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Some contributions to the study of evolution equations describing pseudospherical surfaces, and the theory of zero-curvature representations

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Salvador-Bahia

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To my wife Milena França

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"Imagination is more important than knowl-edge."

Albert Einstein.

Resumo

Este trabalho fornece algumas contribuições originais para o estudo geométrico de equações evolutivas que descrevem superfícies pseudo-esféricas (equações PEs). Por definição, uma equação PE para funções $z = z(x,t)$ é equivalente às equações de estrutura $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$ de uma variedade Riemanniana 2-dimensional com curvatura Gaussiana $K = -1$, com 1-formas $\omega_i = f_{i1}\, dx + f_{i2}\, dt$, $i = 1, 2, 3$, satisfazendo a condição de não-degeneração $\omega_1 \wedge \omega_2 \neq 0$ e com f_{ij} funções suaves de x, t, z e suas derivadas com respeito a x e t. Usando a noção de representação a curvatura nula (RCN), pode-se dizer que toda equação PE admite uma RCN a valores em $\mathfrak{sl}\,(2, \mathbb{R})$.

A primeira contribuição deste trabalho diz respeito a uma classificação completa e explícita de equações PEs evolutivas de segunda ordem da forma $z_t = A(x,t,z)z_2 + B(x,t,z,z_1)$, com $z = z\,(x,t)$ e $z_i = \frac{\partial^i z}{\partial x^i}$, sob as hipóteses que $f_{ij} = f_{ij}\,(x,t,z,z_1,z_2)$ e $f_{21} = \eta$. De acordo com a classificação dada, estas equações subdividem-se em três classes principais (chamadas de Tipos I-III) juntamente com os correspondentes sistemas de 1-formas $\{\omega_1, \omega_2, \omega_3\}$ que, em virtude da hipótese $f_{21} = \eta$, definem para cada tipo uma família a 1-parâmetro de RCNs associadas. Nesta classe de equações PEs encontram-se em particular algumas equações já conhecidas, dentre as quais as equações integráveis classificadas por Svinolupov e Sokolov, a equação de Boltzmann, e equações de reação e difusão como a equação de Murray. Ulteriores novos exemplos explicitos são também apresentados.

A segunda contribuição é relativa ao problema de existência de imersões isométricas locais, no espaço Euclidiano 3-dimensional $\mathbf{E}^3$, para as famílias de superfícies pseudo-esféricas descritas pelas equações PEs da classificação acima. O resultado principal obtido neste caso é que estas imersões existem somente para as equações do Tipo I, que possuem forma de lei de conservação, e isso levou à uma extensão natural deste resultado ao caso das equações evolutivas de ordem k da forma $D_t\,(f(x,t,z)) = D_x\,(\Omega(x,t,z,z_1,\ldots,z_k))$. No âmbito da literatura existente sobre este problema, todos os resultados obtidos nesta parte do trabalho são novos; em particular além de equações de segunda ordem, como por exemplo as equações de Boltzmann, Murray e as equações de Svinolupov e Sokolov, entre os exemplos de equações PEs que admitem este tipo de imersão isométrica há também equações de ordem superior como as equações de Kuramoto-Sivashinsky, Sawada-Kotera, Kaup-Kupershmidt e inteiras hierarquias de equações integráveis como as de Burgers, mKdV e KdV.

Finalmente, nós consideramos o problema de construir famílias a 1-parâmetro não-triviais de RCNs para equações PEs. Este problema é de interesse especial para as aplicações da teoria das RCNs, por exemplo no cálculo de soluções exatas e hierarquias infinitas de leis de conservação, e tem sido resolvido no caso mais geral de RCNs a valores em $\mathfrak{g}$, com $\mathfrak{g}$ uma sub-álgebra de $\mathfrak{gl}\,(n, \mathbb{R})$ ou $\mathfrak{gl}\,(n, \mathbb{C})$, usando a teoria de simetrias clássicas de equações diferenciais.

Os resultados originais deste trabalho são exibidos nos Capítulos 2, 3 e 4. Em particular, os resultados do Capítulo 4 tem sido recentemente publicados no artigo [15].

Palavras-chave: Equações que descrevem superfícies pseudo-esféricas; equações integráveis; representações a curvatura nula; imersões isométricas; simetrias clássicas; geometria das equações diferenciais.

Abstract

This work provides some original contributions to the geometric study of evolution equations which describe pseudospherical surfaces (PS equations). By definition, a PS equation for functions $z = z(x,t)$ is equivalent to the structure equations $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$ of a 2-dimensional Riemannian manifold with Gaussian curvature $K = -1$, and with 1-forms $\omega_i = f_{i1}\, dx + f_{i2}\, dt$, $i = 1, 2, 3$, satisfying the non-degeneracy condition $\omega_1 \wedge \omega_2 \neq 0$ with f_{ij} smooth functions of x, t, z and derivatives of z with respect to x and t. Using the notion of zero-curvature representation (ZCR), one can say that every PS equation admits an $\mathfrak{sl}(2,\mathbb{R})$-valued ZCR.

The first contribution of this work concerns a complete and explicit classification of second order evolution PS equations of the form $z_t = A(x,t,z)z_2 + B(x,t,z,z_1)$, with $z = z(x,t)$ and $z_i = \frac{\partial^i z}{\partial x^i}$, under the assumptions that $f_{ij} = f_{ij}\,(x,t,z,z_1,z_2)$ and $f_{21} = \eta$. According to this classification, these PS equations are subdivided into three main classes (referred to as Types I-III) together with the corresponding systems of 1-forms $\{\omega_1, \omega_2, \omega_3\}$ which, in view of the assumption $f_{21} = \eta$, define for any such equation an associated 1-parameter family of ZCRs. Some already known equations are found to belong to this class of PS equations, like Svinolupov-Sokolov equations admitting higher weakly nonlinear symmetries, Boltzmann equation and reaction-diffusion equations like Murray equation. Other explicit examples are presented, as well.

