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THE INTEREST RATE DERIVATIVE MARKET IN BRAZIL AND ARBITRAGE OPPORTUNITIES

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THIAGO RAMOS ALMEIDA

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Dissertação de Mestrado aprovada como requisito parcial para obtenção do Grau de Mestre em Economia no Programa de Pós-Graduação em Economia da Faculdade de Economia da Universidade Federal da Bahia, pela seguinte banca examinadora:

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“Do not conform to the pattern of this world,
but be transformed by the renewing of your mind.
Then you will be able to test and approve what Gods will is
his good, pleasing and perfect will
(Bible, Romans 12:2)

ABSTRACT

I empirically analyze the performance of a term structure model, in which each instantaneous forward rate is driven by a different stochastic shock, namely, string market model. This model was employed to pricing interest rates derivatives traded in the Brazilian financial market. I find that the market use four risk factors to pricing the option on DI futures. How the option on IDI index has the same underlying variable (interbank deposit rates) to the option on DI futures, both are connected by no-arbitrage relationship through the correlation structure of interest rates. Accordingly, to identify arbitrage opportunities between these two derivatives, the string market model was calibrated with the option on DI futures and then, used to pricing a sample of the option on IDI index. The empirical investigation show that there is no systematic mispricing between these two options. However, the prices of a sub-sample of the "near the money" option on IDI index, deviate considerably from the no-arbitrage values implied by the market prices of the option on DI futures.

Keywords: Interest Rates. Financial Economics. Arbitrage. Derivative.

RESUMO

Eu empiricamente analiso a performance de um modelo de estrutura a termo, em que cada taxa de juros futura possui choques estocásticos diferentes, nomeadamente, "string market model". Este modelo foi empregado para precificar derivativos de taxas de juros negociados no mercado financeiro brasileiro. Foi identificado que o mercado utiliza quatro fatores de risco diferentes para precificar a opção sobre o futuro de DI. Como a opção sobre o índice IDI possui o mesmo ativo-objeto (taxa dos depositos interbancários) que a opção sobre o futuro de DI, ambas estão conectadas por uma relação de não arbitragem devido a estrutura de correlação das taxas de juros. Consequentemente, para identificar oportunidades de arbitragem entre estes dois instrumentos financeiros, o "string market model"foi calibrado com a opção sobre futuro de DI e então, utilizado para precificar uma amostra da opção sobre o índice IDI. A investigação empírica mostrou que não há desvios de preços de forma sistematizada entre as duas opções. Contudo, os preços de uma sub-amostra de opções "próxima do dinheiro"da opção sobre o índice IDI, desviaram-se consideravelemente dos valores requeridos pela relação de não arbitragem.

Palavras-chaves: Taxas de juros, Economia Financeira, Arbitragem, Derivativos.

SUMÁRIO

1 INTRODUCTION

The search for risk-sharing, the financial deregulation and the development of information technology have been leading a fast growth to the derivative market. Many firms around the world manage their risk exposure on fluctuations in interest rates using these type of instruments. According with ISDA (International Swaps and Derivatives Association), almost 94% of the 500 biggest companies of the world use derivatives to protect themselves against undesired fluctuation in the interest rates [1]. In accordance with this institution, the global notional value[2] of the interest rate derivatives market was approximately $ 436.8 trillion in the end of 2017. Then, is of enormous importance a well-specified and efficient pricing model for these type of financial instruments.

Usually, the derivatives are used with the purposes of hedging, to get protection against undesired variations on prices (commodities, energy), currencies, rates (interest, inflation), among others. Adopting a position on the derivative market opposed of the position on the spot market. Meanwhile, there are another types of uses, like leverage (allow the improvement at the portfolio return with a cost lower than buy another assets), speculation (when adopt a position on derivative market without the correspondence position on the spot market, aiming for profit, exclusively) and, finally, arbitrage, that is the main topic of this study. The arbitrageurs seek distortions in financial assets prices, like approximately similar security that are sold in two different markets for different prices, with the objective to buy the cheaper security and sells the more expensive one and when occurs the market prices correction, the arbitrageur dissolve his position with profit.

In Brazil, the most trade derivative is the DI (interbank deposit rate) futures contract and this is an of the most liquidity contract in the world[3] (approximately 1.5 million of contracts are traded daily). Its dynamics are similar to swaps contract, one party operating in DI rate, the floating rate and the counterparty is active in a fixed rate. Another type of contingent claims, that also have as underlying variable the DI rates, are: the option on average one-day interbank deposit

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1 In accordance with the ISDA, 471 companies that was interviewed at 2018, reported the utilization of interest rates derivatives. The report about the data set of derivative markets was obtained in the ISDA site, on 06/30/2019. Available in < www.isda.org/researchnotes/isdaresearch.html>, accessed on 30 jun. 2019.

2 The notional value is the quantity in money concerned in a transaction (HULL et al., 2009). For example, let's assume an interest rate swap, the company A issues R$ 1 mi in debentures to be pay monthly on 120 installments with interest of 10% a.a, the economic analysts of this firm believe that the interest rates will fall soon, then, they deserve exchange a fixed rate by the floating rate, to have less financial outlay. Meanwhile, the investment fund B has a portfolio of bonds with market value of R$ 1 mi, if the interest rates increase the bonds will lose value, because this the fund deserve migrate to a fix rate to get protection against the depreciation of yours bonds. Consequently, the investment fund B and the firm A agree to get in a swap contract of interest rates. A will pay to B in a future date (1 year for example) the SELIC rate on the capital of R$ 1 mi and will receive of B 10% on R$ 1 mi.

3 In accordance with <http://www.b3.com.br/pt_br/market-data-e-indices/servicos-de-dados/market-data/consultas/mercado-de-derivativos/resumo-das-operacoes/resumo-por-produto> accessed on 30 ago. 2019

rate index (henceforth, the option on IDI index) and the option on one-day interbank deposit futures (henceforth, the option on DI futures).

How these two options sharing the same underlying variable, financial theory implies no-arbitrage relation that must be satisfied by the option on IDI index and the option on DI futures. The option pricing theory (MERTON et al., 1973) implies that the relation between these options are drive primarily by the correlation structure of the forward rates, because this, no-arbitrage relationship must be satisfied.

Significant part of the literature about interest rates derivatives, in accordance with Jarrow, Li e Zhao (2007), approaching two subjects. The first one is related to the *unspanned stochastic volatility*, the risk factors that pricing interest rates derivatives are not the same of the term structure. The second one, is about the *relative valuation of Caps e Swaptions*[4], how these two options has the same underlying variable, both prices movements are correlated to the movements of the term structure of interest rate, this involve a no-arbitrage relation between these two derivatives. However, according to Longstaff, Santa-Clara e Schwartz (2001a) and Jagannathan, Kaplin e Sun (2003), there is an important mispricing between caps and swaptions, verified through the use of various multifactor term structure models, what has been called by the “swaptions/caps puzzle”.

For the valuation framework, i adapted for the DI rates, the model of Longstaff, Santa-Clara e Schwartz (2001a) (henceforth LSS model), developed to simulate the evolution the term structure of the London Interbank Offered Rate (Libor rate) in continuous time. One of the mains innovations of my model consist in the development of the genetic algorithm to capture the average implied volatility of the options. The algorithm’s objective is to find an implied volatility that best pricing a set of options that has the same maturity, with the Black (1976) model.

This dissertation has as one of its contributions an empirical analysis about the behavior of the prices of financial instrument traded on the securities exchange. The hypothesis that the investor are rationale bring great implications to the behavior of the asset markets, if this hypothesis are true, there should be no arbitrage opportunities, or at least, a correction very fast on mispricing assets. Therefore, an empirical analysis about the behavior of the prices and an evaluation on its arbitrage opportunities is an important channel to identify how the rational the investors really are. This work evaluated over 61 weeks, 8.396 options and the empirical results provide an evidence that there is not systematic arbitrage opportunities between the option on IDI index and the option on DI futures. Nonetheless, for a little part of the sample, there were significant mispricing, what can lead a gain with arbitrage.

Beyond the empirical contributions, there are theoretical ones, which refers to the specifications

[4] The swaption is an option that confer to his owner the right to enter in a swap contract of interest rates, and is divided in two types: the *payer swaption* and the *receiver swaption*. With the *payer swaption*, the buyer has the right to enter in a swap contract to receive the floating rate and pay the fixed rate. The *receiver swaption* is the opposite. Already the *Caps* is a derivative that the buyer receive a payment at the end of each period in which the interest rate exceed the strike of the option agreed at the moment of the purchase.

of equations that expresses the payoffs of the options and the DI forward rates with continuous time capitalization. Carreira e Jr (2016) and Santos e Silva (2015) exhibit the payoff of the option on DI futures with the annual interest rates and annual capitalization, however, to promote computational tractability and simplify the equations, i adapted these equations to expresses it with annual interest rates and continuous time capitalization, also is present new expressions for the payoff of these derivatives. I present in a sequence of equations, a method to get the forward rate with continuous time capitalization and the bond prices, with the use of the unit price-PU of the DI futures contract, that are published by the São Paulo Securities Exchange (henceforth B3, fantasy name of securities exchange).

The rest of this work is organize as follows. In chapter I, i present some principles of the modern theory of finance, among them: the stochastic discount factor, the risk neutral probability measure, the risk-free rate, the risk correction and the risk sharing. Chapter II report the financial instruments utilized in the empirical investigation and its payoffs equations. In Chapter III, i develop the string market model adapted for the DI interest rates. Chapter IV reports the empirical findings and Section IV concludes.

2 OVERVIEW ABOUT THE MODERN THEORY OF FINANCE

The neoclassic theory of finance is based on assumptions that allows the pricing of uncertainty cash flow. In a model of competitive equilibrium of the asset market, the price are endogenous obtained, from the aggregate decision economic agents, by means of the pricing function that maps future uncertainty payoffs in today's price.

A model is characterized by its assumptions and one of the most important assumption of the pricing function is the no-arbitrage principles (NA). The arbitrage exist when is possible to get an strategy of investment that allow a positive payoff with a probability bigger than zero and a negative payoff with zero probability, without any initial investment (ROSS, 2009).

The minimal requirement is the wealth preference, people prefer more wealth than fewer, therefore, asset that are sold in some market with price smaller than the other, will attract attention of investors that will buy in the market with smaller price to sell in the market with the biggest price, such that this opportunities of gain will be exhausted in competitive markets.

NA is also a necessary condition to the equilibrium of financial market. If there is an arbitrage opportunity, the demand and supply of this asset can be infinite, what is inconsistent with an equilibrium. So, the study of the implication of NA is a fundamental theme in the modern theory of finance.

The principles of NA is consequence of the law of one price (LOOP). The LOOP require that two asset that have the same payoff in every states of the wold in the future, must be sold at the same price, otherwise, we would have an arbitrage opportunity. This assumption is very important in the asset pricing function, to avoid the occurrence of different price of assets that have the same payoff.

The other assumption used at the literature (JARROW, 2002) about asset pricing model, include: i) an economy without friction (without transaction costs, without restriction in the buy and sell of assets); ii) competitive markets (every trader is price taking).

Over time, because of the advances in the information technology and the increase in the competition of banks and brokers, the fee charged in the negotiation of assets sold in financial market have been fallen. Thereby, because of the decreasing impact and the complication that this fee generate in models, we assume an economy without transaction costs.

2.1 THE STOCHASTIC DISCOUNT FACTOR

Consider an economy with asset negotiation beginning at time zero, the uncertainty in this economy is characterized by the probability space (Ω, F, Pr), where $\omega \in \Omega$ represent the states of the world, Pr is a probability measure associated with the states of the world and F is a $\sigma - algebra$ that represent the measurable event. To simplify the exposition, we use only two periods, t and t+1. Let $\pi_{t,t+1}$ be the operator of valuation at date t, this operator point out prices to portfolio of payoffs p_{t+1} in the space P_{t+1}. P_{t+1} is specified here as a set of payoffs at t+1 to portfolio of assets available to acquisition at time t and I_t is the set of all random variables that are measurable with respect to F_t.

Hansen e Richard (1987) formalized the existence of diverse property of the set P_{t+1}. Below, we enumerate the two fundamental assumptions:

Assumption 1: allow that the pricing function $\pi : P_{t+1} \rightarrow I_t$, be value additive: to any p^1_{t+1} and p^2_{t+1} in P_{t+1} and any w_1 and w_2 in I_t,

$$\pi(w_1 p^1_{t+1} + w_2 p^2_{t+1}) = w_1 \pi(p^1_{t+1}) + w_2 \pi(p^2_{t+1}) \tag{2.1}$$

Despite the non-linearity of the SDF, in many models, with relation to the factors, the relation between the SDF and the payoffs is linear. This means that if the payoff of an asset is the double of another in all states of the world, then it price must be the double of the other, then the pricing function is linear.

Assumption 2: Afford the continuity of pricing function. If $\{p^j : j = 1, 2, ...\}$ is a sequence of payoffs in P that converge conditionally to zero, so, to any $\epsilon > 0, lim_{j\rightarrow\infty} Pr\{\mid \pi(p^j) \mid > \epsilon\} = 0$. With this assumption, we can infer that small *payoffs* has small prices.

Hansen e Richard (1987) demonstrated that, under a suitable set of assumption, that include the two outlined above, there exist an single *payoff* p^* in P_{t+1} that satisfies:

$$\pi(p_{t+1}) = \langle p_{t+1}, p^* \rangle_{F_t} \tag{2.2}$$

For all p_{t+1} in P_{t+1}. Moreover, $Pr\{\| p_* \|\}_{F_t} > 0\} = 1$, where $\| p^* \| = [\langle p^*, p^* \rangle_{F_t}]^{\frac{1}{2}}$. The assumption outlined in the equation 2.2, say that there exist a unique *payoff* $p^* \in P_{t+1}$ such that $\pi(p_{t+1}) = E(p^* p_{t+1})$ for all $p_{t+1} \in P_{t+1}$. I demonstrated in equation 2.1, that the pricing function is linear, in equation 2.2, I express the same function as result of inner product, this is possible due of Riesz theorem of representation, according to which, any linear function can be represented by an inner product. Important to say that if the market is not complete, there exist a infinite number of random variable that satisfies $p_t = E[p^* p_{t+1}]$, for example, for any ϵ orthogonal to p^*, we will have $p_t = E[(p^* + \epsilon)p_{t+1}]$, since $E(\epsilon p_{t+1}) = 0$.

If in the space of payoffs P_{t+1} there exist a contingent claim for the S states of the world possible to occur at t+1, then, $P_{t+1} = R^s$ and the market is complete. Consider here a contingent claim, as a financial instrument that pay one unit of the numeraire good, only if arise at t+1 the state s of the world, else occur a state different of s (considering the individual states by s), nothing is payed. With the hypothesis of complete market, then, it would have for all states s of the world a contingent claim.

There is a convenient interpretation to p^*, when π does not have arbitrage opportunities in P_{t+1}. A pricing function π_t at the space of payoff P_{t+1}, is not allow arbitrage opportunities if, to any $p_{t+1} \in P_{t+1}$ such that $Pr(p_{t+1} \geq 0) = 1, Pr(\{\pi_t(p_{t+1}) \leq 0\} \cap \{p_{t+1} \geq 0\}) = 0$. In other words, if the *payoff* is positive, with probability bigger than zero, then, its price must be positive in order that not allow arbitrage opportunities.

