occur. The spectral gap property implies **exponential decay of correlations**. Therefore, if $\phi \in L_1$ and $\psi \in L^\infty$ and $Cov(\phi, \psi \circ f^n) \to 0$ sub-exponentially, then there is no Banach space which contains ϕ such that $\mathcal{L}_{f,\phi}$ has the spectral gap property. Others obstructions are a breakdown of the central limit theorem, non-integrable invariant density and spectral phase transition that we studied in the previous chapters.

3.1 Preliminaries

Let $T : M \to M$ be a continuous transformation of a compact metric space. For each continuous function $\phi : M \to \mathbb{R}$ we define the maximum ergodic average

$$\beta(\phi) := \sup_{\mu \in \mathcal{M}_1(T)} \int \phi d\mu = \sup_{x \in M} \limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k(x))$$

and the set of all maximizing measures of ϕ:

$$\mathcal{M}_{\max}(\phi) := \left\{ \mu \in \mathcal{M}_1(T); \int \phi d\mu = \beta(\phi) \right\}$$

Theorem 3.1.1. *(Morris) [M10] Suppose that $\mathcal{U}$ is a dense open subset of $\overline{\mathcal{M}_{erg}(T)}$. Then the set*

$$U := \left\{ \phi \in C(M); \overline{\mathcal{M}_{erg}(T)} \cap \mathcal{M}_{\max}(\phi) \subset \mathcal{U} \right\}$$

is open and dense in $C(M)$. Conversely, if $U \subset C(X)$ is open and dense, then the set

$$\mathcal{U} := \left\{ \mathcal{M}_{erg}(T) \cap \bigcup_{\phi \in U} \mathcal{M}_{\max}(\phi) \right\}$$

is open and dense in $\mathcal{M}_{erg}(T)$

As we alrady saw, in [BC21], for local diffeomorphism on the circle, spectral gap is equivalent to quasi-compactness as we saw in Lemma 1.3.6, and using estimates of the essential radius it can be shown that ,in our context, the essential radius is strictly less than the spectral radius. In the context of local diffeomorphism on the circle, the *Campbell-Latushkin's* Theorem 1.3.10, reduces to:

Theorem 3.1.2. (Campbell, Latushkin) *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a C^r-local diffeomorphism and let $\phi \in C^r(\mathbb{S}^1, \mathbb{R})$ be given. Then*

$$\rho_{ess}(\mathcal{L}_{f,\phi}|_{C^k}) \leq \exp\left[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi d\mu - k\lambda(\mu)\}\right] \text{ and}$$

$$\rho(\mathcal{L}_{f,\phi}|_{C^k}) \leq \exp\left[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi d\mu\}\right],$$

for $k = 0, 1, \ldots, r$ and $\lambda(\mu)$ is the Lyapunov exponent of the measure μ.

For the next result, we need to point out some notations and results from [BJL96]. Let I be a compact interval. For $0 < \alpha < 1$, let Λ^α the space of α-Hölder functions, i.e. functions $\varphi : I \to \mathbb{C}$ satisfying

$$|\varphi|_\alpha = \sup_{x\neq y\in I} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\alpha} < \infty.$$

Then, Λ^α is a Banach space for the norm $\|\varphi\|_\alpha = \max(\sup_I |\varphi|, |\varphi|_\alpha)$. Take $\mathcal{I}$ as a finite or countable set and $0 \leq \delta < 1$. In [BJL96], a dynamical system is a family of $C^{1+\delta}$ diffeomorphisms, $f_i : I \to J_i$, for $i \in \mathcal{I}$, where the intervals $J_i \subset I$ have disjoint interiors. They assume further that $\sup_i \|f_i'\|_\delta < \infty$, in particular $\lambda := 1/\sup_{i,x} |f_i'(x)| > 0$. Another concept introduced was the **weight**. The weight is a family of functions $g_i : I \to \mathbb{C}, i \in \mathcal{I}$. Such a family g_i is called summably bounded if $\sup^\Sigma |g| = \sum_i \sup |g_i| < \infty$. A summably bounded family is called summably Λ^α if $|g|_\alpha^\Sigma = \sum_i |g_i|_\alpha < \infty$ for some $0 < \alpha \leq 1$; Then, the transfer operator $\mathcal{L}$ acting on functions $\varphi : I \to \mathbb{C}$, is defined by

$$\mathcal{L}\varphi(x) = \sum_{i\in\mathcal{I}} g_i(x)\varphi(f_i(x))$$

For each $n \geq 1$ and $i_l \in \mathcal{I}$, $1 \leq l \leq n$, introduce the maps $f_{\vec{i}}^{(n)} = f_{i_n} \circ \cdots \circ f_{i_1}$ and the weights $g_{\vec{i}}^{(n)}(x) = g_{i_n}(f_{i_{n-1}} \cdots f_{i_1}) \cdots g_{i_2}(f_{i_1}(x)) \cdot g_{i_1}(x)$. Note that for all $n \geq 1$

$$\mathcal{L}^n\varphi(x) = \sum_{\vec{i}\in\mathcal{I}^n} g_{\vec{i}}^{(n)}(x)\varphi\left(f_{\vec{i}}^{(n)}(x)\right) \tag{3.1}$$

With the discussion above, the following results was proved:

Theorem 3.1.3. (Baladi, Jiang, Lanford) *If the family g_i is summably Λ^α for some $0 < \alpha \leq 1$, the essential spectral radius $\rho_{ess}(\mathcal{L})$ of the operator $\mathcal{L}$ acting on Λ^α is equal to*

$$\rho_{\text{ess}}(\mathcal{L}) = \lim_{n\to\infty} \left[\sup_{x\in I} \sum_{\vec{i}\in\mathcal{I}^n} \left| g_{\vec{i}}^{(n)}(x) \right| \left| f_{\vec{i}}^{(n)\prime}(x) \right|^\alpha \right]^{1/n}.$$

Now we have to estimate the spectral radius. We denote the Banach space B of bounded functions on I endowed with the supremum norm. The next theorem was credited by the authors to Ruelle.