As a second contribution we considered, for the families of pseudospherical surfaces described by above class of PS equations, the problem of existence of local isometric immersions into the 3-dimensional Euclidean space $\mathbf{E}^3$. We found that only Type I equations admit such a kind of immersion and, on the base of this result we also provided an extension to the case of k-th order evolution equations in the conservation law form $D_t\,(f(x,t,z)) = D_x\,(\Omega(x,t,z,z_1,\ldots,z_k))$. The results and explicit examples discussed in this part of the work are new, when compared with the existing literature, in particular the examples include equations like Boltzmann, Murray and Svinolupov-Sokolov equations, as well as higher order equations like Kuramoto-Sivashinsky, Sawada-Kotera and Kaup-Kupershmidt equations and also full hierarchies of integrable equations like Burgers, mKdV and KdV.

Finally, we considered the problem of constructing nontrivial 1-parameter families of ZCRs for PS equations. This problem is of special interest for the application of the theory of ZCRs, for instance in the calculation of exact solutions and infinite hierarchies of conservation laws, and has been solved in the more general case of $\mathfrak{g}$-valued ZCRs, with $\mathfrak{g}$ a Lie sub-algebra of $\mathfrak{gl}\,(n,\mathbb{R})$ or $\mathfrak{gl}\,(n,\mathbb{C})$, by using the theory of classical symmetries of differential equations.

The original results of this work are exposed in the Chapters 2, 3 and 4. In particular, the results of Chapter 4 have been recently reported in the paper [15].

Keywords: Equations describing pseudospherical surfaces; integrable equations; zero-curvature representations; isometric immersions; classical symmetries; geometry of differential equations.

Contents

Introduction

Differential equations which describe pseudospherical surfaces (*PS equations*) arise ubiquitously as suitable models in the description of nonlinear physical phenomena as well as in many problems of pure and applied mathematics. Geometrically these equations are characterized by the fact that their generic solutions provide metrics on open subsets of $\mathbb{R}^2$, with Gaussian curvature $K = -1$. The first well known example of such an equation is the sine-Gordon equation $z_{xt} = sin(z)$. This example was discovered by Edmond Bour [3], who realized that in terms of Darboux asymptotic coordinates the Gauss-Codazzi equations for pseudospherical surfaces contained in $\mathbb{R}^3$ reduce to the sine-Gordon equation. Then, the discovery of Bäcklund transformations first, and later the construction by Bianchi of the superposition formula for solutions of this equation, focused even more attention on the sine-Gordon equation, that in the end it turned out to be an important model in the description of several nonlinear phenomena (see for example [31, 35, 59]). However, it was after the early observation [56] that "all the soliton equations in $1+1$ dimensions that can be solved by the AKNS 2×2 inverse scattering method (for example, the sine-Gordon, KdV or modified KdV equations) ... describe pseudospherical surfaces", that the general study of these equations was initiated. In particular, it was with the fundamental paper [20] by S. S. Chern and K. Tenenblat that initiated a systematic study of these equations. The results of [20], together with the considerable effort addressed over the past few decades to the possible applications of inverse scattering method, gave a significant contribution to the discovery of new integrable equations. For instance, Belinski-Zakharov system in General Relativity [8], the nonlinear Schrödinger type systems [19, 24, 27], the Rabelo's cubic equation [6, 46, 47, 55], the Camassa-Holm, Degasperis-Procesi, Kaup–Kupershmidt and Sawada-Kotera equations [11, 14, 50, 51, 52, 53] are some important examples of PS equations which are integrable by inverse scattering method. All these facts prove the relevance of these equations and justify the general interest in their study and classification. This thesis provides some contributions to the geometric study of evolution PS equations.

From a geometric point of view, every PS equation $\mathcal{E}$ satisfies the following remarkable property: to any generic solution (see below) $z = z(x, t)$ of $\mathcal{E}$, defined on an open domain $U \subset \mathbb{R}^2$, it is associated a Riemannian metric defined almost everywhere on the domain U with Gaussian curvature $K = -1$. Indeed, by definition a differential equation $\mathcal{E}$ for a real function $z = z(x,t)$ is a PS equation if it is equivalent to the structure equations $d\omega_1 = \omega_3 \wedge \omega_2$, $d\omega_2 = \omega_1 \wedge \omega_3$, $d\omega_3 = \omega_1 \wedge \omega_2$ of a 2-dimensional Riemannian manifold whose Gaussian curvature $K = -1$, and with 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ satisfying the non-degeneracy condition $\omega_1 \wedge \omega_2 \neq 0$ with f_{ij} smooth functions of x, t, z and derivatives of z with respect to x and t. Notice that according to the definition $\omega_1 \wedge \omega_2$ is generically nonzero on the solutions of a PS equation $\mathcal{E}$. However, this condition does not guarantee the property that, for any solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$, the restriction $(\omega_1 \wedge \omega_2)[z]$ of $\omega_1 \wedge \omega_2$ to z is everywhere nonzero on U. Relatively to a given system of 1-forms $\{\omega_1, \omega_2, \omega_3\}$, we will call *generic* a solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $(\omega_1 \wedge \omega_2)[z]$ is almost everywhere nonzero on U, i.e., it is everywhere nonzero except for a subset of U of measure zero. Thus, for any generic solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ of a PS equation $\mathcal{E}$, the restriction $I[z]$ of $I = \omega_1^2 + \omega_2^2$ to z defines almost everywhere a Riemannian metric $I[z]$ on the domain U with Gaussian curvature $K = -1$. It is in this sense that one can say that a PS equation describes, or parametrizes, a family of non-immersed pseudospherical surfaces.

For instance, one may easily check that sine-Gordon equation $z_{xt} = sin(z)$ is equivalent to the above structure equations for the following system of 1-forms

$$\begin{aligned} \omega_1 &= \tfrac{1}{\eta} sin\,(z)\ dt, \\ \omega_2 &= \eta\, dx + \tfrac{1}{\eta} cos\,(z)\ dt, \\ \omega_3 &= z_x\, dx, \end{aligned} \tag{0.0.1}$$

with $\eta \in \mathbb{R} - \{0\}$. In this case one has that $I = \omega_1^2 + \omega_2^2 = \frac{1}{\eta^2} dt^2 + 2\,cos(z)\,dx\,dt + \eta^2 dx^2$. Notice that, with respect to the system (0.0.1), sine-Gordon equation admits non-generic solutions. For instance, $z = k\pi$, $k \in \mathbb{Z}$, is a non-generic solution of sine-Gordon equation.