According with Hansen e Richard (1987), for each t=1,2,3..., let I_t be the set of all random variables that are measurable with respect to F_t, then I define,

$$P^+ = \{\text{p em} I_{t+1} : E(p^2 \mid F) < \infty\} \tag{2.3}$$

For all p^1 and p^2 in P^+, I consider the following assumption, that imply that the pricing function can be represented by a inner product,

$$\langle p^1, p^2 \rangle_F = E(p^1 p^2 \mid F). \tag{2.4}$$

With the assumption 2.3 and 2.4 satisfied, then π do not have any arbitrage opportunities in P^+ if, and only if, $Pr\{p^* > 0\} = 1$. I can interpret p^* as a equilibrium measure of the marginal rate of intertemporal substitution of consumption between the time t and t+1, I call the random variable p^* as a stochastic discount factor and, thereafter, I will use the notation $p^* = m_{t+1}$. The multi-period version of m_{t+1} to represent a process of stochastic discount is $\{M_{t+1} : t = 1, 2, ...\}$, which can be expressed as:

$$M_{t+1} = \prod_{j=1}^{t+1} m_j \tag{2.5}$$

Accordingly, the price of an asset at the date zero that pay p_{t+1} in units of numeraire good at t+1, is equal to,

$$\pi_{0,t+1}(p_{t+1}) = E(M_{t+1} p_{t+1} \mid F_0) \tag{2.6}$$

2.2 RISK NEUTRAL PROBABILITY MEASURE

Suppose a bond issued at the date t, with face value at t+1 of R$ 1,00, if this bond is risk-free, then for any state of the world that is possible to occur at the maturity, the investor will receive R$ 1,00. Consequently, the price of this bond is given by $E[m_{t+1} \mid F_0]$, that is the discount rate for risk-free assets.

Harrison e Kreps (1979) utilize the variable m (stochastic discount factor) defined above, to introduce a new probability measure that allow that the price of assets can be calculated starting of its expected payoffs. To build the risk neutral probability measure, consider the expression below, with the pricing operator for a single period,

$$\pi_{t,t+1}(p_{t+1}) = E(m_{t+1} \mid F_0)\tilde{E}(p_{t+1} \mid F_t) \tag{2.7}$$

where,

$$\tilde{E}(p_{t+1} \mid F_t) = E\left(\left[\frac{m_{t+1}}{E(m_{t+1}) \mid F_t)}\right] p_{t+1} \mid F_t\right) \tag{2.8}$$

With this, the pricing operator is simplified and becomes the multiplication of the expected value of two random variable, the first is the expected payoff at t+1 obtained with the risk neutral probabilities measure, $\tilde{E}(. \mid_t)$. The second variable is the discount factor for the risk-free assets and have the function to bring out the payoff at time t+1 to time t. In essence, the risk neutral measure aim to capture a type of preference represented by a linear utility function, with these type of preference, the agents will be risk neutral. This means that the investor will not demand risk premium for the more volatile asset, whats matter for the decision-making, with this type of preference, is only the expected value of the payoff (variance is not important).

The risk neutral valuation is very useful in continuous time process, because it works only using the mean, is not necessary use the covariance, as will be demonstrate in the next section.

The stochastic discount factor (SDF) can be decomposed into the following manner,

$$M_{t+1} = \bar{M}_{t+1} D_{t+1} \tag{2.9}$$

where,

$$\bar{M}_{t+1} = \left[\prod_{j=1}^{t+1} E(m_j \mid F_{j-1})\right] \tag{2.10}$$

is the discount factor for risk-free asset and,

$$D_{t+1} = \prod_{j=1}^{t+1} \frac{m_j}{E(m_j \mid F_{j-1})} = \frac{M_{t+1}}{\bar{M}_{t+1}} \tag{2.11}$$

D_{t+1} is a martingale that point out risk neutral probabilities to events in F_{t+1}, then $E(D_{t+1} \mid F_t) = D_t$. The fundamental theorem of asset pricing set up the connection between the equivalent martingal probability measure, cited above, and the absence of arbitrage opportunities.

In an economy where there isn't arbitrage opportunities in none trading strategy, the product of SDF with the value of any self-financing trading strategy V, must be a martingale,

$$V(t) = E_t \left[V_{t+1} \frac{M_{t+1}}{M_t} \right]$$

Based with this expression, I can enunciate the **fundamental theorem of asset pricing**:

Theorem. The two statement below are equivalent to a model X of financial asset pricing:

- X is arbitrage-free (NA);
- Exist a probability measure *Qr* (risk neutral measure) equivalent to *Pr* (physics probability measure) in the space of probability (Ω, F, Pr).

To explain what is a self-financing strategy V, consider below, the set of all assets trading in an economy, with a stochastic process in $\Re^{d+1}$,

$$(S_t^0, S_t^1, ..., S_t^d)_{t \in \mathbb{T}}$$

Where $\mathbb{T} = \{0, 1, ..., T\}$ represent the negotiation dates. S_t^0 means the risk-free bond, it isn't a random variable because independently of states of the world, have the same payoff, on the other hand, $(S_t^1, ..., S_t^d)$ are random variables that represent the price at time t to d different assets.

The trading strategy ϕ denote the quantity of asset purchased by the investor to create his portfolio, represent the separation of capital between different assets, ϕ^0 is the quantity of risk-free bond that compose the portfolio and ϕ^d is the quantity of asset d,

$$\phi = (\phi_t^0, ..., \phi_t^d)_{t=0}^T$$

For a time t, the vector $(\phi_t^0, ..., \phi_t^d)_{t=0}^T$ denote the portfolio in this date and the value of this portfolio (V_t) is obtained by the following manner,

$$V_t(\phi) = \phi_t.S_t = \sum_{i=0}^{d} \phi_t^i S_t^i \quad \forall t \in \mathbb{T}, t \geq 1.$$

An strategy is calling self-financing if, and only if,

$$\phi_t.S_t = \phi_{t+1}.S_t \quad t = 1, ..., T-1$$

The investor, over the trading dates, don't use they resources to consume and also, don't add new values, based with this information until the date t, the investor build they portfolio and in each date he can readjust the quantity of any asset. However, the evolution of his wealth in a self-financing strategy occur only with the appreciation or depreciation of the asset price that belongs to the portfolio.

2.3 THE INVESTOR'S PROBLEM

Suppose an investor with a utility function $U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})]$, such that $U'(c_t, c_{t+1}) > 0$ and $U''(c_t, c_{t+1}) < 0$. Then, he is a risk-averse agent (because the concave utility function), such that, for investment in risk asset (risk in the sense of larger volatility and absence of information about the investment's return), is required a risk premium, in other words, for this type of investor, the utility of the lottery expected value is bigger than the expected utility of a lottery. This relation only becomes an equality when exists a risk premium.

The goal of the investor is maximize his utility function, to accomplish this objective, he must do optimal choices in each period between consumption and savings. To formalize the problem in a simplified way, I consider a temporal horizon with only two times, I assume also that there is not restriction about the acquisition or sale of the payoff p_{t+1} at the prices p_t, in the other words, he can purchase the desired quantity. Thus, this problem are represented by the equation below,

$$Max \quad U(c_t, c_{t+1}) = u(c_t) + \beta E_t[u(c_{t+1})], \; st. \tag{2.12}$$

$$c_t = e_t - p_t\delta, \tag{2.13}$$

$$c_{t+1} = e_{t+1} - p_{t+1}\delta. \tag{2.14}$$

Where c_t is the consume at time t, e_t represent the consume if the investor does not buy any asset, δ is the quantity of asset chosen and p_{t+1} is the payoff at t+1. c_{t+1} is the consume in a future period, which become uncertainty and random, the β denote the impatient rate (a subjective discount rate), the more impatient are the agent, the bigger are the disposal to consume at the present his resources, because his discount rate is high. This mean that the expectation of return must be high in order to make this type of agent save instead of consume.

Replacing the restriction inside the objective function and deriving with respect to δ, I obtain the first order condition to get the optimal choice of consumption and saving,

$$p_t u'(c_t) = E_t[\beta u'(c_{t+1}) p_{t+1}] \tag{2.15}$$

or

$$p_t = E_t\left[\beta \frac{u'(c_{t+1})}{u'(c_t)} p_{t+1}\right] \tag{2.16}$$

Taking the decision of buy an asset, instead of consume, the investor has a loss in his marginal utility, whose magnitude is computed by $u'(c_t)p_t$. However, $E_t[\beta u'(c_{t+1})p_{t+1}]$ is the increase in his expected utility by virtue of the gain the payoff p_{t+1} at $t+1$. The investor will buy or sell an asset until the marginal loss matches the marginal gain.

If the agents are utility maximizer, then, there is a direct connection between his marginal rate of intertemporal substitution of consume and the stochastic discount factor (COCHRANE, 2009),

$$p_t = E_t(m_{t+1} p_{t+1}) \tag{2.17}$$

$$m_{t+1} \equiv \beta \frac{u'(c_{t+1})}{u'(c_t)} \tag{2.18}$$

Therefore, the SDF (m_{t+1}) is used to represent valuation operator in dynamics economies and constitutes an essential part of the pricing function. The pricing operators point out prices for the assets that are trading in competitive markets, with the use of payoffs that may occur between several states of the world.

I demonstrate above, that the basic pricing equation arise by the first order condition of the investor's problem. Accordingly, the main objective is not to obtain bigger returns on investment, but maximize his utility function. In the financial market, there is the desire of larger returns and less variance of the portfolio (the return is a good and the variance is a harm), however, this objective are considered as intermediate.

2.4 THE RISK-FREE RATE

I can represent the return of an investment in the asset I between the time t and t+1, through the ratio $R^i = p_{t+1}/p_t$, then, the present value is 1 and the payoff is the return. If the rate of appreciation of an asset were, for example, 10%, so, the return is represented for 1.1.

The return of the asset i, R^i, can be express by the following manner,

$$1 = E(mR^i). \tag{2.19}$$

The risk-free rate is the theoretical rate of return of an investment that remunerate by a rate r, regardless of the state of the world at the maturity, in other words, is a firmed rate of return of an investment that offer none risk for your creditor. At the American economy, for example, is denoted by the short-term interest rates pays by the U.S treasury. If I utilize the expression 2.19 above to represent the return of the risk-free asset, I will have,

$$1 = E(mR^f) \tag{2.20}$$

$$1 = E(m)R^f \tag{2.21}$$

$$R^f = \frac{1}{E(m)} \tag{2.22}$$

To analyze the implication of equation 2.22 above, I take the utility function with constant relative risk aversion (CRRA),

$$u(c) = \frac{c^{1-\delta} - 1}{1-\delta} \qquad \delta \neq 1$$

Based on this utility function and its derivative $u'(c) = c^{-\delta}$, I sketch the risk-free rate as,

$$R^f = \frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\delta}. \tag{2.23}$$

With the equation 2.23 outlined above, I can say that:

1. If the subjective discount rate β be high and, consequently, it value be low to cause a larger discount, the economy will have a high interest rates to convince individuals with higher impatient rate to saving today, in order to consume more in the future.

2. The higher is the difference between c_{t+1} and c_t, the greater will be the impact on interest rates. If the consume possess a sharp grown rate, the economy also will have higher interest rates. High interest rates increase the intertemporal substitution of consume between today and future period, it is a stimulus for the postponement of the consumption to the future.

3. The interest rates is an increase function of the parameter δ that represent the risk aversion. The higher is δ, the greater will be the curvature of the utility function and the risk aversion, accordingly, this type of investor look for maintain a pattern of consume with smooth shifts, at that fashion, is necessary a high interest rates to convince him to substitute today's consume to tomorrow's one.

2.5 RISK CORRECTION

The covariance can be represented by the following manner: $cov(m, p_{t+1}) = E(mp_{t+1}) - E(m)E(p_{t+1})$ and how $p = E(mp_{t+1})$, isolating to p_{t+1}, I have,

$$p_t = E(m)E(p_{t+1}) + cov(m, p_{t+1})$$

How the risk-free rate is given by $R^f = \frac{1}{E(m)}$, then,

$$p_t = \frac{E(p_{t+1})}{R^f} + cov(m, p_{t+1})$$

The first term is a very useful form of calculation the net present value. If the investors has a linear utility function and consequently, are risk neutral, the price would be obtained based on the expected value of the payoff, carried out until the present time by the risk-free rate. The risk neutrality imply that the agents are indifferent with relation to the volatility and what matter on his decision-making, is the expected value of the payoff.

The second term of the equation above is the risk correction. The risk correction of an asset is given by the covariance between the asset's payoffs with the marginal utility of consumption. An asset with a positive correlation with consumption is less valued than the other that have none or negative correlation. To understand this logic, suppose a recession in which the consumer have his financial situation worsened, if he own an asset whose value fall in recession, then, he will be in the worst situation than if he own an asset that is not correlated with consumption.

Assets that have negative correlation with consume are more valued because it can be employed in the portfolio to diminish its variance, preserving its expected value. This type of asset have a net present value bigger than its payoffs in the future discounted by the risk-free rate, for example, a holder of an insurance might have his purchase power recovered, upon a sinister has affected his wealth, so, the insurance's payoff and the consume must have negative correlation. For this reason, people buy insurance, despite the premium charged by the insurance companies is greater than the expected value of the payoff discounted by the interest rates.

Considering the basic equation of the expected returns: $1 = E(mR^i)$ and applying the decomposition of the covariance, I get,

$$1 = E(m)E(R^i) + cov(m, R^i)$$

Substituting by the risk-free rate,

$$1 = \frac{E(R^i)}{E(R^f)} + cov(m, R^i)$$

And isolating the return,

$$\begin{aligned}
&(1 - cov(m, R^i))R^f = E(R^i) \\
&E(R^i) - R^f = -R^f cov(m, R^i) \\
&\frac{E(R^i)}{R^f} - 1 = -cov(m, R^i)
\end{aligned}$$

How, in general, the investor is risk averse, the assets have an expected rate of return that is equal the risk-free rate plus a risk adjustment, this excess of return is equal to $-cov(m, R^i)$. It is worth mentioning that the risk correction is not a function of the volatility $\sigma^2(p_{t+1})$, what matter here is the covariance between the stochastic discount factor and the payoff, if $cov(m, p_{t+1}) = 0$, then $p_t = E(p_{t+1})/R^f$.

2.6 RISK SHARING

As provided by Ross (2013), only the risks that cannot be avoided with the diversification through assets portfolio are priced, then only aggregate macroeconomic shocks matter for risk prices. Part of the empirical research in macroeconomics seek to identify shocks to quantify its impact on the economics variables, meanwhile, the asset pricing model point out prices for the exposition to this shocks, because its can cause impact on the cash flow of the investment.

In accordance with Cochrane (2009), assume an economy whereby the marginal rate of substitution to any investor is equal to the price of the contingent claim, since the price of the contingent claim is the same for all investor, we get the equality below to investor i and j,

$$\beta^i \frac{u'(c^i_{t+1})}{u'(c^i_t)} = \beta^j \frac{u'(c^j_{t+1})}{u'(c^j_t)} \tag{2.24}$$

If the investor has the homothetic utility function, then, the consume grow up at the same reason among them.

$$\frac{c^i_{t+1}}{c^i_t} = \frac{c^j_{t+1}}{c^j_t}$$

This equality say that the variation rate of the consume between time t and t+1 is the same for all individuals, then, consumer's shocks effect is proportionally equal among them. In a complete market of contingent claim, everybody sharing all risks, at this way, when someone are affected for a shock in his income, this shock affect all the other in the same proportion. I am not saying that the consume is the same for all individuals, but that the variation between the two periods are the same, if one grow up 10%, the other perceive an elevation at 10%.

With this suppositions, any idiosyncratic risk that can affect the income is sharing among the investors, the existence of the contingent claim is that permit this sharing, working as type of insurance. The unique risks that cannot be insurable, are the macroeconomics risks, because it would be impracticable insure an asset's portfolio against all macroeconomics risks, the price of these contingent claim would be extraordinarily high. Like this, only aggregate shocks should matter, such that, the stochastic discount factor m, that priced all assets, is not affected by the idiosyncratic risk.