Theorem 3.1.4. (Baladi, Jiang, Lanford) *If the family g_i is summably bounded, then the spectral radius of $\mathcal{L}_{|g|}$ acting on B is equal to*

$$e^P := \lim_{n\to\infty} \left(\sup_{x\in I} \sum_{\vec{i}\in\mathcal{I}^n} \left| g_{\vec{i}}^{(n)}(x) \right| \right)^{1/n}$$

and the spectral radius of $\mathcal{L}_g$ on B is bounded above by e^P.
If the g_i are summably Λ^α and $\delta \geq 0$, the spectral radius of $\mathcal{L}_{|g|}$ acting on Λ^α is equal to $\max\left(e^P, \rho_{ess}\left(\mathcal{L}_{|g|}\right)\right)$, and the spectral radius of $\mathcal{L}_g$ acting on Λ^α, is bounded above by $\max\left(e^P, \rho_{\text{ess}}\left(\mathcal{L}_{|g|}\right)\right)$.

We now interpret in our setting the estimates in Theorems 3.1.3 and 3.1.4. In our context, $f : \mathbb{S}^1 \to \mathbb{S}^1$ is full branch transitive piecewise monotone map and $\phi \in C^\alpha(\mathbb{S}^1, \mathbb{R})$ is a Hölder continuous potential. Since, by definition

$$f|_{I_i} : I_i \to [0,1]$$

where $I_i = [a_i, a_{i+1}]$ is diffeomorphism, then we set

$$f_i := (f|_{I_i})^{-1} : [0,1] \to I_i,$$

the finite inverse branches and $g = |g| = e^\phi$, which has the summability hypothesis. Thus rewriting as (3.1), we get

$$\mathcal{L}_{f,\phi}^n \varphi(x) = \sum_{y\in f^{-n}(x)} e^{S_n\phi(y)}\varphi(y).$$

Furthermore, from the estimates for the essential and spectral radius, we have:

$$\rho_{ess}(\mathcal{L}_{f,\phi}|_{C^\alpha}) = \lim_{n\to\infty} \|\mathcal{L}_{f,\phi-\alpha\log|Df|}^n 1\|_\infty^{1/n} \text{ and}$$
$$\rho(\mathcal{L}_{f,\phi}|_{C^\alpha}) = \max\{\lim_{n\to\infty} \|\mathcal{L}_{f,\phi}^n 1\|_\infty^{1/n}, \rho_{ess}(\mathcal{L}_{f,\phi}|_{C^\alpha})\},$$

and, by the spectral radius formula, $\lim_{n\to\infty} \|\mathcal{L}^n_{f,\phi}1\|_\infty^{1/n} = \rho(\mathcal{L}_{f,\phi}|_{C^0})$. So the estimate on the essential radius is:

$$\rho_{ess}(\mathcal{L}_{f,t\phi}|_{C^\alpha}) = \rho(\mathcal{L}_{f,t\phi-\alpha\log|Df|}|_{C^0}) \leq \exp\left[\sup_{\mu\in\mathcal{M}_1(f)}\left\{h_\mu(f) + t\int\phi\mathrm{d}\mu - \alpha\lambda(\mu)\right\}\right]. \quad (3.2)$$

To estimate the spectral radius, similarly:

$$\rho(\mathcal{L}_{t\phi}|_{C^\alpha}) = \max\{\rho(\mathcal{L}_{t\phi}|_{C^0}), \rho_{ess}(\mathcal{L}_{t\phi}|_{C^\alpha})\} = \exp\left[\sup_{\mu\in\mathcal{M}_1(f)}\left\{h_\mu(f) + \int\phi\mathrm{d}\mu\right\}\right] \quad (3.3)$$

By the fact that there exists a conformal measure ν_ϕ such that $\rho(\mathcal{L}_{f,\phi}|C^0) \leq \rho(\mathcal{L}_{f,\phi}|C^0)$, we have $\rho(\mathcal{L}_{f,\phi})|_{C^r} = \rho(\mathcal{L}_{f,\phi}|_{C^0})$ and

$$\rho(\mathcal{L}_{f,\phi}|_{C^r}) = \lim_{n\to\infty}\left\|\mathcal{L}^n_{f,\phi}1\right\|_\infty^{1/n} = e^{P_{top}(f,\phi)}. \quad (3.4)$$

Our main objective in this section is to study the behaviour of the topological pressure function $t \mapsto P_{top}(f,\phi)$ for Hölder continuous potentials, equivalent properties and its relation with spectral concepts.

Remark 3.1.5. *A simple case to consider is when ψ is cohomologous to a constant k, in this case $P_{top}(t\varphi) := kt + h_{top}(f)$. By Thompson [Th09], potentials cohomologous to a constant form a first category set. So the topological pressure function does not have thermodynamic phase transition. Hence, when the potential is cohomologous to a constant, it is easy to understand the behaviour of the topological pressure function.*

Before proving the theorems, we need some additional concepts. The first one is notion of **hyperbolic potential**, following [ReR12], which basically says that no equilibrium state has no chaos. The second one is the notion of **expanding potentials** from the work of Pinheiro and Varandas [PV22].