PS equations can also be characterized in few alternative ways (see Section 1.2, of Chapter 1). For instance, above structure equations are equivalent to the integrability condition of an auxiliary first order linear system, and this naturally leads to study some properties of PS equations by using the notion of zero-curvature representations (ZCRs), (see Sections 1.2 and 1.5, of Chapter 1) which originates by the observation that some nonlinear partial differential equations (PDEs) can be interpreted as integrability conditions of an auxiliary linear system [54, 60]. Indeed, since the early applications of the inverse scattering method to the computation of soliton solutions of PS equations like KdV [1, 28], the notion of ZCR has been widely used in the study of PS equations as well as of most general nonlinear PDEs (see for instance [2, 8, 9, 26, 54, 65] and references therein). In particular, it is typical for an integrable system of PDEs to admit a ZCR which depends on some real parameter η, usually referred to as the *spectral parameter*. An example of this is given by the sine-Gordon example (see Section 1.2, of Chapter 1). The presence of such a parameter is crucial not only for the determination of exact solutions, via the inverse scattering method [1, 64] or the finite gap integration method [42], but also to guarantee other remarkable attributes of integrable equations like, for instance, parametric Bäcklund transformations and the existence of infinite hierarchies of conservation laws (see Section 1.6, of Chapter 1, and also [18, 20, 54, 56]). However, only nontrivial parameters are suitable for such applications of 1-parameter families of ZCRs. Hence the problem of deciding whether a parameter is trivial or not is particularly relevant in the theory of PS equations, as well as in the most general theory of ZCRs. This problem has been already studied in the paper [38], by identifying a cohomological obstruction to removability and providing an effective method for the elimination of trivial parameters. In Chapter 4, as discussed below, we consider another important problem which is that of constructing families of ZCRs (or linear problems) depending on nontrivial parameters.

In [20] Chern and Tenenblat obtained characterization results for evolution equations of the form $z_t = F(z, z_1, ..., z_k)$ (from now on we denote $z_i = \partial^i z/\partial x^i$), under the assumptions that $f_{ij} = f_{ij}(z, z_1, ..., z_k)$ and $f_{21} = \eta$, where η is a parameter. In the same paper the authors also considered a similar problem for equations of the form $z_{1,t} = F(z, z_1, ..., z_k)$. A noteworthy result of this study was an effective method for the explicit determination of entire new classes of differential equations that describe pseudospherical surfaces. Motivated by the results of [20], in a series of subsequent papers [30, 46, 47, 48], the same method was systematically implemented and new classes of pseudospherical equations were identified still with the basic assumption that $f_{21} = \eta$. Then in [18] the authors showed how the geometric properties of pseudospherical surfaces may provide infinite number of conservation laws when the functions f_{ij} are analytic functions of the spectral parameter η.

In 1995, Kamran and Tenenblat [34] generalized the results of [20] by giving a complete characterization of evolution equations of type $z_t = F(z, z_1, ..., z_k)$ which describe pseudospherical surfaces, in terms of necessary and sufficient conditions that have to be satisfied by F and the functions $f_{ij} = f_{ij}(z, z_1, ..., z_k)$, with no further additional conditions. Another generalization of [20] came in 1998 by Reyes who considered in [49] evolution equations of the more general form $z_t = F(x, t, z, z_1, ..., z_k)$, allowing x, t to appear explicitly in the equation and assuming that $f_{ij} = f_{ij}(x, t, z, z_1, ..., z_k)$ and $f_{21} = \eta$. Then, in a subsequent series of papers [50]-[52] Reyes also studied other aspects of such equations.

In 2002, differential systems describing pseudospherical surfaces or spherical surfaces (with constant positive curvature metrics) were studied by Ding-Tenenblat in [24].

Such systems include equations such as the nonlinear Schrödinger equation and the Heisenberg Ferromagnet model, and large new families of differential systems describing pseudospherical surfaces were obtained. In particular, these families have relations with those obtained by Fokas in [27].

Also we mention that a higher dimensional geometric generalization of the sine-Gordon equation, characterizing n-dimensional sub-manifolds of the Euclidean $\mathbf{E}^{2n-1}$ with constant sectional curvature $K = -1$, was considered in [62] and its intrinsic version as a metric on open subsets of $\mathbb{R}^n$, with $K = -1$, was studied in [7], by applying inverse scattering method. Other differential n-dimensional systems that are the integrability condition of linear systems of PDEs can be found in the so called *generating system* (see [61] and its references).

The several characterization results obtained in [20, 34, 49] are extremely useful, either in checking if a given differential equation describes pseudospherical surfaces or in generating large families of such equations. For instance, as an application of [34], Gomes [29] and Catalano-Tenenblat [17] classified evolution equations of the form $z_t = z_5 + G(z, z_1, z_2, z_3, z_4)$ and $z_t = z_4 + G(z, z_1, z_2, z_3)$, respectively, under the auxiliary assumptions that f_{21} and f_{31} are linear combinations of f_{11}. More recently, the same assumptions have been used by Silva and Tenenblat in [14] to give a classification of third order equations of the form $z_t = z_{2,t} + \lambda z z_3 + G(z, z_1, z_2)$, with $\lambda \in \mathbb{R}$.

The results of [14, 17, 29] permit the explicit description of huge classes of equations describing pseudospherical surfaces which, apart from the already known examples, represent a great amount of new equations whose physical relevance is highly expected. For example, some applications of equations classified by Rabelo and Tenenblat [6, 30, 46, 47] have been recently discussed by Sakovich in a series of papers (see for instance [55]). Of course, the same should occur in the case of results obtained in [14, 17, 29].