Any payoff can be decomposed in two parts, the first one correlated with the stochastic discount factor (SDF) and the second one that is not correlated. How, in general, the SDF is estimated through macroeconomic variables, then, the part that is not correlated with the SDF account for the idiosyncratic risk, are the suppressed variables in the model that has significant correlation with the payoff.

Only the systemic risk matter in the pricing function, the idiosyncratic risk not affect prices. Consider that ϵ embody the suppressed variables in the model and that has correlation with the payoff, then, $E(m\epsilon) = 0$.

The assumption exposed above, about risk sharing and the null pricing of the idiosyncratic risk, are very strong and cause profound implications on the asset pricing models. Nonetheless, in real world there is no complete markets, as well as complete risk sharing, as a result, this issue motivated a vivid debate at the literature about the empirical evidence related for the pricing of the idiosyncratic risk. In accordance with **??**), this type of risk is priced. Campbell et al. (2001) decomposed the stock market return in tree components: the return originated by the market, a residual return due the firm's industry and the other due merely by the firm's specificities, the authors conclude that there was in the U.S stock market, between 1962 to 1997, a tendency of growth in the idiosyncratic volatility.

An argument that underpin the thesis that the idiosyncratic risk matter, is related of the absence of investment's diversification in the manner recommended by the economic theory. The investor's portfolio are affected by the idiosyncratic volatility, not only because of the incomplete market and the absence of hedge for this type of risks, but due to the absence of the diversification. In accordance with Campbell et al. (2001), the number of stocks necessary to obtain a certain level of diversification have been increased, what become increasingly difficult to individual investor and resource managers stand exempted of the idiosyncratic volatility.

However, the assumption that prices are only affected by macroeconomics factors draws attention to the importance for the financial instruments that aid the risks sharing. The innovation and creativity observed in derivative markets has helped banks, funds and individuals investors to protect their assets and some of the forces that drives this movement is a better risk sharing.

3 INTEREST RATE DERIVATIVES MARKET

In this section, I outline the characteristics of the derivatives evaluated in this study. In the complex of interest rate derivatives traded in the B3[1], I select only the instrument related to the DI rate, that are the most traded. In this category, are included: one-day interbank deposit futures, the option on IDI index and the option on DI futures.

3.1 ONE-DAY INTERBANK DEPOSIT FUTURES

The One-day Interbank Deposit Futures, known as DI Futures, is a derivatives contract whose object of negotiation is the interest rate, comprised between the trading date and the maturity date of the contract. The product has notional value of R$ 100.000,00 at the maturity (each contract correspond to 100.000 points and each point amount to R$ 1.00). This is an operation based on the expectations of the future behavior of the one-day interbank deposit rate (DI rates, calculated by CETIP Custody and Settlement and expressed as a percentage rate per anuum compounded daily based on a 252-day year) verified in the period between the trade date and the last trading date of the contract.

DI futures contract is used to hedge and manage risk exposures of liabilities, like debts and risk of assets, like government bonds. DI rate is based on interbank transactions, as the same way of the Libor rate. Because of the great liquidity that it has, constitutes a reference to the Brazilian economy, which expresses the expectations about the behavior of interest rates for future periods.

DI futures is one of the most volatile interest contract in the world, which bring many opportunities for trading. This allows speculation and arbitrage opportunities across the entire term structure of interest rates, moreover, this type of product are used for the hedging transactions, because the DI futures contract protect investment funds, companies and individuals investor from changes in interest rates, reducing the exposure to floating or fixed rates.

Future contracts has as one of the main characteristics, the margin account, that is used to cover eventual losses of the operation. To buy a future contract of any commodity or index, there is not a cost at the moment of the purchase, therefore, is necessary deposit a guarantee margin the will be used to allow the investment with controlled risk. The daily cash flow of the margin account can be expressed as,

[1] that include: Extended Consumer Price Index (IPCA) Futures, Futures on the Average Rate of One-Day Repurchase Agreements, Options on the Index of the Average Rate of One-Day Repurchase Transactions, DI x IPCA Spread Futures, DI x U.S. Dollar Spread Futures, Structured Transactions of Forward Rate Agreement on DI x U.S. Dollar Spread, DI x U.S. Dollar Swap, U.S. Dollar Spread Futures Contract Referencing One-Day Repurchase Agreements, Forward Rate Agreement on One-Day Repurchase Agreements X U.S. Dollar Spread, U.S. Dollar Swap with Reset Referencing One-Day Repurchase Agreements

$$MCF_t^T = CP_t^T - TP_t^T$$

MCF is the cash flow of the margin account accounted at the date t, but paid on the next day. The CP is the unitary price-PU at the date of entry in the future contract, adjusted by the rate traded at the initial moment (between date t and T). The TP is the PU at the date of entry in the future contract, adjusted by the DI rate (between date t and the date T).

I consider a continuous trading economy with a continuum of default free discount bonds, trade with different maturities. The bonds are trading at the dates $T \in [0, \tau]$. $P(t, T)$ express the bond's price at the time $t \in [0, T]$, that has face value of R$ 1,00 at the maturity, and T is the date of bond's maturity. I require that $P(T, T) = 1$ for all $T \in [0, \tau]$, $P(t, T) > 0$ for all $T \in [0, \tau]$ and $t \in [0, T]$.

The unit price (PU) of the one-day interbank deposit futures contract is defined by,

$$PU(t, T) = \frac{100,000}{(1 + r/100)^{(T-t)/252}} \tag{3.1}$$

Where r is the interest rate contracted; the difference T-t, represent the reserve day, comprised between the trading date (t), inclusive, and the maturity date of the contract (T), exclusive; reserve day is a business day for the purposes of financial market transactions, as established by the National Monetary Council. It is highlighted that the DI futures contract has value of R$ 100.000,00 at the maturity, then, the PU represent the R$ 100.000,00 discounted by the interest rate contracted.

The bank calendar in the Brazil use annual periods of 252 business day and the capitalization of the DI futures contract occurs only at the business day verified between the trade day and the maturity date, inclusive.

To obtains the annual forward interest rates with annual capitalization, I apply the following equations, to $t \leq T_i \leq T_j$,

$$f(t, T_i, T_j)^{annual} = \left(\frac{PU(t, T_i)}{PU(t, T_j)}\right)^{\frac{252}{T_j - T_i}} - 1 \quad \text{for all } T_i \text{ and } T_j \in [0, \tau],\ t \in [0, T] \tag{3.2}$$

Where T_i and T_j represent the maturities dates of the contract. $PU(t, T_i)$ is the unit price at time t of a contract that has maturity at T_i. Thus, $f(t, T_i, T_j)$ is the term rate at the time t between the dates T_i and T_j.

To simplify the equations and facilitate the numerical implementations of the model, I use all interest rates with continuous time capitalization. Therefore, I use the following expressions to get the annual forward rate with continuous time capitalization.

$$f(t,T_i,T_j)^{continuous} = \log(1+f(t,T_i,T_j)^{annual}) \quad \text{for all } T_i \text{ and} T_j \in [0,\tau], \quad t \in [0,T] \quad (3.3)$$

I apply the expression 3.3 for all historic series, therefore, how the maturity of the DI futures contract is always at the first business day of the month, the number of business day up to maturity are different between the several trading dates, such that, becomes necessary perform a interpolation scheme to align the rates to the same number of business day up to maturity. I perform the piecewise polynomial interpolation (cubic spline) to present a smooth curve, already that the curve with breaks and spikes allow the arbitrage opportunities, moreover, is more feasible assume that the forward rates curve does not present discontinuities.

With the DI rates for the diverse maturities of the DI futures contract, $f(t,T_i,T_j)$, one obtains the bond prices that pay R$ 1,00 at the maturity,

$$P(t,T_j) = \frac{1}{\exp\left(f(t,t,T_j)\frac{T-t}{252}\right)} \qquad \text{for all } T_j \in [0,\tau], \quad t \in [0,T] \qquad (3.4)$$

The spot interest rates at time t, $r(t)$ is the forward rate at time t for the maturity t, embody the instantaneous interest rate practiced on the trade date t,

$$r(t) = f(t,t,t) \text{ for all } t \in [0,\tau] \qquad (3.5)$$

The DI futures contract also can be expressed by the following manner, separating the fixed part from the float part,

$$PV\_fixed = \sum_t C\delta_t PO_t^{fixed}\Lambda_t = C\sum_t \delta_t PO_t^{fixed}\Lambda_t$$
$$PV\_float = \sum_t r_t\delta_t PO_t^{float}\Lambda_t$$

How,

$$PV\_fixed = PV\_float$$

Then,

$$C = \frac{\sum_t r_t\delta_t PO_t^{float}\Lambda_t}{\sum_t \delta_t PO_t^{fixed}\Lambda_t}$$

Where:

- PV_fixed is the fixed leg of the future contract and PV_float is the float leg of the future contract;
- PO_t^{fixed} is the the trading price in PU, updated up to time t by the traded interest rate and PO_t^{float} is the PU updated up to time t by the effective DI interest rate;
- C - Contracts rate (fixed leg);
- Λ_t - discount factor for payment date t and r_t - forward rate (floating rate of future payment)
- δ_t - day count fraction;

3.2 OPTIONS ON INTEREST RATES

In this section, I described some of the main characteristics of the options evaluated in this work, including: the option on DI futures and the option on IDI index.

3.2.1 Option on One-Day Interbank Deposit Futures

At the exercise day of the contract, the buyer and the seller receive a position in the one-day interbank deposit futures contract. The buyer receive a long position on the future contract and the seller receive a short position on the future contract. Basically, this is an instrument of negotiation of forward rates, already that the underlying asset of the option is the future contract with maturity at the date subsequent to the maturity of the option.

Let T_1 be the exercise date of the option and T_2 the maturity date of the future contract, the option's *payoff* can be expressed in T_1 by the following manner,

$$payoff\,[T_1] = Q.max\left(cp.\left[FUT_{T_1,T_2} - \frac{100.000}{\exp\left(k.\left(\frac{T_2-T_1}{252}\right)\right)}\right], 0\right)$$

The equation above was adapted of Carreira e Jr (2016), to represent the payoff with the interest rate in continuous time. Where: i) Q: contract quantity; ii) max(A,B): operator that compute the maximum value between A and B; iii) cp: variable that define if the option is of the type *call* or *put*; iv) FUT_{T_1,T_2}: is the notional value of future contract at T_1. Has value of R\$ 100.000 at the maturity, before the maturity, its value is determined by the market, in accordance with the expectation of interest rates between the actual date and the maturity date; v) k: strike[2] price, defined as rate with continuous capitalization; vi) $T_2 - T_1$: business days between T_1 and T_2, counted at the base 252 business days, according to bank calendar.

[2] strike price (or exercise price) is the fixed price at which the owner of the option can buy (in the case of a call), or sell (in the case of a put), the underlying security

Under the classification of B3, the rule of functioning of the option depend of the type: i) Type 1 (D11): when the object of the option is the future contract with maturity in 3 months after the maturity of the option; ii) Type 2 (D12): when the object of the option is the future contract with maturity in 6 months after the maturity of the option; iii) Type 3 (D13): when the object of the option is the future contract with maturity in 1 year after the maturity of the option; iv) Type 4 (D14): when the object of the option is the future contract with maturity specified by the exchange securities.

How the underlying asset has expiration date posterior to the derivative maturity, what is negotiated is a forward rate, between the maturity of the option and the maturity of the future contract. This contract expose the investor to the parallel and slope shift of the term structure of interest rates.

It is worth noting that this options are european type, the exercise occur only at the maturity date, when the price of the future contract of DI are greater than the exercise price, for call options; or below to the exercise price, for put option, of automatic form. It is noteworthy to point out that the price here is defined as rate.

With a little of algebra is possible transform the *payoff* above to a format similar to the *swaption*, in order to simplified its estimations computationally,

$$payoff\left[T_1\right] = Q.100.000.max.\left(cp.\left[\frac{1}{\exp\left(r.\left(\frac{T_2-T_1}{252}\right)\right)} - \frac{1}{\exp\left(k.\left(\frac{T_2-T_1}{252}\right)\right)}\right],0\right) \tag{3.6}$$

Where r is the annual rate with continuous time capitalization between T_1 and T_2, k is the exercise price of the option (published in rates by the B3).

$$payoff\left[T_1\right] = Q.K.max\left(cp.\left[\frac{\exp\left(k.\left(\frac{T_2-T_1}{252}\right)\right) - \exp\left(r.\left(\frac{T_2-T_1}{252}\right)\right)}{\exp\left(r.\left(\frac{T_2-T_1}{252}\right)\right)}\right],0\right) \tag{3.7}$$

Where,

$$K = \frac{100.000}{\exp\left(k.\left(\frac{T_2-T_1}{252}\right)\right)}$$

Is possible to obtain a simplification through the conversion of exponential rates to linear rates,

$$\exp\left(k.\left(\frac{T_2-T_1}{252}\right)\right) = 1 + k_l.\left(\frac{T_2-T_1}{252}\right) \tag{3.8}$$

$$\exp\left(r.\left(\frac{T_2-T_1}{252}\right)\right) = 1 + r_l.\left(\frac{T_2-T_1}{252}\right) \tag{3.9}$$

Substituting (3.8) and (3.9) on the numerator of (3.7),

$$payoff\left[T_1\right] = Q.K.\left(\frac{T_2 - T_1}{252}\right).max\left(cp.\left[\frac{k_l - r_l}{\exp\left(r.\left(\frac{T_2-T_1}{252}\right)\right)}\right], 0\right) \tag{3.10}$$

The last step consist in obtain the *payoff* on T_2 and from the multiplication of the factor of adjustment between the two periods $\exp\left(r.\left(\frac{T_2-T_1}{252}\right)\right)$ by the payoff in T_1,

$$payoff\left[T_2\right] = Q.K.\left(\frac{T_2 - T_1}{252}\right).max\left(cp.\left[k_l - r_l\right], 0\right) \tag{3.11}$$

The rate r_l represent the DI rate linearized, simulated between T_1 and T_2, already the rate k_l, is the conversion of the *strike* of the option to the linear rate. Therefore, the expression 3.11 above, denote the payoff of the option at the maturity date of the future contract.

Below, Ideveloped equations to show the present value of this type of options, separating the fixed from the float part,

$$\begin{aligned}
PV_fixed &= \sum_t C\delta_t PO_t^{fixed}\Lambda_t = k\sum_t \delta_t PO_t^{fixed}\Lambda_t \\
PV_float &= \sum_t r_t\delta_t PO_t^{float}\Lambda_t
\end{aligned}$$

Where:

- PO_t^{fixed} is the trading price in PU at the date T_1, of a contract that has maturity in T_2, updated up to time t by the contracts rate;
- PO_t^{float} is the PU at the date T_1 with the interest rates between T_1 and T_2 equal to strike price (defined as interest rate, updated up to time t by the effective DI interest rate);
- k - strike price defined at interest rate;
- Λ_i - discount factor for payment date i and r_i - forward rate (floating rate of future payment);
- C - Contract rate (fixed leg);
- δ_i - day count fraction.

And,

$$PO_t^{fixed} = \frac{100,000}{\left(1+\frac{C}{100}\right)^{T_2 - T_1/252}}.\exp(C\delta_t)$$
$$PO_t^{float} = \frac{100,000}{\left(1+\frac{k}{100}\right)^{T_2 - T_1/252}}.\exp(r_t\delta_t)$$

3.2.2 Option on The Average One-Day Interbank Deposit Rate Index (IDI)

The object of negotiation is the DI average rate, comprised between the purchasing date of the option and the maturity date. The average one-day interbank deposit rate index (IDI) is an index daily updated by the One-Day Interbank Deposit Rate (DI rate), the index actually used has base date 01/02/2009 and value in this date of R$ 100000.00. The expression 3.12 that follow, express the correction form of index, that have daily capitalization, in accordance with Carreira e Jr (2016),

$$IDI_T = IDI_{basedate} \prod_{t_i=basedate}^{T} (1 + CDI_{t_i})^{\frac{1}{252}} \tag{3.12}$$

This instrument resemble the instruments highly used in international markets to protect against fluctuations in interest rates, known as caplet and floor. Combining many options, are possible create strategical figures that allow the protection or speculation of the monetary policy. Unlike to the future contract, do not exist cash flow between the parts, relating daily adjustment.