3.1.1 Hyperbolic Potentials

Definition 3.1.6 (Hyperbolic Potential). *A potential $\phi : \mathbb{S}^1 \to \mathbb{R}$ is said to be **hyperbolic** if*

$$\sup_{\mu\in\mathcal{M}_1(f)}\int\phi d\mu < P_{top}(f,\phi).$$

Proposition 3.1.7. *If $h_{top}(f) > 0$ and ϕ is not hyperbolic, then:*

1. *ϕ admits maximizing measure ν with zero metric entropy;*

2. *there exists $t_0 \in [0,1]$ such that $P_{top}(f, t\phi) = t\int \phi d\nu$, where ν is an equilibrium state with zero metric entropy, for all $t \geq t_0$. In particular, a thermodynamical phase transition accour at t_0.*

Proof. To prove item 1, by the fact that ϕ is not hyperbolic, let ν an invariant probability measure satisfying

$$\int \phi d\nu = P_{top}(f, \phi) \geq h_\mu(f) + \int \phi d\mu \geq \int \phi d\mu$$

for all $\mu \in \mathcal{M}_1(f)$. In particular, $h_\nu(f) = 0$ and ν is a maximizing measure, $\nu \in \mathcal{M}_{\max}(\phi)$, in the sense of Theorem 3.1.1. Furthermore, for all $t \geq 1$ and any $\mu \in \mathcal{M}_1(f)$

$$\begin{aligned} h_\mu(f) + t\int \phi d\mu &= h_\mu(f) + \int \phi d\mu + (t-1)\int \phi d\mu \\ &\leq P_{top}(f, \phi) + (t-1)\int \phi d\nu \\ &= \int \phi d\nu + (t-1)\int \phi d\nu = t\int \phi d\nu \end{aligned}$$

showing that $P_{top}(f, t\phi) = t\int \phi d\nu$, for all $t \geq 1$. We define now

$$t_0 := \inf\{t \in (0,1]; t\phi \text{ is not hyperbolic}\}.$$

Then, $P_{top}(f, t\phi) = t\int \phi d\nu$ for all $t \geq t_0$, where ν is the maximizing measure for ϕ. By definition, t_0 is a thermodynamical phase transition for ϕ. Indeed, if $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi)$ was analytic, the we would have $P_{top}(f, t\phi) = t\int \phi d\nu$, in particular

$$h_{top}(f) = P_{top}(f, 0\phi) = 0$$

which contradict the fact that $h_{top}(f) > 0$. □

3.1.2 Expanding Potentials

Following Pinheiro and Varandas, [PV22], we define another class of potentials satisfying a stronger property than being hyperbolic. We prove that in our context it will be equivalent to hyperbolicity.

Definition 3.1.8 (**Expanding Potential**). *We say that a continuous potential* $\phi : \mathbb{S}^1 \to \mathbb{R}$ *is an* ***expanding potential*** *if*

$$\sup_{\lambda(\mu)=0} \{h_\mu(f) + \int \phi d\mu\} < \sup_{\lambda(\mu)>0} \{h_\mu(f) + \int \phi d\mu\}$$

This is equivalent to say that there are no equilibrium states with respect to ϕ *with zero Lyapunov exponent.*

Remark 3.1.9. *If a potential* ϕ *is hyperbolic, then it is expanding due to the Margulis-Ruelle inequality 1.2.3.*

We note that, in our context, f is strongly transitive [CM86]. Thus, if the transfer operator is quasi-compact, then it has the spectral gap property. The proof is analogous to Lemma 1.3.6, which original proof can be found in [BC21]:

Lemma 3.1.10. *Let* ϕ *be a continuous potential. If* $\mathcal{L}_{f,\phi}|_E$ *is quasi-compact, then it has the spectral gap property.*

We have all the ingredients to prove a result that connects expanding potential with spectral gap property. Later on this fact will be very important in characterizing the topological pressure function.

Proposition 3.1.11. *If a Hölder continuous or* C^r *potential* $\phi : \mathbb{S}^1 \to \mathbb{R}$ *is expanding, then the transfer operator* $\mathcal{L}_{f,\phi}|_E$ *associated to* f *and* ϕ *has spectral gap property.*

Proof. We estimate the essential and spectral radius and show that if the transfer operator $\mathcal{L}_{f,\phi}|_E$ does not have the spectral gap property, then ϕ is not expanding. Suppose that $\mathcal{L}_{f,\phi}|_E$ does not have the spectral gap property. This means that the spectral radius and the essential radius are equal as, in our context, spectral gap is equivalent to quasi-compactness by Lemma 3.1.10. Furthermore, by the estimates of the essential radius and spectral radius (holds for C^r potential by Campbell Laturskin's Theorem 3.1.2 and estimate (3.4), and holds also for Hölder continuous potentials by estimates (3.2), (3.3) and (3.4), we have that there exists $k > 0$, such that:

$$\rho_{ess}(\mathcal{L}_{f,\phi}|_E) \leq \exp\left[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi \mathrm{d}\mu - k\lambda(\mu)\}\right]$$

and

$$\rho(\mathcal{L}_{f,\phi}|_E) = \exp\left[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi \mathrm{d}\mu\}\right].$$

As $\mathcal{L}_{f,\phi}|_E$ does not have spectral gap property, we get

$$\exp\left[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi \mathrm{d}\mu - k\lambda(\mu)\}\right] = \exp\left[\sup_{\mu\in\mathcal{M}_1(f)} \{h_\mu(f) + \int \phi \mathrm{d}\mu\}\right]$$

which means that

$$\sup_{\mu\in\mathcal{M}_1(f)}\{h_\mu(f)+\int\phi \mathrm{d}\mu-k\lambda(\mu)\}=P_{top}(f,\phi)$$

Since f is expansive, the entropy is upper semi-continuous and the Lyapunov exponent is upper semi-continuous as well. Hence, there exist a measure $\nu\in\mathcal{M}_1(f)$ that realizes the supremum, so the last expression is equivalent to say that:

$$h_\nu(f)+\int\phi \mathrm{d}\nu-k\lambda(\nu)=P_{top}(f,\phi)\geq h_\mu(f)+\int\phi \mathrm{d}\mu,\ \forall\mu\in\mathcal{M}_1(f).$$