In **Chapter 2** we give a classification of PS equations of the form

$$z_t = A\,(x, t, z)\,z_2 + B\,(x, t, z, z_1)\,, \qquad A \neq 0,$$

with associated 1-forms

$$\omega_1 = f_{11}\,dx + f_{12}\,dt, \quad \omega_2 = f_{21}\,dx + f_{22}\,dt, \quad \omega_3 = f_{31}\,dx + f_{32}\,dt,$$

such that $f_{ij} = f_{ij}\,(x, t, z, z_1)$ and

$$f_{21} = \eta, \qquad \eta \in \mathbb{R}.$$

The main result of this classification shows that these evolution equations fall into three classes, further referred to as types. In each type, differential equations and associated linear problems can be easily obtained by choosing some arbitrary differentiable functions. Examples of such equations are the already known Svinolupov-Sokolov equations admitting higher weakly nonlinear symmetries [43], Boltzman equation, Marvan equation [39] and reaction-diffusion equations like Murray equation. Many other examples are presented forward the end of Section 2.2 and in Section 2.5.

In **Chapter 3** we study the problem of determining local isometric immersion of the families of pseudospherical surfaces described by the PS equations classified in Chapter 2, as well as for that described by some simple generalizations.

From the classical theory of Monge-Ampère equations of the form $f_{,xx}f_{,tt} - f_{xt}^2 = K$, it follows that surfaces of constant Gaussian curvature K always admit local isometric immersions in $\mathbf{E}^3$. However, due to Hilbert theorem, there exists no complete isometric immersion of bidimensional Riemannian metrics with Gaussian curvature $K = -1$ in $\mathbf{E}^3$. Hence, in particular, any given pseudospherical surface described by a PS equation $\mathcal{E}$ admits a local isometric immersion.

Hence, in view of the Bonnet theorem, to any generic solution z of $\mathcal{E}$, it is associated a pair $(I[z], II[z])$ of first and second fundamental forms, which solves the Gauss-Codazzi equations and describes a local isometric immersion into $\mathbf{E}^3$ of the associated pseudospherical surface. However, the dependence of $(I[z], II[z])$ on z may be quite complicate and in general it is not guaranteed the existence of a pair (I, II) which satisfies Gauss-Codazzi equations and smoothly depends on the generic solutions z of $\mathcal{E}$. In particular, the domain of the local immersion of the pseudospherical surface associated to a generic solution z is in general a subset of the domain of z, and by passing to generic solution z' these domains could change as well.

Nevertheless such a pair (I, II), which satisfies Gauss-Codazzi equations and smoothly depends on the solutions z, may still exist for some very special equations. An example is provided by the sine-Gordon equation with ω_1, ω_2 and ω_3 given by (0.0.1): indeed in this case one has $I = \frac{1}{\eta^2}dt^2 + 2\,cos(z)\,dx\,dt + \eta^2 dx^2$ and it is known (see for instance [61], Theorem 2.4) that Gauss-Codazzi equations are integrable and admit the second fundamental form $II = \pm 2\,sin(z)\,dx\,dt$ as an explicit solution. Hence one can always find local isometric immersions of pseudospherical surfaces corresponding to generic solutions of sine-Gordon equation.

Hence, in view of sine-Gordon example, it is natural to ask whether are there other PS equations which admit such a local isometric immersion for the described family of pseudospherical surfaces.

Recently this question has been investigated by T. Castro Silva, N. Kahouadji, N. Kamran and K. Tenenblat in the papers [13, 32, 33], under the assumption that the coefficients of the second fundamental form II depends on finitely many derivatives of z and does not explicitly depend on x and t. In [32, 33] they provided an answer in the case of k-th order evolution PS equations $z_t = F(z, z_1, ..., z_k)$ and second order hyperbolic PS equations $z_{1,t} = F(z, z_1)$, by restricting the study to the classes of 1-forms $\{\omega_1, \omega_2, \omega_3\}$ classified in [20] and [47]. Analogously, in [13] they provided an answer in the case of PS equations of the form $z_t - z_{2,t} = \lambda z z_3 + G(z, z_1, z_2)$, $\lambda \in \mathbb{R}$, by restricting the study to the classes of 1-forms $\{\omega_1, \omega_2, \omega_3\}$ classified in [14].

The results of these papers prove that, in the class of PS equations, the property of admitting local isometric immersions of the type considered above is exceptional since it holds only for some special classes of PS equations. In particular it turns out that sine-Gordon equation occupies a particularly special place amongst all these PS equations. Indeed, in view of the second fundamental form $II = \pm 2\, sin(z)\, dx\, dt$, for the sine-Gordon equation the restriction $(I[z], II[z])$ of the pair (I, II) to a given generic solution z, defined on a domain $U \subset \mathbb{R}^2$, is still defined on the same domain U without additional requirements. On the contrary, for all the other examples identified in the papers [13, 32, 33] the second fundamental form is only defined on a strip contained in the domain of a considered generic solution. Moreover, on the immersed pseudospherical surface defined by any given generic solution z of sine-Gordon equation, the normal curvatures a, c and the geodesic torsion b in the directions $\mathbf{e}_1$ and $\mathbf{e}_2$ dual to ω_1 and ω_2 (see sub-Section 1.1 of Chapter 1) depend explicitly on the particular solution z: indeed one can prove that for the sine-Gordon equation $a = \pm 2/tg(z)$, whereas $b = \pm 1$ and $c = 0$. On the contrary, for all the other examples identified in the papers [13, 32, 33] one has that a, b and c are independent of z, and only depend on x and t. Hence we can say that the local isometric immersions of pseudospherical surfaces described by sine-Gordon equation have the property of having "z-dependent" functions a, b and c.