For an option with maturity at the date T, the payoff is given by,

$$payoff_{T+1*} = Q.max\left(cp.\left[IDI_T - K\right], 0\right) \tag{3.13}$$

Where i) Q: quantity of contracts; ii) max(A,B): operator that compute the maximum value between A and B; iii) cp: variable that define if the option is of the type *call* or *put*; iv) IDI_T : is the IDI index value with maturity at the date T. It is worth highlighting that the DI rate of any day is released at the end of the same day, therefore, the index of the date T is obtained only in T+1; v) K: option strike, established in value.

The *payoff* also can be express by the following manner,

$$payoff_{T+1*} = Q.max\left(cp.\left[IDI_t.\exp\left(r.\left(\frac{T-t}{252}\right)\right) - K\right], 0\right) \tag{3.14}$$

The equation above was adapted of Carreira e Jr (2016), to represent the payoff with the interest rate in continuous time. Where IDI_t is the index value at the date t and r is the annual interest rate with continuous time capitalization, comprised between t and T. Already $T - t$, are the business day comprised between the dates t and T.

4 THE VALUATION MODELS

4.1 BLACK MODEL

The basic model of option valuation in the fixed income market is the Black (1976), where the price is quoted in terms of the implied volatility. For illustration, below follows a description of the model for the option on DI futures, assuming that the linear interest rates between T_1 and T_2, r_l, has log-normal distribution with volatility expressed of annual form, denoted by σ.

$$c = D^{CDI}_{t,T_2}.K.\left(T_2 - T_1\right).\left(r^l_{T_1,T_2}.N(d1) - k^l_{T_1,T_2}.N(d2)\right) \tag{4.1}$$

$$d1 = \frac{ln\left(\frac{r^l_{T_1,T_2}}{k^l_{T_1,T_2}}\right) + 0.5.\sigma^2.\left(T_2 - T_1\right)}{\sigma.\sqrt{\left(T_2 - T_1\right)}}$$

$$d2 = \frac{ln\left(\frac{r^l_{T_1,T_2}}{k^l_{T_1,T_2}}\right) - 0.5.\sigma^2.\left(T_2 - T_1\right)}{\sigma.\sqrt{\left(T_2 - T_1\right)}}$$

Where,

- c: option premium
- D^{CDI}_{t,T_2}: discount factor between the date t and T_2, based on the curve of DI future;
- K = $\frac{100.000}{\exp(k.(T_2-T_1))}$;
- k: option strike, defined in rate;
- σ: volatility;
- $T_2 - T_1$: business day, considering the bank calendar, between the maturity of the option and the future contract subjacent for the option;
- $k^l_{T_1,T_2} = \frac{\left(\exp\left(\left(\log\left(1+k\right).\frac{(T_2-T_1)}{252}\right)\right) - 1\right)}{\frac{(T_2-T_1)}{252}}$;
- $r^l_{T_1,T_2}$: linear interest rates between T_1 and T_2. How it is a rate based on DI curve and that comprehend a future period, this rate is obtained from the curve of future interest rates.

To obtain $r^l_{T_1,T_2}$, firstly is necessary adjust, of t to T_1, the bond price at time t that has face value of R\$ 1,00 in T_2, namely, $P(t, T_2)$,

$$P(T_1, T_2) = P(t, T_2)\exp\left(f(t,t,T_1).\left(\frac{T_2 - T_1}{252}\right)\right) \tag{4.2}$$

What the expression 4.2 make is adjust a bond that expire in T_2, from the date t up to date T_1. Based on the bond price at the date T_1, is possible calculate the interest rate, with continuous capitalization, between T_1 and T_2,

$$r^{continuous}_{T1,T_2} = \log\left(\frac{1}{P(T_1,T_2)}\right)\cdot\frac{252}{(T_2-T_1)} \tag{4.3}$$

Finally, to get the linear rate that are used in the expression 4.1, are necessary applies the equation below,

$$r^{linear}_{T_1,T_2} = \left(\exp\left(r^{continuous}_{T1,T_2}\cdot\left(\frac{T_2-T_1}{252}\right)\right)-1\right)\cdot\frac{252}{T_2-T_1} \tag{4.4}$$

4.1.1 Implied Volatility

Using the Black model, is possible to get the implied volatility from the traded prices. How the Black model quote the prices depart of the implied value of σ, which sets the model price equal to the market price, to obtain the implied volatility is necessary invert the model. To do this, I used a genetic algorithm that is characterized by having three main stages: selection, crossover and mutation. The appendices contain more details about the algorithm.

4.2 STRING MARKET MODEL

For Longstaff, Santa-Clara e Schwartz (2001b), is necessary the development of economically well-specified models, grounded on the real dynamic of the term structure and that embody multiples factors, for the optimal strategies of hedging and investments can be achieved. To implement a model that capture the dynamic of the term structure of DI interest rates, I adapted the LSS model (that are based on the Libor rate) to fit the entire DI rate curve. Accordingly, the term structure is driven by multiple factors, then, the use of models with a single factor to pricing interest rates derivatives can produce hedging and investment strategies sub-otimals. The widely famous Black (1976) model, that employ a single factor, is unable to capture the volatility smile. This expression is employed to denote the plotting result of the implied volatility and the strike, that look like a smile, however, The Black model predict that the implied volatility curve will be flat.

Goldstein (2000), Longstaff, Santa-Clara e Schwartz (2001a) and Santa-Clara e Sornette (2001) shaped the evolution the term structure as a stochastic string. In this approach, that is a generalization of the model of Heath, Jarrow e Morton (1992), is possible to generate a dynamic and shape of the interest rate curve much more rich. The main innovation consist in which each forward rate $f(t, T_i, T_j)$ behave as a distinct random variable, that has its own dynamic, although must be correlated with the other points of the curve.

Below follow the dynamic of the evolution of the forward rate on the risk neutral measure,

$$dF_i = \alpha_i F_i dt + \sigma_i F_i dZ_i \tag{4.5}$$

Where α_i is a not specified *drift*, σ_i is a deterministic volatility function and dZ_i is the brownian motion.

Although the model is specified in terms of the forward rate, I consider more efficient implement it using the vector of the bonds prices that pay R$ 1 at the maturity. With this prices, is possible obtain the forward rate with continuous time capitalization in accordance with the expression below,

$$F_i = log\left(\frac{P(t, T_i + \tau)}{P(t, T_i)}\right) . \frac{\tau}{252} \tag{4.6}$$

Where τ denote a certain quantity of business day. How I choose a discretization of the data based on the difference between the expiration date of the DI futures, τ express the difference in business day between the maturity $T_i + \tau$ and T_i.

The model has as fundamental characteristic the evolution of the bonds prices, based in a stochastic differential equation with two terms, the *drift* extracted from the spot interest rates and a second term, *diffusion*, basically composed by the multiplication of a jacobian matrix with a matrix of implied covariance (both will better explained below). Applying the Itô's rule to the vector P of the bond prices, obtains its dynamic evolution (in appendices are demonstrated the derivation), that is estimated under the neutral risk measure,

$$dP = rPdt + J^{-1}\sigma F dZ \tag{4.7}$$

Where r is the spot interest rate[1] (is the rate comprised between t and t+1, is the equivalent to the yield of the bond with the shorter maturity in each trading date), $\sigma F dZ$ is a vector formed by the terms $\sigma_i F_i dZ_i$ of the equation 4.5 above, J^{-1} is the inverse jacobian matrix obtained through the derivation of the forward $(F_1, F_2 ... F_{29})$ with respect to the discount bond $(P_1, P_2 ... P_{30})$. How the maturity of the option evaluated in this study occurring up to thirtieth maturity of the DI future contract, are necessary only thirty different maturities to build the model, then, the jacobian matrix is represented below, with diagonal values, since each forward rate depend only of the two discount bond to be obtained.

[1] T_0 is considered the initial maturity date, then $\sigma(T_i - T_0) = 0$ for $T_i \leq T_1$, where T_1 is the first maturity of the bond, therefore, it don't affect the diffusion term at the equation 4.7

$$J = \begin{bmatrix} -\frac{P(1)}{P^2(2)} & 0 & 0 & ... & 0 & 0 & 0 \\ \frac{1}{P(3)} & -\frac{P(2)}{P^2(3)} & 0 & ... & 0 & 0 & 0 \\ 0 & \frac{1}{P(4)} & -\frac{P(3)}{P^2(4)} & ... & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{P(29)} & -\frac{P(28)}{P^2(29)} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{P(30)} & -\frac{P(29)}{P^2(30)} \end{bmatrix}$$

The dynamic of P, provided above by the equation 4.7, give a complete specification to the evolution of the term structure and is based with this expression that I simulated the bond prices for future periods and from this prices, I extract the forward rates between each period. Therefore, the expectation of return of all bonds is equal to the spot rate under the risk neutral measure (at this measure, there is no return associating with the risk premium, the expected return of an asset is equal to the risk-free rate).

From the vector of the negotiated rates at a particular date in the DI futures market, is possible draw the vector of the bond prices, the term rate between the maturity of the future contract, the Jacobian matrix and the spot rate. The only matrix that is necessary to be estimated for the other source of data is the instantaneous matrix of variance-covariance, which is estimated with the use of the implied volatility of the traded options.

4.2.1 The Model is Arbitrage-Free

In this economy there exist an accumulation factor that begin with a value corresponding to R$ 1.00 and is adjusted by the short term interest rate.

$$B(t) = \exp\left(\int_0^t r(y)dy\right) \tag{4.8}$$

Let $Z(t,s) = \frac{P(t,s)}{B(t)}$ be the relative price of the bond. In agreement with Harrison e Kreps (1979), if exist a equivalent martingale measure P^*, hence, there is not arbitrage opportunities. In this equivalent probability measure P^*, $(Z(t,s_1), Z(t,s_2)...Z(t,s_n))$ are martingales.

Now, consider the system of equations below, $b(t,T)$ represent the excess return above the risk-free rate of the bond with maturity at T, γ_i represent the risk price of the factor "i" and the $a_i(t,T_j)$ represent the covariance between the bond's (that has maturity in T_j) return with the i-th factor. If exist a equivalent martingale measure P^*, then, the necessary condition denoted by the equation below remains valid, even more, if the covariance matrix is nonsingular, then, the probability measure P^* is unique, for a complete demonstration to this association, see Heath, Jarrow e Morton (1992).

$$\begin{bmatrix} b(t,T_1) \\ \vdots \\ b(t,T_n) \end{bmatrix} = \begin{bmatrix} a_1(t,T_1) & \cdots & a_n(t,T_1) \\ & \vdots & \\ a_1(t,T_n) & \cdots & a_n(t,T_n) \end{bmatrix} \times \begin{bmatrix} \gamma_1(t;T_1,...,T_n) \\ \vdots \\ \gamma_n(t;T_1,...,T_n) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

In a world with risk neutral investors, the risk price that is represented by γ is null, considering that P^* represent the risk neutral measure, is necessary to clean up of the evolution of the bond prices the risk premium. In the expression 4.7, that represent the evolution of the prices under P^*, the *drift* only contain the spot interest rate, considered here as the risk-free rate, then, the expected rate of return on all bonds is equal to the spot rate under the risk neutral measure. Moreover, the string market model is able to replicate of the exact form the initial term structure, with this, the model is arbitrage free.

4.2.2 Implied Covariance Matrix

The instantaneous variance-covariance matrix used in the model are obtained in a date t, with the use of the prices of the options on one-day interbank deposit futures traded at the same date t. Any covariance matrix is symmetric and positive semidefinite[2] by virtue of this necessary and sufficient condition, I use the procedure below to estimate the matrix.

The historic correlations matrix can be decomposed by the following manner $H = U\Lambda_0 U'$ where Λ_0 is a diagonal matrix of eigenvalues (non-negative) and the N columns of U correspond to the eigenvectors. On the assumption that the instantaneous variance-covariance matrix of the interpolated interest rates percent variations[3] share the same eigenvectors of the historic correlations matrix, the implied instantaneous covariance matrix, Σ_t, can be expressed of the following manner,

$$\Sigma_t = U_t \Psi_t U_t' \tag{4.9}$$

Where Ψ is a diagonal matrix with non-negatives values, that represent the eigenvalues of the Σ matrix that best fit the model to the market date. The i-th diagonal element of Ψ can be interpreted as the instantaneous variance of the i-th factor that command the evolution of the interest rates at the date t.

---

2 A matrix is positive semidefinite if, and only if, its eigenvalues are all positive.

3 the percent variations matrix is obtained through the matrix of interpolated forward rates, which is drawing for the variation of the rate at the date T_j subtracted of the rate in T_{j-1}, divided by the rate at the date T_{j-1}

Considering that the eingevectors of the historic correlation matrix are view as factors, then, the same factors that generate the historic correlation matrix, also generate the implied covariance matrix, Σ_t, nonetheless, the variance of these factors differ from the historic values.

To obtain the implied covariance matrix, is necessary build the matrix Ψ, with values that best fit the model prices with the market prices. The elements that make up the diagonal matrix Ψ are values that minimize the root mean squared error (RMSE) of the objective function, in other words, values that minimize the percent difference between the market prices and model prices of these options.

An analytic solutions to Ψ becomes quite complex, given the stochastic nature of the model. The solutions via simulations is a inviable alternative because of the time required to reach out the convergence. To obtains Ψ, I use a numeric solutions that imply on obtain the historic covariance matrix of the forward rate percent variations and, henceforth, substitute the diagonal elements of this matrix by the square of the implied volatility for the option on DI futures, after this, I have the new covariance matrix Ω.

By the inversion of the Black model, is possible achieve different values for volatility of options with the same maturity. Because this, I developed a genetic algorithm to find a unique volatility that minimize the difference between the Black model prices and the market prices for many options that have the same maturity. After the attainment of these volatility, I substitute its by the respective element in the diagonal of the historic covariance matrix.

Therefore, through an iterative process that compare, for a particular maturity, the real prices of the options and the prices obtained by the Black model, the volatility are adjusted, such a way that the price obtained must be closest as possible of the market price. This iterative process are realized for each maturity separately, with the objective to find for each maturity date of the option on DI futures, the implied volatility that best fit the market price.

Subsequently, after the substitution of the historic variance by the variance that best fit the Black model to market prices, I use the singular value decomposition-SVD to extract the eigenvalues of the new covariance matrix, $\Omega_t = USV^T$ where S is the diagonal matrix with the eigenvalues. The decomposition SVD consist to find the eigenvectors and eigenvalues of $\Omega_t\Omega_t'$ and $\Omega_t'\Omega_t$. The eigenvectors of $\Omega_t\Omega_t'$ are in the columns of V and the eigenvalues of $\Omega_t'\Omega_t$ are in the columns of U, already the singular values of S represent the square root of the eigenvalues of $\Omega_t\Omega_t'$ and $\Omega_t'\Omega_t$.

I utilize the larger eigenvalues of the diagonal matrix S to build the Ψ matrix, although 30 eigenvalues are necessary to obtain a full rank matrix, the implied covariance matrix with lower rank can be applied, only is need to define N eigenvalues to form Ψ and leave the 30-N elements of the diagonal with zero value.