In particular this estimate is true when $\mu=\nu$ and that happens if, and only if, $k\lambda(\nu)\leq 0$. As f is transitive, $\lambda(\nu)<0$ can not occur by Lemma 3.0.6. Therefore, since $k>0$, we get $\lambda(\nu)=0$. Our conclusion is that, if the transfer operator does not have the spectral gap property, then there exists an equilibrium state ν with respect to f and ϕ with zero Lyapunov exponent. Furthermore, by the Margullis-Ruelle inequality, Theorem 1.2.3,

$$h_\nu(f)\leq\max\{0,\lambda(\nu)\}=0.$$

From this we have:

$$\int\phi \mathrm{d}\nu\geq h_\mu(f)+\int\phi \mathrm{d}\mu\geq\int\phi \mathrm{d}\mu,\forall\mu\in\mathcal{M}_1(\mathbb{S}^1).$$

So $\nu\in\mathcal{M}_{\max}(\phi)$, where $\mathcal{M}_{\max}(\phi)$ is given in the statement of Theorem 3.1.1. Our conclusion is that

$$\sup_{\mu;\lambda_(\mu)=0}\int\phi d\mu\geq\sup_{\nu;\lambda(\nu)>0}\int\phi d\nu,$$

implying that ϕ is not expanding. $\square$

Next, we will show that, among others properties, having spectral gap property implies the uniqueness of equilibrium states. We will use the following results whose proof follows closely the proof of similar result in [BC21].

Lemma 3.1.12. *Let $\phi\in E$ be given. If $\mathcal{L}_{f,\phi}|_E$ has the spectral gap property, then there exists a unique probability ν_ϕ and $h_\phi\in C^\alpha(\mathbb{S}^1)$ such that $(\mathcal{L}_{f,\phi}|_{C^\alpha})^*\nu_\phi=\rho(\mathcal{L}_{f,\phi}|_{C^\alpha})\nu_\phi$, $\mathcal{L}_{f,\phi}h_\phi=\rho(\mathcal{L}_{f,\phi}|_{C^\alpha})h_\phi$, $h_\phi>0$ and $\int h_\phi d\nu_\phi=1$. Furthermore, supp$(\nu_\phi)=\mathbb{S}^1$.*

Remark 3.1.13. *We denote the f-invariant probability $h_\phi\nu_\phi$ by μ_ϕ. We still have supp$(\mu_\phi)=\mathbb{S}^1$.*

Proposition 3.1.14. *Let $\phi : \mathbb{S}^1 \to \mathbb{R}$ be a continuous potential such that the transfer operator $\mathcal{L}_{f,\phi}|_E$ has the spectral gap property. Then*

1. *$P_{top}(f, \phi) = \log(\rho(\mathcal{L}_{f,\phi}))$ and μ_ϕ is the unique equilibrium state associated to ϕ.*

2. *The topological pressure function $t \mapsto P_{top}(f, t\phi)$ is analytic in a neighbourhood of 1;*

Suppose additionally that ϕ is not cohomologous to a constant. Then, $t \mapsto P_{top}(f, t\phi)$ is strictly convex in a neighbourhood of 1.

Proof. As, by Remark 3.0.2 f has dense pre-images and admit partition by domains of invertibility, the proof of item 1 is similar to Theorem 1.2.12 and the fact that spectral gap property is open by Corollary 1.2.6. To show the uniqueness of the equilibrium state, fix a Hölder potential ψ. As $\mathcal{L}_{f,\phi}$ has the spectral gap property, by Corollary 1.2.6, for sufficiently small values of t close to zero, we still have that $\mathcal{L}_{f,\phi+t\psi}$ has the spectral gap property. Suppose that μ is an equilibrium state with respect to ϕ. Then

$$\begin{aligned} h_\mu(f) + \int (\phi + t\psi) d\mu &\le h_{\mu_{\phi+t\psi}} + \int \phi d\mu_{\phi+t\psi} + t \int \psi d\mu_{\phi+t\psi} \\ &\le h_\mu(f) + \int \phi d\mu + t \int \psi d\mu_{\phi+t\psi} \\ \text{implying that} \quad t \int \phi d\mu &\le t \int \phi d\mu_{\phi+t\psi}. \end{aligned}$$

Taking subsequences $t_k \searrow 0, s_k \nearrow 0$ such that $\nu_1 := \lim\limits_{t_k \searrow 0} \mu_{\phi+t_k\psi}$ and $\nu_2 := \lim\limits_{s_k \nearrow 0} \mu_{\phi+t_k\psi}$ and, by the previous inequality, we have that

$$\int \psi d\nu_2 \le \int \psi d\mu \le \int \psi d\nu_1.$$

As f is expansive, the function $\mu \mapsto h_\mu(f)$ is upper semi-continuous, which implies that ν_1, ν_2 are equilibrium states for f with respect to ϕ. In particular, as ψ is arbitrary and $\int \psi d\mu_\phi = \int \psi d\mu$, which implies $\mu = \mu_\phi$. The additional part follows like in Proposition 1.3.5. □

Corollary 3.1.14.1. *Let $\phi : \mathbb{S}^1 \to \mathbb{R}$ be a Hölder continuous potential. Then ϕ is hyperbolic if, and only if ϕ is expanding.*

Proof. If ϕ is hyperbolic, then it is expanding by definition. Suppose that ϕ is expanding. Then there are two possibilities: either there exists $u \in C^0$ and a constant k such that $\phi = k + u \circ f - u$, i which case

$$P_{top}(f, \phi) = k + h_{top}(f) = \int \phi d\mu + h_{top}(f) \; \forall \; \mu \in \mathcal{M}_1(f).$$