The aim of Chapter 3 is that of continuing the investigations of papers [13, 32, 33] in the case of evolution PS equations classified in Chapter 2, and for a simple k-th order generalization of equations of Type I. Indeed, by first considering PS equations of the form $z_t = A(x, t, z)z_2 + B(x, t, z, z_1)$ with $f_{21} = \eta$ classified in Chapter 2, we found that only Type I equations admit these kind of local isometric immersions. Then, on the base of this result, we found an extension to the case of k-th order evolution equations in conservation law form $D_t\,(f(x, t, z)) = D_x\,(\Omega(x, t, z, z_1, \dots, z_k))$. As a result, we found that in the class of PS equations admitting local isometric immersions one also has second order equations like Boltzmann, Murray and Svinolupov-Sokolov equations, as well as higher order equations like Kuramoto-Sivashinsky, Sawada-Kotera and Kaup-Kupershmidt equations and also full hierarchies of integrable equations like Burgers, mKdV and KdV, which were not covered by the results of previous papers [32, 33]. However, it is noteworthy to observe that the special character of sine-Gordon equation is still confirmed by these results: sine-Gordon equation is the unique known example of a PS equation where the pair (I, II) has "z-dependent" functions a, b and c (see Section 1.1).

Finally in **Chapter 4** we discuss a method which uses the theory of classical symmetries of differential equations to construct nontrivial 1-parameter families of ZCRs, for $\mathfrak{g}$-valued ZCRs with $\mathfrak{g}$ a Lie sub-algebra of $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C})$. The case of ZCRs of PS equations corresponds to the special case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$.

While studying a differential equation, it is not unusual to know only a non-parametric ZCR or even a trivial 1-parameter family of ZCRs [21, 37, 54, 60]. Hence, the problem of constructing nontrivial 1-parameter families of ZCRs is of special interest for the application of the theory of ZCRs. In such cases one is naturally faced with the embedding problem of a given nonparametric ZCR into a nontrivial 1-parameter family of ZCRs.

Due to the importance of this problem, various attempts have been already made to provide any effective embedding method. Among these the symmetry method, first suggested in [37, 56] and further developed in the papers [22, 21, 36], is particularly representative.

In its original formulation, the symmetry method allows one to embed a given ZCR α into a 1-parameter family of ZCRs α_λ of $\mathcal{E}$, via the action on α of a 1-parameter group A_λ of projectable point symmetries of $\mathcal{E}$. However, in general, a 1-parameter group A_λ may be not "good" in the sense that the induced embedding may result in a trivial 1-parameter family α_λ. Hence, to solve this problem, the authors of [21] suggested to compare the symmetry algebras of $\mathcal{E}$ and its covering, and conjectured that "good" symmetry groups A_λ can be identified by a mismatch of these algebras. However, that conjecture remained unproved.

The aim of Chapter 4 is that of further developing the symmetry method, by taking into consideration the action of any kind of classical symmetry, and prove an infinitesimal criterion which is particularly effective in the identification of "good" infinitesimal classical symmetries, i.e., those symmetries which can be used to embed α into a nontrivial family α_λ of ZCRs of $\mathcal{E}$. According to that criterion we show that, relatively to α, one may distinguish classical infinitesimal symmetries of $\mathcal{E}$ into *gauge-like symmetries* and *non gauge-like symmetries*. The first type of symmetries form a Lie sub-algebra of the Lie algebra of symmetries of $\mathcal{E}$ and only produce trivial 1-parameter families of ZCRs. On the contrary, any 1-parameter family α_λ constructed with the flow of a non gauge-like symmetry is nontrivial. These results are illustrated with some examples in Section 4.3 and have been recently reported in the paper [15].

We note that Marvan also formulated in [40] an embedding method which is alternative to the symmetry method discussed in Chapter 4. Both methods may be considered completely algorithmic, however the symmetry method is computationally more simple than Marvan's method, when a non gauge-like symmetry exists.

Chapter 1

Preliminaries

For the reader's convenience we collect here some useful facts, and notations used throughout the thesis. The interested reader should refer to the general references [12, 20, 34, 38, 41, 44, 45, 53, 61, 63] for further details.

In particular, in Section 1.1 we review some useful elements of classical theory of surfaces in terms of moving frame formalism. Then, in Sections 1.2 and 1.3 we collect the material on PS equations used in the Chapters 2 and 3. Finally, in Sections 1.4, 1.5 and 1.6 we review the basic material on the geometric theory of differential equations and zero-curvature representations (ZCRs) in the form which is used in the Chapter 4.

1.1 Elements of surfaces theory with moving frames

In the 3-dimensional Euclidean space $\mathbf{E}^3$, with the canonical scalar product $<\,,\,>$, let $\mathbf{r} = \mathbf{r}(x^1, x^2)$ be a local chart of a regular surface M. By naturally extending the scalar product to $\mathbf{E}^3$-valued 1-forms, the first and second fundamental forms of M are respectively defined by

$$I :=< d\mathbf{r}, d\mathbf{r} >= \sum_{i,j=1}^{2} g_{ij} dx^i \cdot dx^j, \qquad II := - < d\mathbf{r}, d\mathbf{n} >= \sum_{i,j=1}^{2} a_{ij} dx^i \cdot dx^j,$$

with

$$\mathbf{n} := \frac{\mathbf{r}_{,1} \wedge \mathbf{r}_{,2}}{|\mathbf{r}_{,1} \wedge \mathbf{r}_{,2}|},$$

denoting the unit normal to M and

$$g_{ij} =< \mathbf{r}_{,i}, \mathbf{r}_{,j} >, \qquad a_{ij} = - < \mathbf{r}_{,i}, \mathbf{n}_{,j} >=< \mathbf{r}_{,ij}, \mathbf{n} > .$$

The principal curvatures and principal directions of M are the eigenvalues and eigenvectors of the shape operator P defined as $II(X, Y) = I(PX, Y)$, for any pair X, Y of vector fields tangent to M.