After obtains the elements of the diagonal matrix Ψ, I build the covariance matrix $\Sigma = U\Psi U'$ and then simulate 2.000 paths of the vector of bond prices until the maturity, where these prices evolve accordingly with equation 4.7. To reduce the variance of the simulated values, I utilize antithetic variates. To simulate the evolution of the bond prices between the beginning date and the first maturity of the DI futures contract, I use the complete matrix Σ and J in the version 30 by 30, afterward, the first forward rate becomes the spot rate, leaving only 29 bonds to being simulate for the next period, whereas the bond with the shorter maturity achieve its expiration date and obligatorily has face value of R$ 1.00 at this date. Therefore, to maintain the time homogeneity of the model, is necessary exclude the first line and columns of Σ and J, this process is repeated up to maturity of the longer bond.

With the matrix of the bond prices simulated, is possible found the forward rates between the various maturities. To obtain the price of the option on DI futures, are necessary achieve the forward rate between T_1 and T_2, in accordance with 4.6, and this rate is taken from the bond price that has it expiration date in T_2 at the date T_1. With this price is possible found the forward rate between T_1 and T_2, that comprehend the underlying traded object of these options. With the forward rate achieved accordingly with expression 4.6, is possible obtain the rate r_l that compose the equation 3.11.

After to get the payoff at T_2, are necessary bring it to the date t, the trading date. Then, I multiply the payoff at T_2 by the stochastic discount factor accumulated among the dates t and T_2: $\prod_{i=0}^{T_2-1} P(i, i+1)$. Afterwards, I simulate 2.000 paths, in accordance with the steps described above, and finally, I take the average of the prices over all paths.

5 EMPIRICAL RESULTS

In this chapter are present the empirical results about the time series of the derivatives evaluated in this study, as well as, the likelihood ratio test and evidences about the performance of the string market model. In conducting this study, I used tree types of different financial instruments: the DI futures contract, the option on IDI index and the option on DI futures. All data are obtained from the B3 site[1].

5.1 EMPIRICAL RESULTS ABOUT THE TIME SERIES OF THE DERIVATIVES

The sample of the DI futures contract consist on weekly observations, from oct/15 to jan/19, preferentially at friday after the market has closed. Contain 164 different days and each day has, approximately, 40 maturity date. The data are available by B3 in unit price-PU.

The sample of the option on DI futures comprehend the period of 11/03/2017 to 01/18/2019 (61 weeks), with all the options of the sample at the money[2]. The sample contain 2.196 options, being 36 for each week. The sample of the option on IDI index also comprehend the period of 11/03/2017 to 01/18/2019 (61 weeks). Altogether, were 6.100 options evaluated, being 100 for each day. Also all the options are at the money.

With the times series of interpolated DI interest rates, the graph 1 describe the futures interest curve between oct/15 until jan/19. In the graph is possible to note the fall of the short-term interest rates, however, despite the fall of the long-term interest rate, this decline was significant lower. The Central Bank usually handles the short part of the curve, then, the long-term rates tend to be less susceptible to interventions of the monetary authority, which contribute for its volatility be less than the volatility of the short-term interest rates.

Based on the data set mentioned above, the histogram of the forward rate percent variation of a particular maturity are presented in figure 2. These histograms offer an estimate of the true probability distribution of the behavior of DI futures contract.

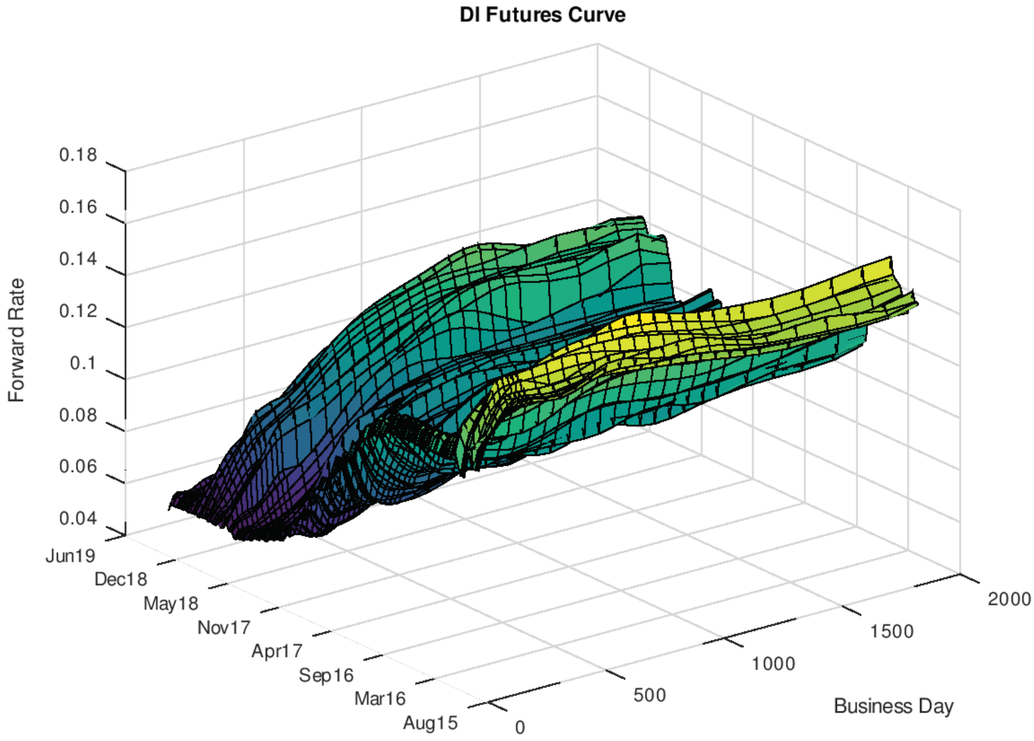
The objective is build a model of bond prices evolution underpinned from the forward rate (under the equation 3.4, the bond prices are function of the forward rate) in such a way that this model provide a reasonable approximation of the true probability distribution. A model that are enough simple to be computable, but sufficient rich to be realistic (JARROW, 2002).

The figure 3 present the interest rate curve at six different rates, between the end of 2015 and the beginning of the 2019. On the period considered, occurred different types of curves, what help to understand the macroeconomic instability among these period. In 10/23/2015, the short-term rates was at a high level, above the 13.5% a.a, the interest curve was upward sloping until the

[1] available at <http://www.b3.com.br> consults made through 05/01/2018 to 03/20/2019.

[2] An option is said at the money if the spot price of the underlying asset of the derivative be equal or above the strike price.

Figure 1 – Evolution of the future interest rates - DI futures contract. The data set consist of observations weekly extracted from oct/15 to jan/19, preferentially to friday closing rates. The sample contain data of 164 different days and for each day there are approximately 40 maturity dates. The rates was interpolated with the cubic spline



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second year, but the long-term rates were reporting a downturn. The scenario was of the high inflation, the IPCA (Broad Consumer Prices Index) reach at the end of 2015, 10.67% over the last 12 months. To control the inflation, the Central Bank raised the SELIC rate to 14.15%, in 07/29/2015.

The curve, in 12/09/2016, was with downward sloping (inverted), that is uncommon slope. This type of position denote a expectation of the interest rates reduction at the medium and long term. In 2016, the IPCA was of 6.29% and the SELIC closed the year at 13.65%. In 2017 and 2018, the curve again presented upward sloping with the drop of the inflation (the IPCA, in 2017, was of 2.95% and, in 2018, was of 3.75%) and the SELIC (in 2017, finish at 6.90% and, in 2018, was 6.40%).

Figure 2 – Histogram of the percent variation of the DI rate future contract, represent annual rate with continuous capitalization. The graph are divided based on the business day until the maturity of the contract, considering a year with 252 business day.

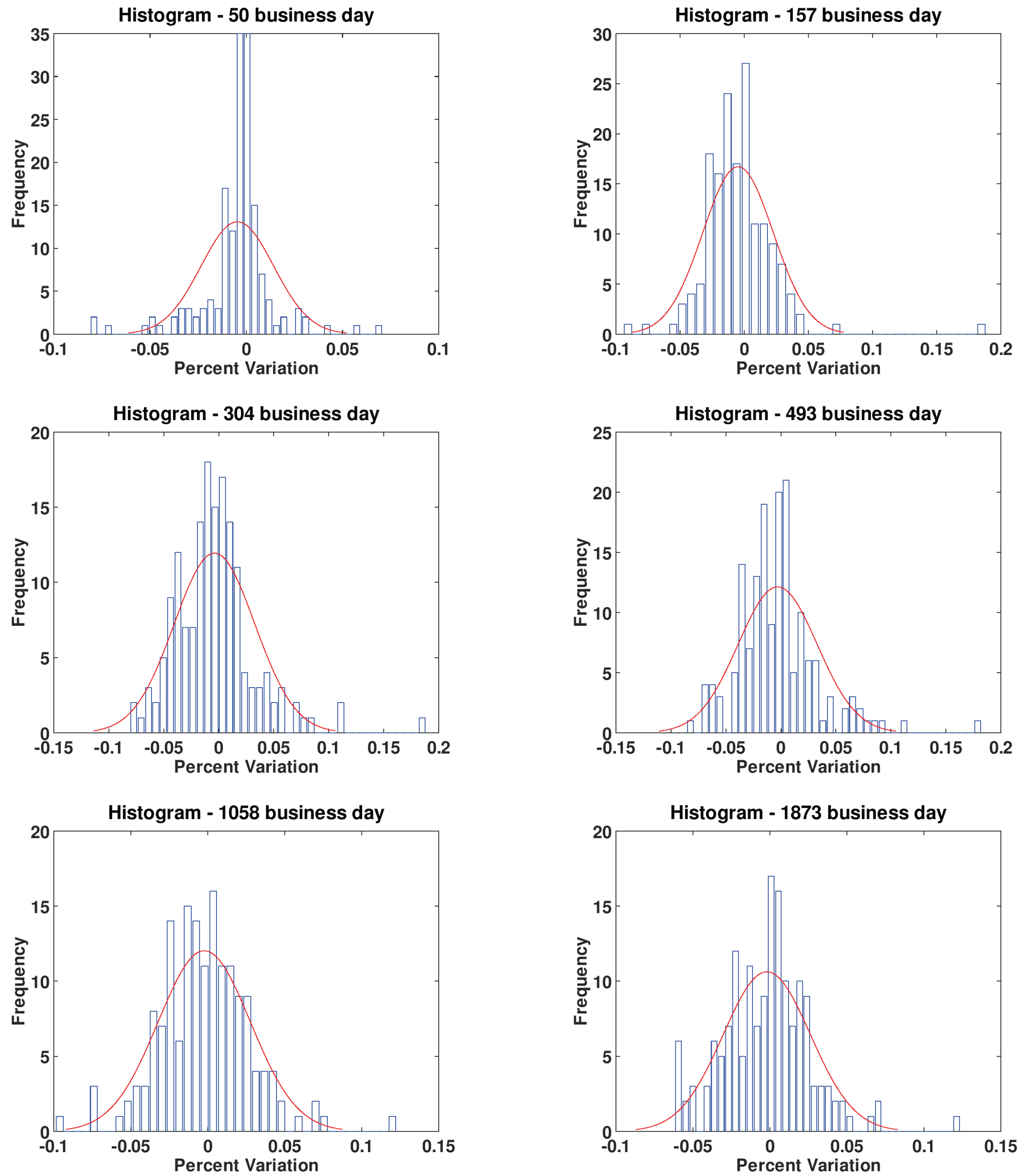

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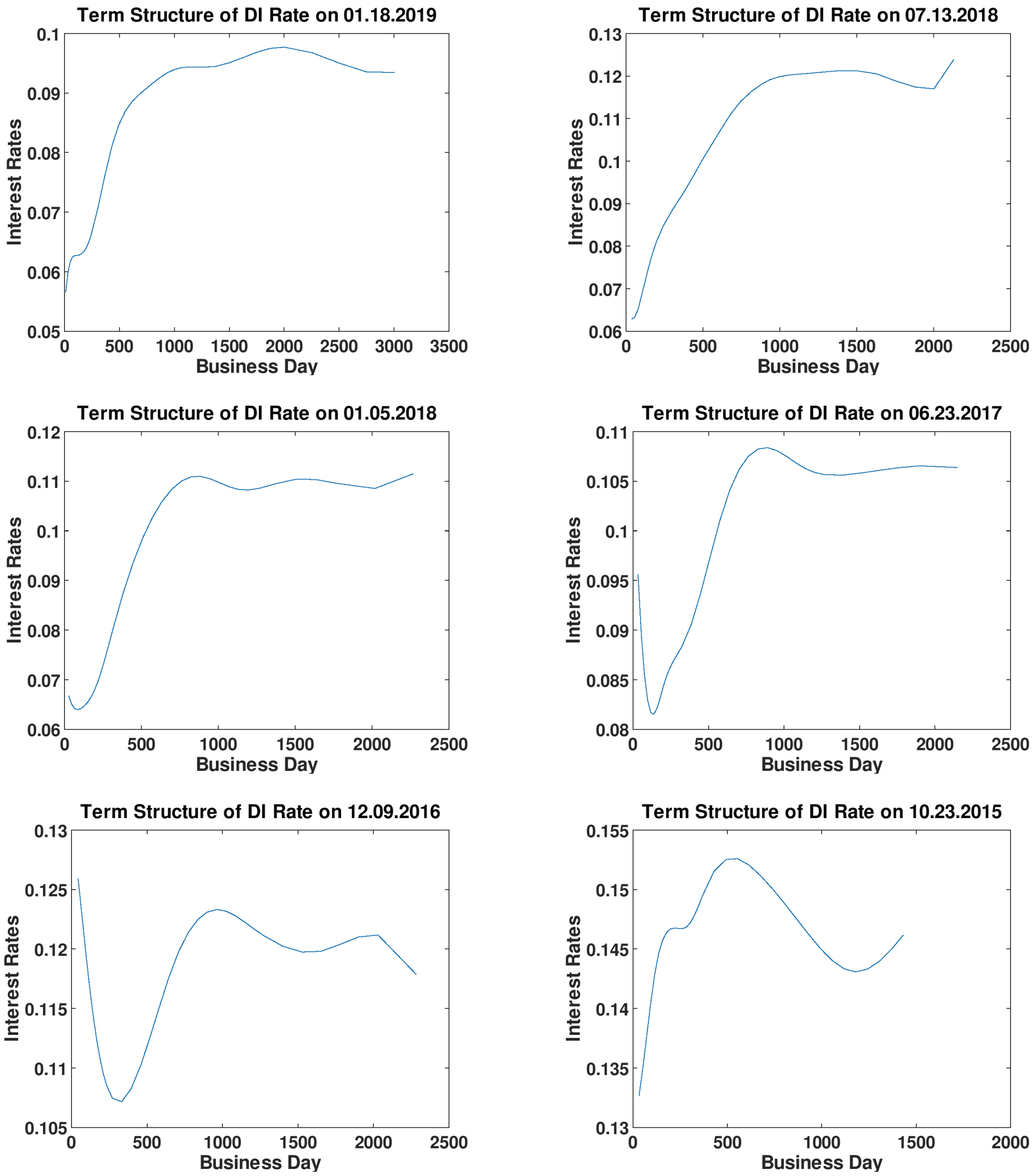
Figure 3 – The future interest curve for many dates, the rates were interpolated using a cubic spline. The business day is on horizontal axis and are obtained accordingly to the Brazilian bank calendar



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5.2 LIKELI-HOOD RATIO TEST

To estimate the number of factors necessary to pricing the option on DI futures with the string market model, i used an incremental likelihood ratio test. With the times series of 61 weeks, with 36 options of each week, i compared models with 1 to 5 factors to verify if there are statistical difference among them. The covariance matrix is generate by N factors when the diagonal matrix Ψ has N eigenvalues at the first elements along the diagonal and the remainder contain zero.