Since $h_{top}(f) > 0$, ϕ is hyperbolic. Otherwise, by Proposition 3.1.11, the transfer operator has the spectral gap and, by Proposition 3.1.14, the topological pressure function is strictly convex in a neighbourhood of 1(supposing ϕ is not cohomologous to a constant). If ϕ was not hyperbolic, by Proposition 3.1.7, there exists $t_0 \in (0, 1]$ such that the topological pressure function is affine for all $t \geq t_0$, and it could not be strictly convex in any neighbourhood of 1. Therefore, ϕ is hyperbolic. □

3.2 Proof of Theorems C, D, E and Corollaries C, E

We now define the following set:

$$\mathcal{R} := \Big\{\phi : \mathbb{S}^1 \to \mathbb{R} \in C^0; \sup_{\mu;\lambda(\mu)=0} \int \phi d\mu < \sup_{\nu;\lambda(\nu)>0} \int \phi d\nu \Big\}.$$

Remark 3.2.1. *$\mathcal{R}$ is a set of continuous and expanding potentials $\phi : \mathbb{S}^1 \to \mathbb{R}$ such that the transfer operator associated to f and ϕ has the spectral gap property, by Proposition 3.1.11. Furthermore, if $\phi \in \mathcal{R}$, then $t\phi \in \mathcal{R}$ for all $t > 0$, which means that $t\phi$ has spectral gap for all $t > 0$.*

Lemma 3.2.2. *$\mathcal{R}$ is open and dense in $C^0(\mathbb{S}^1)$.*

Proof. Firstly, we define $\mathcal{U} := \Big\{\mu \in \mathcal{M}_{erg}(f); \lambda_\mu(f) > 0\Big\}$. We already know that $\overline{\mathcal{M}_{erg}(f)} = \mathcal{M}_1(f)$ by [S74], since f satisfies the periodic specification property. We first show that $\mathcal{U}$ is an open and dense subset in the closure of family of ergodic probability measures and then, using Morris's Theorem 3.1.1, we are going to conclude that $\mathcal{R}$ is open and dense in the space of all continuous potentials.

Openness: It is a consequence of $\mu \mapsto \lambda(\mu)$ being a continuous function. So if $\lambda(\nu) > 0$ for some measure ν, then there exist a neighbourhood V of ν such that for each $\mu \in V$, we have $\lambda(\mu) > 0$.

Density: Take any measure $\nu \in \mathcal{M}_1(f)$ with $\lambda(\nu) > 0$. Then, for each $\mu \in \mathcal{M}_1(f)$ with $\lambda(\mu) = 0$, define $\nu_n(f) := \frac{1}{n}\nu + \Big(1 - \frac{1}{n}\Big)\mu$. For any continuous potential, we have $\varphi : \mathbb{S}^1 \to \mathbb{R}$:

$$\int \varphi d\nu_n = \frac{1}{n}\int \varphi d\nu + \Big(1 - \frac{1}{n}\Big)\int \varphi d\mu \xrightarrow{n\to\infty} \int \varphi d\mu$$

Hence, $\nu_n \xrightarrow[n\to\infty]{weak^*} \mu$. Furthermore,

$$\lambda(\nu_n) = \frac{1}{n}\lambda(\nu) + \Big(1 - \frac{1}{n}\Big)\lambda(\mu) = \frac{1}{n}\lambda(\nu) > 0.$$

By Morris's Theorem 3.1.1, the set:

$$U := \left\{ \phi \in C^0(\mathbb{S}^1); \overline{\mathcal{M}_{erg}(f)} \cap \mathcal{M}_{\max}(\phi) \subset \mathcal{U} \right\}.$$

is open and dense in $C^0(\mathbb{S}^1)$. We have $U = \mathcal{R}$, which means that $\mathcal{R}$ is open and dense.

□

We define now

$$\mathcal{R}^- := \left\{ \phi : \mathbb{S}^1 \to \mathbb{R}; \inf_{\mu;\lambda(\mu)=0} \int \phi d\mu > \inf_{\nu;\lambda(\nu)>0} \int \phi d\nu \right\}$$

Remark 3.2.3. *Note that if $\phi \in \mathcal{R}^-$, we have:*

$$\inf_{\mu;\lambda(\mu)=0} \int \phi d\mu > \inf_{\nu;\lambda(\nu)>0} \int \phi d\nu = - \sup_{\mu;\lambda(\mu)=0} \int (-\phi) d\mu > - \sup_{\nu;\lambda(\nu)>0} \int (-\phi) d\nu$$

$$\text{and consequently} \quad \sup_{\mu;\lambda(\mu)=0} \int (-\phi) d\mu < \sup_{\nu;\lambda(\nu)>0} \int (-\phi) d\nu.$$

Hence, $\mathcal{R}^-$ is a set of continuous potentials ϕ such that $-\phi \in \mathcal{R}$, which means that $-\phi$ is expanding and so it has the spectral gap property, by Proposition 3.1.11. Furthermore, for all $t < 0$ we have $t\phi \in \mathcal{R}$ as well. As a consequence, the transfer operator $\mathcal{L}_{f,t\phi}$ has spectral gap for all $t < 0$.

Lemma 3.2.4. *$\mathcal{R}^-$ is open and dense in $C^0(\mathbb{S}^1, \mathbb{R})$.*

Proof. Take the set $\mathcal{U}$ as in Lemma 3.2.2. Then, with the same argument, $\mathcal{U}$ is open and dense in $\overline{\mathcal{M}_{erg}(f)}$. Again, by Morris's Theorem 3.1.1 the set

$$U' := \left\{ \phi \in C^0; \overline{\mathcal{M}_{erg}(f)} \cap \mathcal{M}_{\max}(-\phi) \subset \mathcal{U} \right\}$$

is open and dense in $C^0(\mathbb{S}^1, \mathbb{R})$. □

From Lemma 3.2.2 and Lemma 3.2.4, we conclude that

$$\mathcal{H} := \mathcal{R} \cap \mathcal{R}^- \tag{3.5}$$

is an open and dense subset of $C^0(\mathbb{S}^1, \mathbb{R})$. If E is given as in Remark 3.0.2, then E is a dense subset of $C^0(\mathbb{S}^1, \mathbb{R})$, and so

$$\mathcal{H} \cap E$$

is a dense subset of $C^0(\mathbb{S}^1, \mathbb{R})$. By definition, if $\phi \in \mathcal{H}$ the associated transfer operator $\mathcal{L}_{f,t\phi}$ has the spectral gap property for all $t \in \mathbb{R}$.