According to G. Darboux and E. Cartan the geometry of surfaces can be conveniently described by using the formalism of moving frames on M, which in the context considered here are orthonormal frames $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \mathbf{n}\}$ of vector fields with $\mathbf{e}_1$ and $\mathbf{e}_2$ tangent to M, and locally parametrized on the domain $U \subset \mathbb{R}^2$ of the chart $\mathbf{r}$. Indeed, since $d\mathbf{r}$ takes values in the tangent plane to M, one has that

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2, \tag{1.1.1}$$

with ω_1 and ω_2 differential 1-forms defined on U. On the other hand, in view of $< \mathbf{e}_i, \mathbf{e}_j >= \delta_{ij}$, one has

$$\begin{cases} d\mathbf{e}_1 = \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3, \\ d\mathbf{e}_2 = \omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3, \\ d\mathbf{e}_3 = \omega_{31}\mathbf{e}_1 + \omega_{32}\mathbf{e}_2, \end{cases} \tag{1.1.2}$$

with differential 1-forms ω_{ij} defined on U and such that

$$\omega_{ij} = -\omega_{ji}.$$

Hence, in terms of these 1-forms the first and second fundamental forms read

$$I = \omega_1^2 + \omega_2^2, \qquad II = \omega_1 \cdot \omega_{13} + \omega_2 \cdot \omega_{23},$$

where $\omega_i(\mathbf{e}_j) = \delta_{ij}$. In particular, this means that $\{\omega_1, \omega_2\}$ is a coframe on M dual to the orthonormal frame $\{e_1, e_2\}$.

Equations (1.1.1-1.1.2) are the Gauss-Weingarten equations of classical theory of surfaces, in the form of a first order system, whose compatibility conditions can be easily obtained in view of $d^2 = 0$. Indeed, from $d^2\mathbf{r} = d^2\mathbf{e}_i = 0$ one easily gets the Cartan's structure equations

$$d\omega_1 = \omega_{12} \wedge \omega_2, \qquad d\omega_2 = \omega_1 \wedge \omega_{12}, \tag{1.1.3}$$

$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0, \tag{1.1.4}$$

and

$$\begin{cases} d\omega_{12} = \omega_{13} \wedge \omega_{32}, \\ d\omega_{13} = \omega_{12} \wedge \omega_{23}, \\ d\omega_{23} = \omega_{21} \wedge \omega_{13}. \end{cases} \tag{1.1.5}$$

It follows that, in view of (1.1.3), the *connection* 1*-form* ω_{12} is completely determined by

$$\omega_{12}(\mathbf{e}_i) = d\omega_i(\mathbf{e}_1, \mathbf{e}_2), \qquad i = 1, 2, \tag{1.1.6}$$

whereas, equation (1.1.4) entails that $\omega_1 \wedge \omega_2 \wedge \omega_{13} = \omega_1 \wedge \omega_2 \wedge \omega_{23} = 0$ and hence one can write ω_{13} and ω_{23} as

$$\omega_{13} = a\omega_1 + b\omega_2, \qquad \omega_{23} = b\omega_1 + c\omega_2, \tag{1.1.7}$$

with a, b, c differentiable functions on U, whose geometric interpretation is as follows (see for instance [12]): functions a and c are the normal curvatures of M in the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively; b (resp., $-b$) is the geodesic torsion in the direction of $\mathbf{e}_1$ (resp., $\mathbf{e}_2$).

Therefore equations (1.1.5) reduce to

$$d\omega_{12} = -K\omega_1 \wedge \omega_2, \tag{1.1.8}$$

with

$$K = ac - b^2, \qquad \text{(Gauss equation)} \tag{1.1.9}$$

being the Gaussian curvature of M in terms of its extrinsic geometry, and

$$\begin{cases} d\omega_{13} = \omega_{12} \wedge \omega_{23}, \\ d\omega_{23} = \omega_{21} \wedge \omega_{13}. \end{cases} \qquad \text{(Codazzi equations)} \tag{1.1.10}$$

This way one easily gets the Gauss and Codazzi equations of the classical theory of surfaces.

Equations (1.1.8-1.1.9) and (1.1.10) are the compatibility conditions of Gauss-Weingarten equations (1.1.2).

It follows that the 1-forms ω_1, ω_2 and the connection form

$$\omega_3 = \omega_{12}$$

satisfy the equations

$$\begin{cases} d\omega_1 = \omega_3 \wedge \omega_2, \\ d\omega_2 = \omega_1 \wedge \omega_3, \\ d\omega_3 = -K\omega_1 \wedge \omega_2. \end{cases} \tag{1.1.11}$$

In particular, the structure equations (1.1.11) describe the intrinsic geometry of the surface M. Moreover, since in view of (1.1.6) the 1-form ω_3 is completely determined by $\{\mathbf{e}_1, \mathbf{e}_2\}$

and $\{\omega_1, \omega_2\}$, and hence it only depends on the intrinsic geometry of M, then the third equation of (1.1.11) provides a proof of Gauss' *teorema egregium,* which states that K does not depend on the extrinsic geometry of M and is completely determined by first fundamental form I. The extrinsic geometry of M, on the other side, is described by the Gauss-Weingarten equations (1.1.1-1.1.2), provided that their compatibility conditions (1.1.10) and (1.1.9) are satisfied.

Finally we remember here the classical Bonnet theorem which states that given two symmetric bilinear forms I and II on $U \subset \mathbb{R}^2$, which satisfy (1.1.8-1.1.9) and (1.1.10) and with I positive-definite, for every $p \in U$ there exists a neighborhood $V \subset U$ of p and a diffeomorphism $\mathbf{r} : V \to \mathbf{r}(V) \subset \mathbb{R}^3$ such that the regular surface $\mathbf{r}(V) \subset \mathbf{E}^3$ has I and II as first and second fundamental forms, respectively, and $\mathbf{r}$ is unique up to isometries of $\mathbf{E}^3$. The mapping $\mathbf{r}$ is a local isometric immersion of the Riemannian manifold (U, I), and in the particular case when I is pseudospherical, i.e. when I is such that $K = -1$, it is also classically known that such a local isometric immersion of (U, I) in $\mathbf{E}^3$ always exists. Observe that, in the case of a pseudospherical surface, equations (1.1.11) reduce to the structure equations of pseudospherical surfaces used throughout this work.