The test work of the following manner: for a model with N factors, i compute the difference between the market prices and model prices, in the other words, the errors. These errors is square and then, added. This procedure are applied for each week and then, the squared errors over the 61 weeks are added, thus is obtained the total sum of the squared errors. This same procedure is applied for a model with N+1 factors. Under the null hypothesis of equality, 61x36=2.196 times the difference between the logarithm sum of the squared errors with N and N+1 factors has a Chi-Square distribution, with 61 degrees of freedom.

The table below reports the results of the comparisons between the models with 1 to 5 factors, arranged in the form to allow pairwise comparisons. The difference between the sum of the squared errors are asymptotically χ^2_{61} under the null hypothesis of equality for the full sample with 61 weeks and χ^2_{30} for the two half samples. The critical value of χ^2_{61} is 89,59 at the 99% level of confidence and for χ^2_{30} is 50,89 for the same level of confidence.

The results present in the upper part of the table 1, show that the relation is statistically significant between the model with one versus two factors, two versus three factors and three versus four factors, namely, the statistical test tell us that these models are different. There are not statistical relevance only in the comparisons between the model with four versus five factors, these results imply that the market employ fours factors to pricing the option on DI futures.

Varga (2007) and Almeida et al. (2008), report that three factors (associated with the level, slope and curvature of the interest rates curve) are necessary to make forecast about the term structure of interest rates-TSIR in Brazil. Therefore, by applying the string market model to pricing interest rates derivatives, i find that four risk factors are present on the TSIR, that are employed by the market to pricing derivatives.

For a more detailed analysis, i realize another likelihood ratio test with the first half of the sample and after that, with the second half of the sample. By virtue of the electoral year with a political scenario *ex-ante* quite uncertain, the division of the sample are adequate to catch up the pre-electoral period with the first half of the sample. Already the second half of the sample catch up the post-electoral period. These two sub-samples also provide evidence about the existence of the four factor.

In accordance with Longstaff, Santa-Clara e Schwartz (2001a), the eigenvalues are interpreted

Table 1 – The table reports results of the likelihood ratio test, that make pairwise comparisons between the models with N and N+1 factors. The difference between the sum of the squared errors is asymptotically χ^2_{61} under the null hypothesis of equality for the sample with 61 weeks and χ^2_{30} for the two half samples. The critical value of χ^2_{61} is 89,59 at the 99% level of confidence and for χ^2_{30} is 50,89 for the same level of confidence.

N Factors	N+1 Factors	Statistical Test	P-Value
	A. Full Sample		
1	2	6,261,40	0.00
2	3	391.05	0.00
3	4	632.97	0.00
4	5	69.31	0.23
	B. First Half of the Sample		
1	2	2,295.60	0.00
2	3	161.22	0.00
3	4	354.51	0.00
4	5	13.43	1.00
	B. Second Half of the Sample		
1	2	3,565.70	0.00
2	3	225.59	0.00
3	4	268.18	0.00
4	5	53.52	0.01

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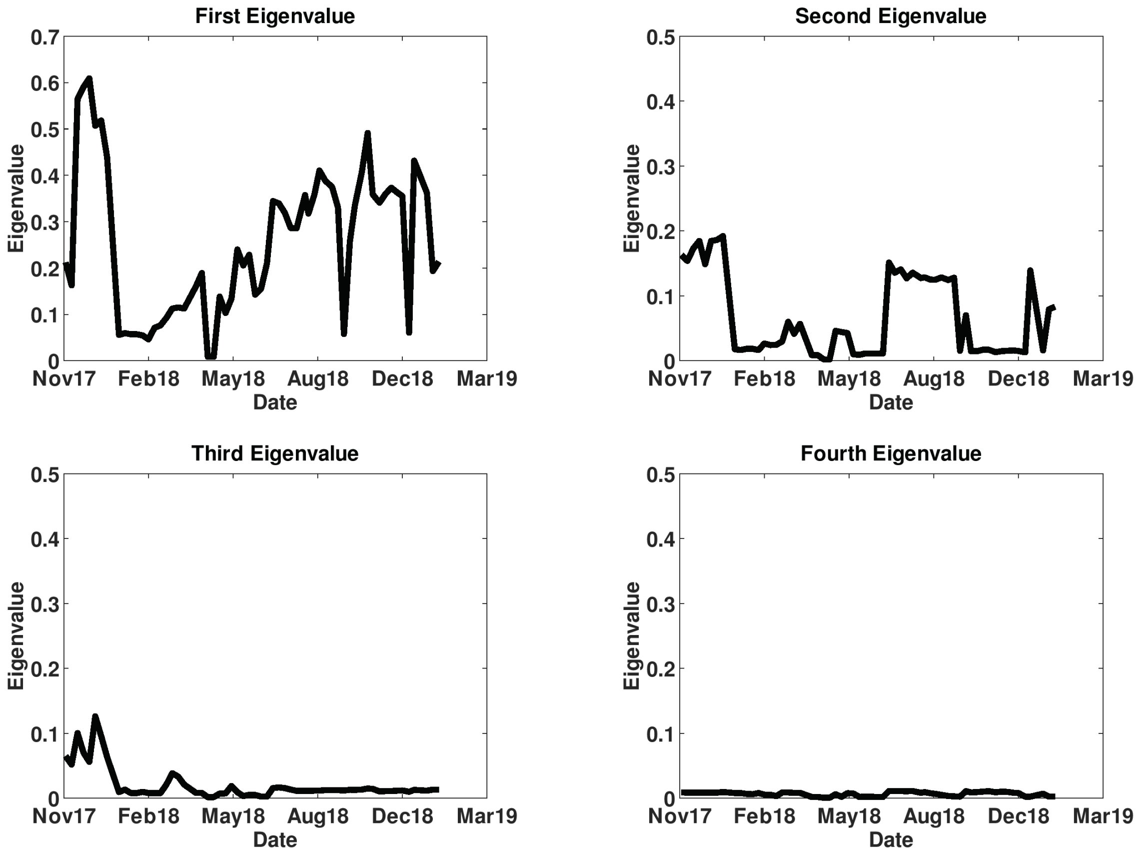
as the implied variance of the factors, already the eigenvectors of the implied covariance matrix are understood as factors of the TSIR.

5.3 EVIDENCES ABOUT THE PERFORMANCE OF THE STRING MARKET MODEL

With the statistical evidence that four factors pricing the option on DI futures, the figure 4, show four subplots with the time series for each one of the four eigenvalues, from 11/03/2017 to 01/18/2019.

The behavior of the first eigenvalue, that refers to the volatility of the first factor (related with the parallel shift of the TSIR) had many pattern over the period analyzed. In the end of 2017, the

Figure 4 – Time series of the eigenvalues related to the 61 weeks, from 11/03/2017 to 01/18/2019, all options of the sample are *at the money*. The eigenvalues are obtained from the implied volatility of the options on one-day interbank deposit futures



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volatility was very high, the scenario was of many political uncertainty and investor's deceptions with the lack of structural reforms approvals (for example, the social security reform).

Curiously, in the beginning of 2018, the volatility fall, but had been growing gradually over the year, as far as, the electoral campaign was gathering strength. Even after the election, the scenario of low volatility registered at the beginning of the year wasn't restored, the new govern elected also bring the uncertainty in relation to its promises at the economic field.

The volatility of the second eigenvalue, associated with the slope of the interest rates curve, was into a threshold of 0.2 in the end of 2017, and in the beginning of 2018, suddenly fall to approximately 0.03. In July/2018, close to the election, the volatility had raised for a threshold close to the end of 2017. Only after the conclusion of the first round of elections is that the volatility return to fall.

The volatility of the third eigenvalue, associated with the curvature, presented values more pronounced at the end of 2017. In 2018, this value had fallen significantly and was maintained at this level until the end of the sample period.

With relation to the fourth eigenvalue, its implied volatility is close to zero. This value fluctuate around 0.007, despite the low value of the average, in the moment of market stress this volatility can increase and become an important source of TSIR movements, for example, in the electoral period this volatility reached a value close to 0.012.

5.3.1 Convergence of the Model

The string market model sketched above is calibrate with the use of the option on DI futures. The objective function, when calibrating the model, is the root mean squared error (RMSE) that is calculate as the average percentage differences between the market prices and the model prices. Moreover, i used the model with four factor to measure the RMSE.

In relation to the option on DI futures, the sample contain 36 options for each one of the 61 weeks analyzed and the median RMSE of this sample is 5.8 percent and the standard deviation is 1.4 percent. The RMSE presented are inside the acceptable expected value, considering the *bid-ask spread*[3] and trading rates. At the moments of larger volatility, as the end of 2017 and the beginning of 2018, the RMSE was more higher than the average, accordingly to graph 5, then, pricing at moments of greater volatility in the market are more challenging.

The input of the string market model are the time series of the DI futures contract and the prices of the option on DI futures, therefore, i use this model to pricing the option on IDI index and verify if there are arbitrage opportunities between these two options. The median RMSE is 3.7 percent and the standard deviation is 1,6 percent, the figure 6 present the historic series of the RMSE of this option.

5.3.2 The "Near the Money" Options and Arbitrage Opportunities

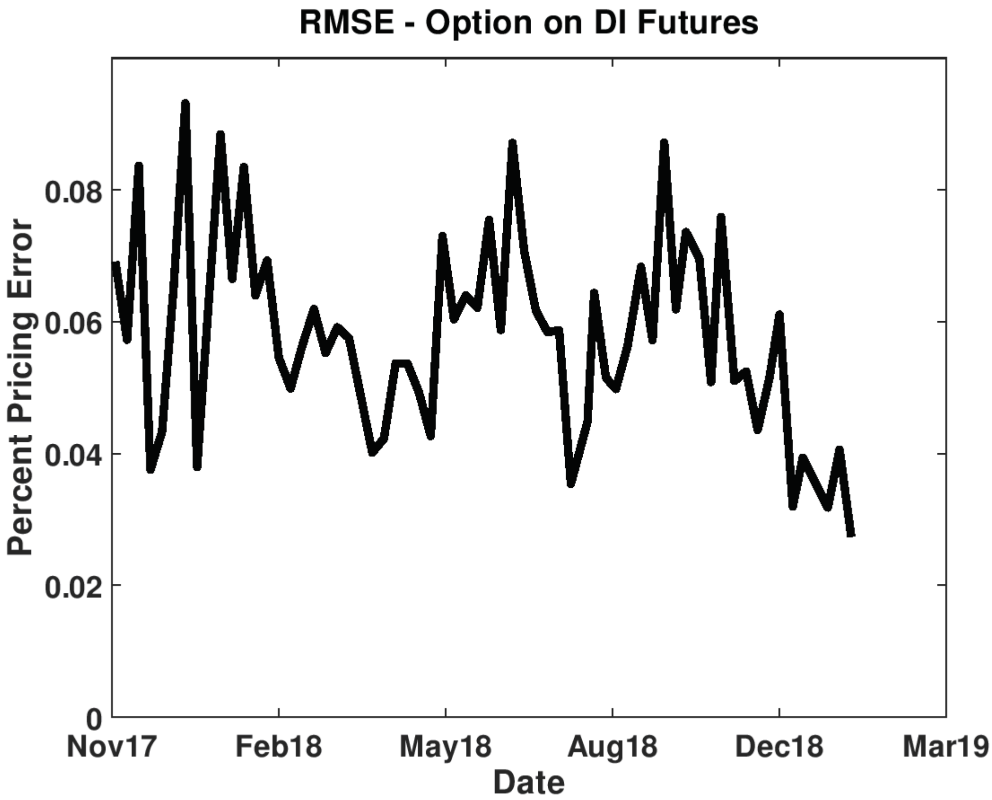
Considering that the model calibrated with the option on DI futures pricing the option on IDI index with high degree of accuracy, is possible state that there is not arbitrage opportunity of a systematized manner in this market. Therefore, extracting a sub-samples only with the options "near the money"[4] the RMSE of this sub-samples are significant higher than the RMSE of the full sample.

The sub-sample of the option on IDI index, with strike price below to 7.5 percent (which i call here of "near the money"), has a RMSE of 11.35 percent and standard deviations of 45 percent. The full sample of 6.100 options of this type (100 options for each one of the 61 weeks), 2.013 has strike below to 7.5 percent. Based with this sub-sample, in figure 8 is plotted RMSE, the analysis of this graph show that for many dates, the pricing error was considerable higher, when

[3] The *bid-ask spread* represent the difference in price that the seller and buyer cast into the trade systems, essentially is the difference between the highest price that the buyer is willing to pay and the smallest price that the seller is willing to sell. The individual that are willing to sell get the bid and the individual that are willing to buy get the ask

[4] The "near the money" options is the option that has the strike price close to the current market price of the corresponding underlying security.

Figure 5 – Time series of the RMSE of the option on DI futures, from 03/11/2017 to 18/01/2019. All options of the sample are at the money and there are 36 options for each date, preferentially obtained on friday after the market has closed. The simulated prices was obtained with the string market model using four factors.



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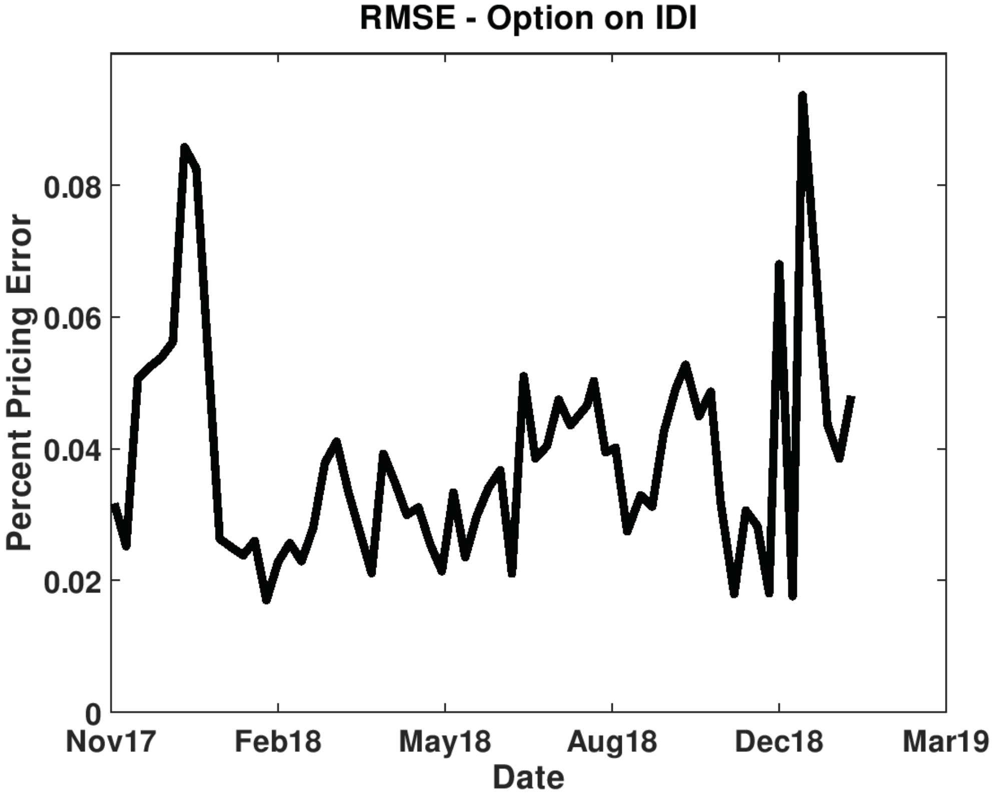
the same analysis are realized with the sub-samples of the option on DI futures, a different result is obtained.