3.2.1 Proof of Theorem C and Corollary C

We recall that we want to prove the following:

Theorem C. *There exist an open and dense subset $\mathcal{H} \subset C^0(\mathbb{S}^1, \mathbb{R})$ in the uniform topology, such that if $\phi \in \mathcal{H}$ is Hölder continuous, then ϕ has **no** thermodynamic phase transition and $\mathbb{R} \ni t \mapsto P_{top}(f, \phi)$ is strictly convex.*

Proof. Take the open and dense subset $\mathcal{H}$ as in (3.5). Note that, by construction, if $\phi \in \mathcal{H}$ is Hölder continuous, then $\mathcal{L}_{f,t\phi}|_E$ has the spectral gap property for all $t \in \mathbb{R}$. Applying Proposition 3.1.14, we have that $t \mapsto P_{top}(f, t\phi)$ is analytic and, since ϕ is not cohomologous to a constant, it is strictly convex. With that we complete the proof of Theorem C.

□

Next we prove that a consequence of the previous theorem is the prevalence of no phase transitions, as follows.

Corollary C. *1. The set of smooth potential such that $t \mapsto P_{top}(f, t\phi)$ is strictly convex and has no thermodynamic phase transition is dense, in the uniform topology;*

*2. The set of Hölder continuous potential having thermodynamic phase transition is **not** dense, in the uniform topology.*

Proof. (1) follows immediately from the fact that $\mathcal{H} \cap E$ is dense in the uniform topology. To prove (2), let $\phi \in \mathcal{H}$ and $\phi_n \to \phi$ be a sequence of C^r(Hölder continuous) potentials converging uniformly. As $\mathcal{H}$ is open, there exists $n_0 \in \mathcal{N}$ such that ϕ_n has no phase transition for all $n \geq n_0$, which means that the set of potentials with phase transition is not dense in the uniform topology.

□

3.2.2 Proof of Theorem D

We recall that we want to characterize phase transitions:

Theorem D. *Let ϕ be a Hölder continuous potential. The following items are equivalent:*

1. *ϕ does not have thermodynamic phase transition;*
2. *ϕ does not have spectral phase transition, i.e., $\mathcal{L}_{f,t\phi}|_E$ has spectral gap for all $t \in \mathbb{R}$;*

If in addition ϕ is not cohomologous to a constant, then the previous items are equivalent to:

3. *the topological pressure function $t \mapsto P_{top}(t\phi)$ is strictly convex.*

Furthermore, if any of the previous items holds, then $t\phi$ has an unique equilibrium state, for all $t \in \mathbb{R}$.

Proof. We show $(i)1 \implies 2$, $(ii)2 \implies 1$ and finally $(iii)3 \Leftrightarrow 1$.

(i) Suppose that ϕ does not have thermodynamic phase transition. Then ϕ must be expanding. Indeed, if ϕ was not expanding it could not be hyperbolic and, applying Proposition 3.1.7, there would exist a thermodynamical phase transition $t_0 \in (0, 1]$, which contradicts the assumption. Furthermore, $t\phi$ is expanding for all $t \in \mathbb{R}$, otherwise there would exist $t_0 \in \mathbb{R}$ with $t_0\phi$ not hyperbolic and, again by Proposition 3.1.7, we would have a thermodynamical phase transition. Applying Proposition 3.1.11, $\mathcal{L}_{f,t\phi}|_E$ has the spectral gap property for all $t \in \mathbb{R}$;

(ii) Is a direct consequence of Proposition 3.1.14;

(iii) Suppose that ϕ is not cohomologous to a constant. On the one hand, by Lemma 3.1.7, if ϕ has no phase transition, then it is hyperbolic and thus expanding. Furthermore, $t\phi$ is hyperbolic and expanding as well, for all $t \in \mathbb{R}$. On the other hand, if $t\phi$ is hyperbolic and expanding, by Proposition 3.1.11, $t\phi$ has the spectral gap property for all $t \in \mathbb{R}$ and, by the fact that $1 \Leftrightarrow 2$, ϕ does not have thermodynamic phase transition. Then, $(1 \implies 3)$ due the fact that $t\phi$ is hyperbolic and expanding for all $t \in \mathbb{R}$, and this implies that $t\phi$ has the spectral gap property. By Proposition 3.1.14, $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi)$ is strictly convex. To show that $(3 \implies 1)$, suppose that $\mathbb{R} \ni t \mapsto P_{top}(f, t\phi)$ is strictly convex and, arguing by contradiction, suppose that ϕ has a thermodynamical phase transition in t_0. Then, by Proposition 3.1.14, $t_0\phi$ cannot have spectral gap, which implies that $t_0\phi$ is not expanding and so not hyperbolic. It follows from the Proposition 3.1.7 that there exists a maximizing invariant probability measure ν_1 such that $P_{top}(f, t\phi) = t\int \phi d\nu_1$ for all $t \geq t_0$, contradicting the assumption.