1.2 PS equations

If (M, g) is a 2-dimensional Riemannian manifold and $\{\omega_1, \omega_2\}$ is a coframe, dual to an orthonormal frame $\{e_1, e_2\}$, then $g = \omega_1^2 + \omega_2^2$ and ω_i satisfy the structure equations: $d\omega_1 = \omega_3 \wedge \omega_2$ and $d\omega_2 = \omega_1 \wedge \omega_3$, where ω_3 denotes the connection form defined as $\omega_3(e_i) = d\omega_i(e_1, e_2)$. The Gaussian curvature of M is the function K such that $d\omega_3 = -K\omega_1 \wedge \omega_2$.

We say that a k-th order differential equation $\mathcal{E}$ for a real-valued function z=$z\,(x, t)$, describes pseudospherical surfaces, or that it is a *PS equation,* if it is equivalent to the structure equations of a surface with Gaussian curvature $K = -1$, i.e.,

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3, \quad d\omega_3 = \omega_1 \wedge \omega_2, \tag{1.2.1}$$

where $\{\omega_1, \omega_2, \omega_3\}$ are 1-forms

$$\omega_1 = f_{11}dx + f_{12}dt, \quad \omega_2 = f_{21}dx + f_{22}dt, \quad \omega_3 = f_{31}dx + f_{32}dt, \tag{1.2.2}$$

such that $\omega_1 \wedge \omega_2 \neq 0$ and f_{ij} are functions of x, t, $z(x, t)$ and derivatives of $z(x, t)$ with respect to x and t.

Notice that according to the definition $\omega_1 \wedge \omega_2$ is generically nonzero on the solutions of a PS equation $\mathcal{E}$. However, this condition does not guarantee the property that, for any solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$, the restriction $(\omega_1 \wedge \omega_2)[z]$ of $\omega_1 \wedge \omega_2$ to z is everywhere nonzero on U. Relatively to a given system of 1-forms $\{\omega_1, \omega_2, \omega_3\}$, we will call *generic* a solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ such that $(\omega_1 \wedge \omega_2)[z]$ is almost everywhere nonzero on U, i.e., it is everywhere nonzero except for a subset of U of measure zero. Thus, for any generic solution $z : U \subset \mathbb{R}^2 \to \mathbb{R}$ of a PS equation $\mathcal{E}$, the restriction $I[z]$ of $I = \omega_1^2 + \omega_2^2$ to z defines almost everywhere a Riemannian metric $I[z]$ on the domain U with Gaussian curvature $K = -1$. It is in this sense that one can say that a PS equation describes, or parametrizes, a family of non-immersed pseudospherical surfaces.

A classical example is the KdV equation $z_t = z_{xxx} + 6zz_x$, which corresponds to

$$\begin{aligned}
\omega_1 &= (1-z)\ dx + (-z_{xx} + \eta z_x - \eta^2 z - 2z^2 + \eta^2 + 2z)\ dt,\\
\omega_2 &= \eta\, dx + (\eta^3 + 2\eta z - 2z_x)\ dt,\\
\omega_3 &= -\,(1+z)\ dx + (-z_{xx} + \eta z_x - \eta^2 z - 2z^2 - \eta^2 - 2z)\ dt,
\end{aligned}$$

with $\eta \in \mathbb{R}$. Another classical example is the sine-Gordon equation $z_{xt} = sin\,(z)$, which corresponds to

$$\begin{aligned}
\omega_1 &= \tfrac{1}{\eta} sin\,(z)\ dt,\\
\omega_2 &= \eta\, dx + \tfrac{1}{\eta} cos\,(z)\ dt,\\
\omega_3 &= z_x\, dx,
\end{aligned}$$

with $\eta \in \mathbb{R} - \{0\}$.

Another example is the nonlinear dispersive wave equation (Camassa-Holm)

$$z_t - z_{xxt} = zz_{xxx} + 2z_x z_{xx} - 3zz_x - mz_x,$$

which corresponds to

$$\begin{aligned}
\omega_1 &= \left(z - z_{xx} + \tfrac{m+\eta^2}{2} - 1\right) dx + \left[-z\,(f_{11}+1) \pm \eta z_x - \tfrac{m+\eta^2}{2} + 1\right] dt,\\
\omega_2 &= \eta\, dx + (-\eta z \pm z_x - \eta)\ dt,\\
\omega_3 &= \pm\left(z - z_{xx} + \tfrac{m+\eta^2}{2}\right) dx + \left[\mp z\left(z - z_{xx} + \tfrac{m+\eta^2}{2}\right) + \eta z_x \mp z \mp \tfrac{m+\eta^2}{2}\right] dt,
\end{aligned}$$

with $\eta \in \mathbb{R}$.

PS equations can also be characterized in few alternative ways. For instance, the system of equations (1.2.1) is equivalent to the integrability condition of the linear system

$$\begin{pmatrix} dv^1 \\ dv^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{1.2.3}$$

where $v^i = v^i\,(x,t)$.

Another interpretation comes by the use of the $\mathfrak{sl}(2,\mathbb{R})$-valued 1-form

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega_2 & \omega_1 - \omega_3 \\ \omega_1 + \omega_3 & -\omega_2 \end{pmatrix} = X dx + T dt, \tag{1.2.4}$$

with X and T being the $\mathfrak{sl}(2,\mathbb{R})$-valued smooth functions (also known as Lax pair in matrix form)

$$X = \frac{1}{2}\begin{pmatrix} f_{21} & f_{11} - f_{31} \\ f_{11} + f_{31} & -f_{21} \end{pmatrix}, \quad T = \frac{1}{2}\begin{pmatrix} f_{22} & f_{12} - f_{32} \\ f_{12} + f_{32} & -f_{22} \end{pmatrix}. \tag{1.2.5}$$

Indeed, equations (1.2.1) are equivalent to

$$d\Omega - \frac{1}{2}[\Omega, \Omega] = 0.$$

This means that, for any solution $z = z(x,t)$ of $\mathcal{E}$, defined on a domain $U \subset \mathbb{R}^2$, Ω is a Maurer-Cartan form defining a flat connection on a trivial principal $SL(2,\mathbb{R})$-bundle over U (see for instance [25, 58]).