With the full sample of 2.196 options on one-day interbank deposit futures, i draw a sub-sample of 545 instruments with strike below to 7.5 percent. The RMSE of this sub-sample was of 5.52 percent and the standard deviations of 9.8 percent, differently that was occurred with another option (option on IDI), the model had a good performance on this sub-samples of the “near the money” options. Therefore, arbitrage opportunities might appear at specifies cases, where for the same strike and maturity, a type of options has been well priced while for another type has been considerable mispricing between the model price and market price.

This is the first work, according with my knowledge, that analyze the relative valuation between the option on IDI index and the option on DI futures. In relation to the “swaption/caps puzzle”, Zhao (2010) argument that the unspanned stochastic volatility-USV is an essential factor to understand why the traditional dynamic term structure models-DTSM has difficulties in the conciliation of the Caps prices with the Swaption prices. In accordance with the author, the interest rate derivatives are not redundant financial instruments and cannot be hedging only

Figure 6 – Time series of the RMSE of the option on IDI index. Comprise the period between 11/03/2017 to 01/18/2019, with all options of the sample at the money, being 100 options for each date, preferentially obtained on friday after the market has closed. The simulated prices was obtained with the string market model using four factors.



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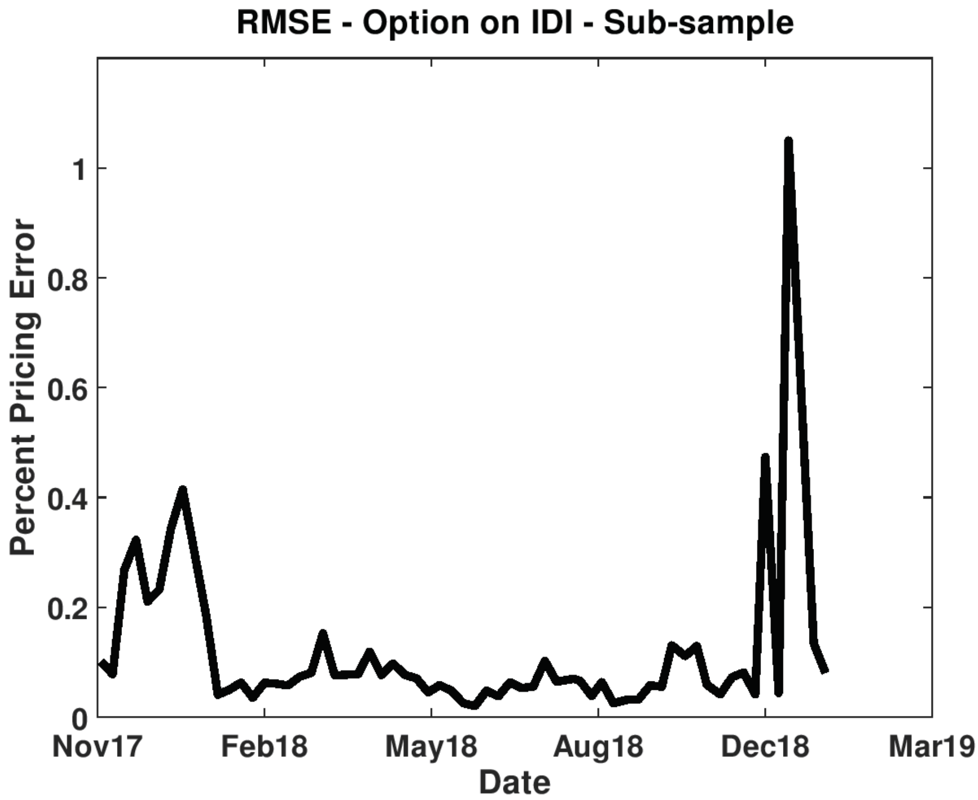
with bonds, in other words, the derivatives contain unique information about the term structure.

Jarrow, Li e Zhao (2007) developed a model with stochastic volatility and jumps on the Libor forward rates, and with this model was possible conciliate the prices between the Caps and swaptions. Model with stochastic volatility, in accordance with Zhao (2010), are able to better capture the volatility of these options.

The string market model depicted at this study has been able to conciliate the prices of the option on IDI index with the prices of the option on One-Day Interbank Deposit. One of the main difference between the original LSS model with the model presented here, is relate with the eigenvectors of the historic correlations matrix. In accordance with Longstaff, Santa-Clara e Schwartz (2001a), the historic correlations matrix of the percent variations on the forward rate was obtained taken from a five-year period before the beginning of the sample period. The correlation matrix was decomposed into its spectral representation $H = U\Lambda U'$, and U, that denote the eigenvectors of the historic correlation matrix, was kept fixed on all dates of the empirical evaluation. Thus, the instantaneous covariance matrix can be expresses by the following manner,

Figure 7 – RMSE of the sub-sample of 2.013 options on one-day interbank deposit rate index (IDI) classified as “near the money”, from 11/03/2017 to 01/18/2019, with all options at the money. The simulated price was obtained using the string market model with four factors.



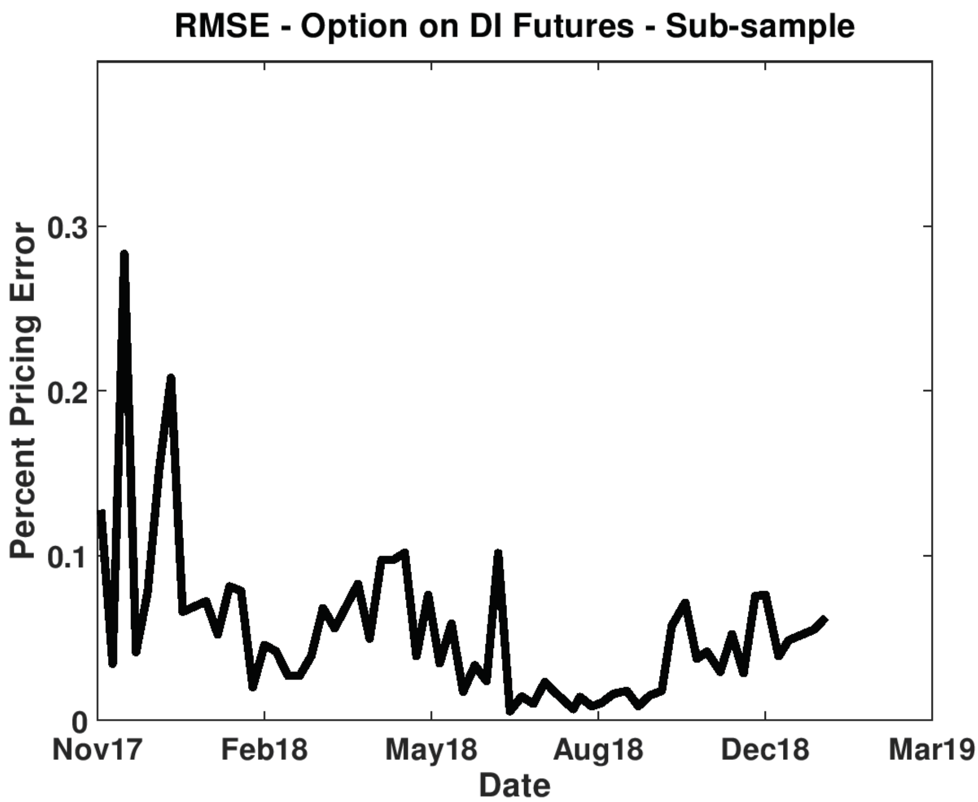
Font: compilation of the author. Dataset obtained from the B3 site, available in b3.com.br.

$$\Sigma_t = H\Psi_t H' \tag{5.1}$$

In the model presented at this work, the eigenvectors are time-varying, in other words, as far as the time t pass, the recent data are incorporated into the historic correlation matrix.

$$\Sigma_t = H_t\Psi_t H_t' \tag{5.2}$$

Figure 8 – RMSE of the sub-sample of 545 options on one-day interbank deposit futures classified as “near the money”, from 11/03/2017 to 01/18/2019, with all options at the money. The simulated price was obtained using the string market model with four factors.



Font: compilation of the author. Dataset obtained from the B3 site, available in b3.com.br.

6 CONCLUSIONS

In this study, I examine the ability of the string market model with one until five factors, to describe the dynamics of the DI rate. Then, investigate the relative valuation of the option on IDI index and the option on DI futures, using the estimate model. This investigation was undertaken because these two options have the same underlying variable, therefore, arbitrage opportunities can exist between these two financial instruments. The resulting pricing error, between the model prices and market prices, were computed using the four factors model.

With the use of a genetic algorithm, which capture the volatility that minimize the difference between the market price of a set of options with the same maturity and the Black model prices, the instantaneous covariance matrix among forward percent variations were implied from the market prices of the option on DI futures. Then, the model calibrated with this option data was used to pricing the option on IDI index.

Evidences about the behavior of the market prices and model prices of these two derivatives are describe below:

Four risk factors was found. Five versions of the model was simulated, with one to five factors and then, an incremental likelihood ratio test was applied. The test provide evidences that the market employ four risk factors in the valuation of the option on DI futures. However, in accordance with the mentioned literature, two or three factors captures the historical behavior of term-structure of interest rates movements.

The prices obtained through the string market model converge to market prices. The average pricing error is inside to the typical bidask spread. With relation to the sub-sample of "near the money" options on DI futures, the model had a good convergences. However, with relation to the option on IDI index, despite the full sample present a low RMSE, with their sub-sample, the model not had a good performance. These option were overvalued, because theirs market prices was, in general, higher than the model prices.

Arbitrage opportunities can exist between these two derivatives. However, there is no arbitrage opportunity of a systematized manner. I found that the "near the money" options are more likely to present deviations between the model and market prices. In future research, can be analyze if the deviations that generate arbitrage opportunities are quickly adjust for the fundamental value or, the existence of *noise traders* promote deviations for a long time.

For future research, to better capturing the volatility smile of the interest rate derivatives, innovations can be implemented to improve the model performance. These developments can involve more complex and time-consuming computational methods, like: stochastic volatility for each one of the eigenvalues used in the implied covariance matrix and addition of jumps process (type of stochastic process that is characterized for discrete movements, random arrival times, that is called jumps) in the dynamic evolution of the DI rate.

The times series of factor's volatility also can be use in macroeconomics models. The uncertainty in the economy is highly associated with the measure of the volatility, moreover, in the growing literature about the relation between the financial market and the macroeconomic variables, one of the main factors used to verify this relation is the volatility of the assets prices.

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Appendices

A DYNAMIC EVOLUTION OF THE BOND PRICES

Let $P(t,T)$ be the price at time t of a bond that has maturity at the date T, $P(T,T) = 1$ and $P(t,T) > 0$. The instantaneous interest rates at time t for a period comprised between t and $T > 0$, are denoted by $f(t,T)$. The spot interest rates r(t), is the instantaneous interest rates available at time t, with time to maturity t, namely, r(t) = f(t,t). Then,

$$f(t,T) = -\log \frac{\partial P(t,T)}{\partial T}$$
$$P(t,T) = \exp(-\int_0^T f(t,T)df)$$
$$r(t) = f(t,t)$$

The LSS dynamic of interest rates (i adopt for the DI rate, the same dynamic of the Libor rate), in the risk neutral measure are,

$$\frac{dF(t,T_{j-1})}{F(t,T_{j-1)})} = \hat{\alpha}_{j-1}(t)dt + \bar{\sigma}_{j-1}(t)d\tilde{Z}_{j-1}(t) \tag{A.1}$$

with,

$$\hat{\alpha}_{j-1}(t) = \frac{1+\delta F(t,T_{j-1})}{\delta F(t,T_{j-1})} \int_{y=T_{j-1}-t}^{T_j-t} \int_{u=0}^{T_j-t} R_t(u,y)dydu \tag{A.2}$$

$$\bar{\sigma}_{j-1}(t)d\tilde{Z}_{j-1}(t) = \int_{y=T_{j-1}-t}^{T_j-t} d\tilde{Z}(t,y)\bar{\sigma}(t,y)dy \tag{A.3}$$

$$R_t(u,y) = \frac{cov[df(t,u),df(t,y)]}{Ft} = c(t,x,y)\sigma(t,x)\sigma(t,y) \tag{A.4}$$

where

$$c(t,x,y) = \frac{d[Z(.,x),Z(.,y)]_t}{dt} = corr[d[Z(t,x),Z(t,y)]$$

LSS volatility:

$$\sigma^2_{i-1}(t) = \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{i-1}-t}^{T_i-t} R_t(x,y)dxdy \tag{A.5}$$

Historic Covariance Matrix:

$$\Theta_{ij}(t) = \int_{x=T_{i-1}-t}^{T_i-t} \int_{y=T_{j-1}-t}^{T_j-t} R(x,y)dxdy \tag{A.6}$$

Implied Covariance Matrix:

$$\Sigma_{ij}(t,T_0) = \int_{x=T_0-t}^{T_i-t}\int_{y=T_0-t}^{T_j-t} R(x,y)dxdy \tag{A.7}$$

Consider the interval $[T_0, T_n]$ and the partition $\{T_j = T_0 + \delta j\}_{j=1}^{n}$, where $\delta = \frac{T_n - T_0}{n}$, then:

$$P(T_{j-1}, T_j) = [1 + \delta F(T_{j-1})]^{-1} \tag{A.8}$$

and,

$$1 + \delta F(T_{j-1}) = \frac{P(t,T_{j-1})}{P(t,T_j)} \tag{A.9}$$

The bond prices evolve in accordance with:

$$dP(t) = r(t)P(t) + J^{-1}(t)\bar{\sigma}(t)F(t)d\tilde{Z}(t) \tag{A.10}$$

where:

$$P(t) = (P(t,T_1), ..., P(t,T_{n-1}))'$$
$$\bar{\sigma}(t)F(t)d\tilde{Z}(t) = (\bar{\sigma}_0 F(t,T_0)d\tilde{Z}_0(t), ..., \bar{\sigma}_{n-2}F(t,T_{n-2})d\tilde{Z}_{n-2}(t))$$

$J(t)$ is the Jacobian matrix, with $J_{ii}(t) = -\frac{1}{\delta}\frac{P(t,T_{i-1})}{P^2(t,T_i)}$ and $J_{i,i-1}(t) = \frac{1}{\delta P(t,T_i)}$, for i=1,...,n-1 and zero for the remainder terms. The introduction of the spot interest rates at the expression A.10 is a manner of impose the no-arbitrage condition. Let apply this assumption at the equation A.1 and verify if the expression A.10 are correct.