∎

3.2.3 Proof of Theorem E

We recall that we want to describe the topological pressure function and the transfer operator for potentials with phase transition:

Theorem E. *Let $\phi \in E$ be a potential having phase transition. Then, there exist $t_2 < 0 < t_1$, with at least $t_1 \in \mathbb{R}$ or $t_2 \in \mathbb{R}$, such that:*

1. *for $t \geq t_1$ or $t \leq t_2$, then the topological pressure function is an affine map and for $t_2 < t < t_1$ the topological pressure function $t \to P_{top}(t\phi)$ is analytic and strictly convex;*

2. *for $t_2 < t < t_1$ the associated transfer operator $\mathcal{L}_{f,t\phi}|_{C^\alpha}$ has spectral gap property, and for $t \geq t_1$ or $t \leq t_2$ the associated transfer operator $\mathcal{L}_{f,t\phi}|_{C^\alpha}$ does not have spectral gap property.*

Proof. By assumption, ϕ has thermodynamical phase transitions, which means that the topological pressure function $t \mapsto P_{top}(f, t\phi)$ is not analytic. Following the proof of Theorem D, there should exist parameters $t_1 > 0, t_2 < 0$, with at least $t_1 < \infty$ or $t_2 > -\infty$, such that $t_1\phi$ is not hyperbolic and $t_2\phi$ is not hyperbolic as well. We define

$$0 < \hat{t}_1 := \inf\{t > 0; t\phi \text{ is not hyperbolic}\}$$

and

$$0 > \hat{t}_2 := \sup\{t < 0; t\phi \text{ is not hyperbolic}\}.$$

Then, following the proof of Lemma 3.1.7, the topological pressure function $t \mapsto P_{top}(f, t\phi)$ is affine for all $t \leq \hat{t}_1$ and $t \geq \hat{t}_2$. Furthermore, for $t \in (\hat{t}_1, \hat{t}_2)$ we have that $t\phi$ is hyperbolic and so $t\phi$ is expanding. By Lemma 3.1.11, $\mathcal{L}_{f,t\phi}|_E$ has spectral gap for all $t \in (\hat{t}_1, \hat{t}_2)$ and, by Lemma 3.1.14, the topological pressure function is analytic in $(\hat{t}_1, \hat{t}_2)$ and strictly convex. Finally, for $t \geq t_1$ or $t \leq t_2$ the transfer operator $\mathcal{L}_{f,t\phi}|_E$ does not have the spectral gap property, by Lemma 3.1.14, because the topological pressure is not strictly convex. ∎

The consequence for Maneville-Pomeau-like maps is the following:

Corollary E. *If f is a Maneville-Pomeau like map, then there exists at most one thermodynamical phase transition for Hölder continuous potentials.*

Proof. Let f be a Maneville-Pomeau-like map and ϕ a Hölder continuous potential. Arguing by contradiction, suppose that ϕ has two phase transitions. By item (1) of Theorem E, there are $t_1 < 0$ and $t_2 > 0$ such that the topological pressure function $t \mapsto P_{top}(f, t\phi)$ is not analytic in t_1 and t_2. This implies that $t_1\phi$ and $t_2\phi$ are not hyperbolic. Otherwise, they would be expanding and so the transfer operator would have the spectral gap property in those parameters. In particular, by Proposition 3.1.7, there are ν_1 and ν_2 maximizing probabilities in $\mathcal{M}_1(f)$ associated to $-\phi$ and ϕ, respectively, with $\lambda(\nu_1) = \lambda(\nu_2) = 0$. We show that there is only one measure for Maneville-Pomeau-like maps with zero Lyapunov exponent. In fact, suppose that $\nu \in M_{erg}(f)$ and $\lambda(\nu) = 0$. By Birkhoff's Ergodic Theorem, there exists $x \in \mathbb{S}^1$ such that

$$\nu = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

By the non-uniform expansion property of f, given $\epsilon > 0$, let $\delta > 0$ be such that

$$d(y, 0) \geq \epsilon \implies |Df(f(y))| \geq \delta.$$

For each n, define $A_n := \{0 \leq i \leq n-1; d(f^i(x), 0) \geq \delta\}$. Since $\lambda(\nu) = 0$, then:

$$\frac{1}{n} \log |Df^n(x)| = \frac{1}{n}\Big[\sum_{i\in A_n} \log |Df(f^i(x))| + \sum_{i\notin A_n, i\leq n-1} \log |Df(f^i(x))|\Big] \geq \frac{1}{n}\delta \# A_n$$

which implies $\quad 0 = \lambda_\nu(f) = \lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)| \geq \delta \lim_{n\to\infty} \frac{\# A_n}{n}$

and so $\quad \lim_{n\to\infty} \frac{\# A_n}{n} = 0.$

Thus, taking any continuous potential $\varphi : \mathbb{S}^1 \to \mathbb{R}$, we have:

$$0 = \lim_{n\to\inf} \frac{\# A_n}{n} \sup_{x\in\mathbb{S}^1}\{\varphi(x)\} \geq \lim_{n\to\infty} \frac{1}{n} \sum_{i\in A_n} \varphi(f^i(x))| \geq \lim_{n\to\infty} \frac{\# A_n}{n} \inf_{x\in\mathbb{S}^1}\{\varphi(x)\} = 0$$

In conclusion, by the Birkhoff's Ergodic Theorem:

$$\begin{aligned}
\Big|\int \varphi d\nu - \int \varphi d\delta_0\Big| &= \Big|\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) - \varphi(0)\Big| \\
&= \Big|\lim \frac{1}{n} \sum_{i\notin A_n; i\leq n-1} \varphi(f^i(x)) - \varphi(0)\Big| \\
&\leq \sup_{d(y,0)<\epsilon} |\varphi(y) - \varphi(0)|.
\end{aligned}$$

As ϵ is arbitrary, letting $\epsilon \to 0$, we get $\int \varphi d\nu = \int \varphi d\delta_0$. By definition of convergence in the *weak** topology, $\nu = \delta_0$. From that, we must have $\nu_1 = \nu_2 = \delta_0$. So, we must have

$$\mu \mapsto \int \phi d\mu$$

is constant, and by Thompson [Th10] ϕ is cohomologous to a constant, which contradicts the fact that we assumed ϕ having phase transitions. $\square$

Chapter 4

Further comments and more questions

4.1 Differentiability of the topological pressure function

An interesting question is the existence of examples where the topological pressure function is not analytic but it is differentiable at the transition parameter. Suppose that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a C^2-local diffeomorphism Maneville-Pomeau-like map. According to Pianigiani [Pi80], f does not admit a finite a.c.i.p.. In his proof, Pianigiani studied non-uniformly expanding maps $T : [0,1] \to [0,1]$ and consider the first return map $R_{T,A}$ on a set A in which T is uniformly expanding, so $R_{T,A}$ is uniformly expanding as well. Then, he shows that the existence of an a.c.i.p. μ_A for $R_{T,A}$ induces an a.c.i.p. μ for T. For example, consider

$$T : [0,1] \to [0,1]$$

given by

$$T(x) = \begin{cases} \frac{x}{x-1}, & \text{if } x \in [0, \frac{1}{2}] \\ 2x - 1, & \text{if } x > \frac{1}{2} \end{cases}$$

Then, $|T'(x)| > 1$ on $(0,1]$ and $T'(0) = 1$. Pianigiani showed that the a.c.i.p. μ_T is not finite. Following Bomfim and Carneiro [BC21], for those classes of dynamics, non differentiability of the pressure function is equivalent to have a finite a.c.i.p.. More specific, taking the geometric potential, $-\log|DT|$, the topological pressure function is differentiable at the phase transition. It tells us that the lack of analiticity of the pressure function does not imply lack of differentiability, even if the potential is regular. On other hand, if f is a $C^{1+\alpha}$-local diffeomorphism and in a neighbourhood of the indifferent fixed point f is not C^2, then f admits a unique finite a.c.i.p (see [Pi80]). It follows from [BC21] that

$$t \mapsto P_{top}(f, -\log|Df|)$$

is not differentiable in $t = 1$. We propose the following question:

Question 3. *If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a $C^{1+\alpha}$ local-diffeomorphism, then f is expanding or there is a Hölder continuous potential $\phi : \mathbb{S}^1 \to \mathbb{R}$ such that the topological pressure function $t \mapsto P_{top}(f, t\phi)$ is not differentiable?*

4.2 Uniqueness of equilibrium states

We proved in Theorem D that the lack of phase transition for a potential ϕ implies uniqueness of equilibrium states for $t\phi$, for all $t \in \mathbb{R}$. The inverse implication is not true. For each $\alpha \in [0, 1]$ define the constants:

$$b(\alpha) = \left(\left(\frac{1}{2}\right)^{3+\alpha} - \frac{4+\alpha}{4+2\alpha}\left(\frac{1}{2}\right)^{2+\alpha}\right) \text{ and } a(\alpha) = -\frac{-b(4+\alpha)}{4+2\alpha}$$

and the polynomial $g_\alpha : [0, 1/2] \to [0, 1]$, given by $g_\alpha(y) = y + ay^{3+\alpha} + by^{4+\alpha}$. Then we define the family of intermittent maps on the circle

$$f_\alpha(y) = \begin{cases} g_\alpha(y), if \quad 0 \leq y \leq 1/2 \\ 1 - g_\alpha(1-y), if \quad 1/2 \leq y \leq 1. \end{cases}$$

Each member of the family will be a C^2-local diffeomorphism. It follows from [Pi80] that the geometric potential $\phi = -\log|Df_\alpha|$ has an unique equilibrium state. On the other hand, it follows from Bomfim-Carneiro [BC21], that f_α has phase transition with respect to the ϕ.

Suppose now that $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a transitive $C^{1+\alpha}$-local diffeomorphism. By Bomfim and Carneiro [BC21], we know that f is expanding if, and only if, f does not have phase transition with respect to geometric potential $\phi = -\log|Df|$. With a view to the previous discussion, we propose the following question:

Question 4. *If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is a $C^{1+\alpha}$ local-diffeomorphism, then f is expanding or there is a Hölder continuous potential $\phi : \mathbb{S}^1 \to \mathbb{R}$ such that ϕ has at least two equilibrium states?*

4.3 More than one phase transition

As we saw in Theorem E, if f is a piecewise monotone interval map like in Definition 0.2.7, each regular potential ϕ can have at most two phase transition. We may ask ourselves if there exists a regular potential having exactly two phase transitions, $t_2, t_1 > 0$. As we saw in the proof of Theorem E, then it should exist two maximizing measures μ_1, μ_2 associated to $t_1\phi$ and $-t_2\phi$, respectively. However, every known example of phase transitions for regular potentials occurs with respect to geometric potential $\phi = -\log|Df|$, in fact exhibits a unique phase transition (see e.g. [Lo93, PS92, BC21]). Thus, we propose the following question:

Question 5. *Is there a Holder continuous potential $\phi : \mathbb{S}^1 \to \mathbb{R}$ such that the topological pressure function $t \mapsto P_{top}(f, t\phi)$ has two phase transitions?*

4.4 Higher Dimensional Case

In the first two Chapters of this preset work we proved existence of phase transition for skew-product $F \in \mathcal{D}^r$ with respect to the geometric potential in the central direction, and the consequences for multifractal analysis. Taking in account our results for transitive piecewise monotone dynamics in Chapter 3, we present the following question:

Question 6. *Let $F \in \mathcal{D}^r$ be fixed. Is there a dense subset of regular potentials $\phi : \mathbb{T}^d \times \mathbb{S}^1 \to \mathbb{R}$ in the uniform topology, such that $t \mapsto P_{top}(F, t\phi)$ has no phase transition and it is strictly convex? Is there a characterization of potentials without phase transition, as in Theorem D or with transition, as in Theorem E?*

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