Consider that:

$$P(t,T_{j-1}) = \frac{P(t,T_0)}{\prod_{k=0}^{j-2}[1+\delta F(t,T_k)]}, \quad j = 2, ..., n \tag{A.11}$$

So, applying the Itô's rule to $P(t, T_{j-1})$,

$$\begin{aligned} dP(t,T_{j-1}) = \sum_{i=0}^{j-2} \frac{\partial P(t,T_{j-1})}{\partial F(t,T_i)} dF(t,T_i) + \frac{1}{2}\sum_{i,l=0}^{j-2} \frac{\partial^2 P(t,T_{j-1})}{\partial F(t,T_i), \partial F(t,T_l)} d[F(.,T_i), F(.,T_l)]_t) \\ + \frac{\partial P(t,T_{j-1})}{\partial P(t,T_0)} dP(t,T_0) + \sum_{i=0}^{j-2} \frac{\partial^2 P(t,T_{j-1})}{\partial F(t,T_i), \partial P(t,T_0)} d[F(.,T_i), P(.,T_0)]_t \\ + \frac{1}{2}\frac{\partial^2 P(t,T_{j-1})}{\partial^2 P(t,T_0} d[P(.,T_0), P(.,T_0)]_t \end{aligned}$$

Expanding the first term of $dP(t, T_{j-1})$

$$\sum_{i=0}^{j-2} \frac{\partial P(t, T_{j-1})}{\partial F(t, T_i)} [\hat{\alpha}_i(t) F(t, T_i) dt + \bar{\sigma}_i(t) F(t, T_i) d\tilde{Z}_i(t)] \tag{A.12}$$

where,

$$\frac{\partial P(t, T_{j-1})}{\partial F(t, T_i)} = \frac{-\delta P(t, T_{i+1})}{P(t, T_i)} P(t, T_{j-1}) \tag{A.13}$$

Then,

$$\sum_{i=0}^{j-2} \frac{-\delta P(t, T_{i+1})}{P(t, T_i)} P(t, T_{j-1}) [\hat{\alpha}_i(t) F(t, T_i) dt + \bar{\sigma}_i(t) F(t, T_i) d\tilde{Z}_i(t)] \tag{A.14}$$

Substituting the expression A.2 into the expression above,

$$\begin{aligned} \sum_{i=0}^{j-2} \frac{-\delta P(t, T_{i+1})}{P(t, T_i)} P(t, T_{j-1}) F(t, T_i) \frac{1 + \delta F(t, T_i)}{\delta F(t, T_i)} \int_{y=T_i-t}^{T_{i+1}-t} \int_{u=0}^{T_{i+1}-t} R_t(u, y) dy du \\ = -P(t, T_{j-1}) \sum_{i=0}^{j-2} \int_{y=T_i-t}^{T_{i+1}-t} \int_{u=0}^{T_{i+1}-t} R_t(u, y) dy du \, dt \end{aligned} \tag{A.15}$$

By the symmetry of $R_t(x, y)$, the expression A.15 is equal to,

$$-P(t, T_{j-1}) \int_{y=T_0-t}^{T_{j-1}-t} \int_{u=0}^{T_0-t} dy du R_t(u, y) dt \tag{A.16}$$

With relation to the second part of the equation A.12,

$$\sum_{i=0}^{j-2} \frac{-\delta P(t, T_{i+1})}{P(t, T_i)} P(t, T_{j-1}) \bar{\sigma}_i(t) F(t, T_i) d\tilde{Z}_i(t) \tag{A.17}$$

For the remaining terms, introducing the equation A.3 in A.1,

$$\frac{dF(t, T_{j-1})}{F(t, T_{j-1)})} = \hat{\alpha}_{j-1}(t) dt + \int_{T_{j-1}-t}^{T_j-t} dy \bar{\sigma}(t, y) \tilde{Z}(t, y) \tag{A.18}$$

Henceforth, using the following rule: $dt^2 = 0$, $dt.dZ = 0$ and $dZ^2 = dt$.

Expanding the second term of $dP(t,T_{j-1})$

$$\frac{1}{2}\sum_{i,l=0}^{j-2}\frac{\partial^2 P(t,T_{j-1})}{\partial F(t,T_i,\partial F(t,T_l)}[\hat{\alpha}_i(t)F(t,T_i)dt+\bar{\sigma}_i(t)F(t,T_i)d\tilde{Z}_i(t),$$
$$\hat{\alpha}_l(t)F(t,T_l)dt+\bar{\sigma}_l(t)F(t,T_l)d\tilde{Z}_i(t)]$$

Applying the rules above:

$$\frac{1}{2}\sum_{i,l=0}^{j-2}\frac{\partial^2 P(t,T_{j-1})}{\partial F(t,T_i,\partial F(t,T_l)}\int_{T_{i-1}-t}^{T_i-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y)\int_{T_{l-1}-t}^{T_l-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y)$$

Once again, by the symmetry of $R_t(x,y)$, the term above reaches zero.

Expanding the third term of $dP(t,T_{j-1})$

Considering the follow dynamic of the bonds prices, according with Bueno-Guerrero, Moreno e Navas (2016),

$$\frac{dP(t,T_0)}{P(t,T_0)}=r(t)dt-\int_{y=0}^{T_0-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y) \tag{A.19}$$

Then, with relation to the third term,

$$P(t,T_{j-1}).\left(r(t)dt-\int_{y=0}^{T_0-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y)\right) \tag{A.20}$$

Expanding the fourth term of $dP(t,T_{j-1})$

$$\begin{aligned}
&\sum_{i=0}^{j-2}\frac{\delta}{1+\delta F(t,T_i)}.\frac{1}{\prod_{k=0}^{j-2}[1+\delta F(t,T_k)]}.P(t,T_0).F(t,T_i)\\
&\int_{T_i-t}^{T_{i+1}-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y).\int_{y=0}^{T_0-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y)\\
=\sum_{i=0}^{j-2}\frac{\delta F(t,T_i)}{1+\delta F(t,T_i)}.P(t,T_{j-1}).&\int_{T_i-t}^{T_{i+1}-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y).\int_{y=0}^{T_0-t}dy\bar{\sigma}(t,y)\tilde{Z}(t,y)\\
&=P(t,T_{j-1})\int_{y=T_0-t}^{T_{j-1}-t}\int_{u=0}^{T_0-t}dyduR_t(u,y)dt
\end{aligned} \tag{A.21}$$

Expanding the fifth term of $dP(t,T_{j-1})$

How,

$$\frac{1}{2}\frac{\partial^2 P(t,T_{j-1})}{\partial^2 P(t,T_0)} = 0 \tag{A.22}$$

Then, this term are dropped.

Aggregation of the terms

Adding the equations A.16, A.17, A.20 e A.21, and after that, dividing everything by $P(t,T_{j-1})$,

$$\frac{dP(t,T_{j-1})}{P(t,T_{j-1})} = r(t)dt - \sum_{i=0}^{j-2}\frac{-\delta P(t,T_{i+1})}{P(t,T_i)}\bar{\sigma}_i(t)F(t,T_i)d\tilde{Z}_i(t) - \int_{y=0}^{T_0-t} dy\bar{\sigma}(t,y)\tilde{Z}(t,y) \tag{A.23}$$

In accordance with Longstaff, Santa-Clara e Schwartz (2001a) (footnote 12), for $y \leq (T_0-t)$ the volatility $\sigma(t,y)=0$, in other words, the process are not stochastic and not affect the diffusion term in equation A.10. Therefore, the last term of the expression A.23 are dropped. Take into account that:

$$[\mathbf{J}^{-1}(t)]_{ij} = \begin{cases} \frac{-\delta P(t,T_{i+1})}{P(t,T_i)} & if\ j \leq i \\ 0 & if\ j > i \end{cases}$$

i get,

$$dP(t) = r(t)P(t) + J^{-1}(t)\bar{\sigma}(t)F(t)d\tilde{Z}(t)$$

B GENETIC ALGORITHM

Below i outline the genetic algorithm structure, that are responsible to obtains the implied volatility of the options.

1. Starts with a value ϕ_1 for the volatility.
2. Randomly generate a population set ϕ of solutions around ϕ_1.
3. Let maxit be the maximum number of interactions.
4. iter=1
5. Assess the initial population set ϕ in the cost function.
6. **while** item $<$ maxit
7. Select the survivors for the next generation, in this case, the 50% with the best score in the set ϕ.
8. **for** t = 1 to n **do**
9. The offspring are create by the cross-over between the survivors of the prior generation (random linear combination).
10. **end for**
11. Substitute the worst solutions on ϕ by the offsprings, forming a new set ϕ.
12. Randomly mutate ϕ
13. Assesses the new population and classify in increasing order.
14. iter = iter+1
15. **end while**

Below, i present the codes (language MATLAB), with the explanations. The first part consist to establish the initial value for the volatility, as close as possible to observed data. Thereafter, it seeks to find in the vector of business day until the maturity of the option, the line number that split the different days.

```
variance = 0.3.*ones(1,1)    % Initial value for the volatility
    .

```

```
% find the line number that split different dates of option's
    maturities.

for i=1:rows(opcao_DIa)
  if opcao_DIa(i,6) ~= opcao_DIa(i+1,6)
       row1=i
  break;
endif
endfor

    for i=row1+1:rows(opcao_DIa)
      if opcao_DIa(i,6) ~= opcao_DIa(i+1,6)
          row2=i
          break
        else
          row2=0
    endif
    endfor

    if row2>0
      for i=row2+1:rows(opcao_DIa)
        if i+1 >rows(opcao_DIa)
          break
          else
        if opcao_DIa(i,6) ~= opcao_DIa(i+1,6)
            row3=i
            break
          else
            row3=0
      endif
      endif
      endfor
    endif

    if row3>0
      for i=row3+1:rows(opcao_DIa)
        if i+1 >rows(opcao_DIa)
          break
          else
```

```
            if opcao_DIa(i,6) ~= opcao_DIa(i+1,6)
                row4=i
                break
              else
                row4=0
          endif
          endif
          endfor
        endif
```

The second part consist in find the implied volatility that minimize the pricing error for a set of options with the same maturity. Then, this volatility are inserted in the diagonal of the covariance matrix.

```
if row3>0
clear pop
[pop] = genetic2(variance, 1, row1)
COVV(find(datas1(:,1) == opcao_DI_du1(row1,1)), ...
find(datas1(:,1) == opcao_DI_du1(row1,1))) =pop(1,1)

[pop] = genetic2(variance, row1+1, row2)
COVV(find(datas1(:,1) == opcao_DI_du1(row2,1)), ...
find(datas1(:,1) == opcao_DI_du1(row2,1))) =pop(1,1)

[pop] = genetic2(variance, row2+1, row3)
COVV(find(datas1(:,1) == opcao_DI_du1(row2,1)), ...
find(datas1(:,1) == opcao_DI_du1(row2,1))) =pop(1,1)

[pop] = genetic2(variance, row3+1, rows(opcao_DIa))
COVV(find(datas1(:,1) == opcao_DI_du1(row2+1,1)), ...
find(datas1(:,1) == opcao_DI_du1(row2+1,1)))=pop(1,1)

elseif row2>0

[pop] = genetic2(variance, 1, row1)
COVV(find(datas1(:,1) == opcao_DI_du1(row1,1)), ...
find(datas1(:,1) == opcao_DI_du1(row1,1))) =pop(1,1)

[pop] = genetic2(variance, row1+1, row2)
COVV(find(datas1(:,1) == opcao_DI_du1(row2,1)), ...
```

```
find(datas1(:,1) == opcao_DI_du1(row2,1))) =pop(1,1)

[pop] = genetic2(variance, row2+1, rows(opcao_DIa))
COVV(find(datas1(:,1) == opcao_DI_du1(row2+1,1)), ...
find(datas1(:,1) == opcao_DI_du1(row2+1,1)))=pop(1,1)

endif
```

At the third part, i show the steps that the algorithm use to find the optimal value. First, randomly generate a population set ϕ of solutions around ϕ_1, then, each value of these population are available in the cost function (this function will be demonstrated in the fourth part).

```
function [pop] = genetic2(variance, a, b)
  %% Setup the GA
   ff='bls_price2'; % filename objective function
   nbits=1; % # of optimization variables
   Nt=nbits; % # of columns in population matrix
   maxit=150; % max number of iterations
  % GA parameters
   popsize=20; % set population size
   mutrate=0.3; % set mutation rate
   selection=0.5; % fraction of population kept
   keep=floor(selection*popsize); % #population members that survive
  M=ceil((popsize-keep)/2); % number of matings/cruzamentos

   iga=0; % generation counter initialized

%%%%%%%%%%%%%% CREATE THE POPULATION %%%%%%%%%%%%%%%%%%%%%%

        pop(:,1) = variance

   for k = 2:popsize
     for j=1:nbits
         if k<5
       pop(j,k) = 0.8*abs(stdnormal_rnd (1))
   elseif k<10
     pop(j,k) = 0.6*abs(stdnormal_rnd (1))
  elseif k<15
     pop(j,k) = 0.1*abs(stdnormal_rnd (1))
```

```
  elseif k<=popsize
  pop(j,k) = 0.05*abs(stdnormal_rnd (1))
 endif
      endfor
    endfor
%%%%%%%%%%%%%%%%%%%%

  %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% THE COST FUNCTION
   for i=1:popsize
   cost(i,1) =feval(ff,pop(:,i),a, b) % calculates population cost using ff
   endfor

   [cost, ind1] = sort(cost) % min cost in element 1
   pop=pop(:,ind1'); % sort population with lowest cost first
   minc(1)=min(cost); % minc contains min of population
   meanc(1)=mean(cost); % meanc contains mean of population

%%%% DO THE SELECTION, SORT THE BETTER TO WORSE
%

%% Iterate through generations (MAIN LOOP)

while iga<maxit  %| RMSE < 0.05
iga=iga+1; % increments generation counter

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%fazer o crossover   %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
     for ic=1:M  %M = popsize - keep /2

     xp = ceil(nbits*rand(1,1))               %numero do cromossomo
     dad = ceil(rand(1, 1)*popsize*selection)
     mon = ceil(rand(1, 1)*popsize*selection)

     indx=2*(ic-1)+1; % odd numbers starting at 1

```

```
    pop(:,keep+indx)= vertcat( pop(1:xp,dad) , pop(xp+1:nbits, mon) )
    pop(:,keep+indx+1)= vertcat(0.6.*pop(1:xp,dad)+0.4.*pop(1:xp,mon) ,...
    0.6.*pop(xp+1:nbits, mon)+0.4.*pop(xp+1:nbits, dad) )

    endfor
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Mutate the population
    nmut=ceil(popsize*nbits*mutrate);
    for i=1:nmut;
    col=randi(popsize);
    row=randi(nbits);
        if col ~=1
        pop(row,col)= rand(1,1);
        endif
    end
%------------------------ EVALUATE THE NEW GENERATION ------------------------------%


  for i=1:popsize
% pop1=reshape (pop(:,i), 32, 31)
  cost(i,1) =feval(ff,pop(:,i),a, b) % calculates population cost using ff
  endfor

  [cost, ind1] = sort(cost) % min cost in element 1
  pop=pop(:,ind1'); % sort population with lowest cost first

    % Do statistics
  minc(iga)=min(cost);
  meanc(iga)=mean(cost);

endwhile
%_______________________________________________________
% Sort the costs and associated parameters
```

```
  [cost, ind1] = sort(cost) % min cost in element 1
  pop=pop(:,ind1'); % sort population with lowest cost first

endfunction
```

Finally, i present the objective function. In the matrix with the data of the options on DI futures, the lines that appear the options with the same maturity will be selected and these options will all be evaluated in this function.

```
function RMSE = bls_price2(variance, a, b)

global opcao_DI P_real K taxa_anual opcao_DI_du2...
 opcao_DI_du3 opcao_DI_du1 opcao_DI_taxa P_t datas1 pu_titulo K_linear ...
index_v1 index_v2 index_v3 ajuste factor Rcdi Rlinear factor1

   %fórmula de Black 1976.
 d1 = (log(Rlinear(a:b,1)./K_linear(a:b,1))+0.5*variance.*(opcao_DI_du2(a:b,1)./252) )./...
  (sqrt(variance).*sqrt((opcao_DI_du2(a:b,1)./252)))

  d2 = (log(Rlinear(a:b,1)./K_linear(a:b,1))-0.5.*variance.*(opcao_DI_du2(a:b,1)./252) )./...
  (sqrt(variance).*sqrt((opcao_DI_du2(a:b,1)./252)))

 c = K(a:b,1).*(opcao_DI_du2(a:b,1)./252).*(Rlinear(a:b,1).*normcdf (d1) -...
  K_linear(a:b,1).*normcdf(d2)).*pu_titulo(index_v3(a:b,1),1)

%Room mean square error - avaliar o erro de precificação
 RMSE = ((P_real(a:b,1)-c)./P_real(a:b,1) ).^2
RMSE = sum(RMSE) / (b-a-1)

  endfunction
```