Moreover, by using the notation $V := (v^1, v^2)^T$, (1.2.3) can be written as the linear problem

$$\frac{\partial V}{\partial x} = XV, \quad \frac{\partial V}{\partial t} = TV. \tag{1.2.6}$$

It is easy to show that equations (1.2.1) are equivalent to the integrability condition of (1.2.6), namely

$$D_t X - D_x T + [X, T] = 0, \tag{1.2.7}$$

where D_t and D_x are the total derivative operators with respect to t and x, respectively.

In the literature [23] 1-form Ω, and sometimes the pair (X,T) or even (1.2.7), is referred to as an $\mathfrak{sl}(2,\mathbb{R})$-valued *zero-curvature representation* for the equation $\mathcal{E}$. Moreover, the linear system (1.2.3) or (1.2.6) is usually referred to as *the linear problem associated to $\mathcal{E}$.*

Remark 1.2.1. It is noteworthy to remark that saying that an equation $\mathcal{E}$ admits an $\mathfrak{sl}(2,\mathbb{R})$-valued zero-curvature representation is not equivalent to say that $\mathcal{E}$ is a PS equation. Indeed, for $\mathcal{E}$ describing pseudospherical surfaces it is required that the functions f_{ij} in (1.2.5) satisfy the non-degeneracy condition $\omega_1 \wedge \omega_2 = (f_{11}f_{22} - f_{12}f_{21})\, dx \wedge dt \neq 0$, which guarantees that $\omega_1^2 + \omega_2^2$ is generically non-degenerated.

It is this linear problem that, in some cases, is used in the construction of explicit solutions of PS equations, by means of inverse scattering method [4, 5, 28]. In particular, when $f_{21} = \eta$, where η is a parameter and f_{11}, f_{31} are independent of η, the linear problem (1.2.6) is the so called AKNS system [1].

We notice that, under the gauge transformation $X \to X^S = S\,X\,S^{-1} + D_x S\,S^{-1}$, $T \to T^S = S\,T\,S^{-1} + D_t S\,S^{-1}$, where S is an $SL(2,\mathbb{R})$-valued smooth function, left hand side of (1.2.7) transforms to

$$D_t X - D_x T + [X, T] = S\,\left(D_t X^S - D_x T^S + \left[X^S, T^S\right]\right)\,S^{-1},$$

and hence (1.2.7) is invariant. However, one should be aware of the fact that such a gauge transformation may not preserve the condition $\omega_1 \wedge \omega_2 = (f_{11} f_{22} - f_{12} f_{21})\,dx \wedge dt \neq 0$.

For instance, with $f_{11} = e^{\eta x} z = -f_{31}$, $f_{12} = e^{\eta}\,(z_x + z^2) = -f_{32}$, $f_{21} = \eta$, $f_{22} = 0$, the pair X, T given by (1.2.5) defines an $\mathfrak{sl}\,(2,\mathbb{R})$-valued zero-curvature representation of Burgers equation $z_t = z_{xx} + 2 z z_x$, which transforms to

$$X^S = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \quad T^S = \begin{pmatrix} 0 & z_x + z^2 \\ 0 & 0 \end{pmatrix},$$

under the transformation given by $S = \begin{pmatrix} e^{\frac{\eta x}{2}} & 0 \\ 0 & e^{-\frac{\eta x}{2}} \end{pmatrix}$.

Therefore under the transformation defined by S, f_{ij} transforms as $f^S_{ij} = e^{-\eta x} f_{ij}$, for $(i,j) \neq (2,1)$, and $f^S_{21} = 0$. Hence $f^S_{11} f^S_{22} - f^S_{12} f^S_{21} = 0$, whereas $f_{11} f_{22} - f_{12} f_{21} \neq 0$.

Throughout this thesis, partial derivatives of $z = z(x,t)$ of order i with respect to x will be denoted by z_i, i.e.,

$$z_i = \frac{\partial^i z}{\partial x^i}.$$

Hence, an evolution equation of order k will be written in the form

$$z_t = F\,(x, t, z, z_1, \ldots, z_k)\,. \tag{1.2.8}$$

It is noteworthy to remark that equations in conservation law forms are PS equations, as stated by the following easy to prove

Theorem 1.2.2. *The k-th order evolution equations of the form*

$$D_t\,(f) = D_x\,(\Omega)\,, \tag{1.2.9}$$

where $f = f\,(x, t, z)$, $\Omega = \Omega\,(x, t, z, z_1, \ldots, z_{k-1})$ are arbitrary differentiable functions, such that $f_{,z}\,\Omega_{,z_{k-1}} \neq 0$ on a nonempty open set, is a PS equation with associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$ of one of the following two alternative types:

a)

$$
\begin{aligned}
f_{11} &= e^{-\epsilon(\eta x+g)} f, & f_{12} &= e^{-\epsilon(\eta x+g)} \Omega, \\
f_{21} &= \eta, & f_{22} &= g', \\
f_{31} &= \epsilon e^{-\epsilon(\eta x+g)} f, & f_{32} &= \epsilon e^{-\epsilon(\eta x+g)} \Omega,
\end{aligned}
\tag{1.2.10}
$$

where $\epsilon = \pm 1$, $g = g(t)$ *is an arbitrary differentiable function and* $\eta^2 + (g')^2 \neq 0$*;*

b)

$$
\begin{aligned}
f_{11} &= cosh\,(\eta x + g)\, f, & f_{12} &= cosh\,(\eta x + g)\, \Omega, \\
f_{21} &= \eta, & f_{22} &= g', \\
f_{31} &= -sinh\,(\eta x + g)\, f, & f_{32} &= -sinh\,(\eta x + g)\, \Omega,
\end{aligned}
\tag{1.2.11}
$$

where $g = g(t)$ *is an arbitrary differentiable function and* $\eta^2 + (g')^2 \neq 0$*.*

Remark 1.2.3. The class of PS equations described by Theorem 1.2.2 may be thought as being a generalization to k-th order of the Type I class of PS equations obtained in Chapter 2.

1.3 Finite-order local isometric immersions of surfaces described by PS equations

In view of (1.2.2) and (1.1.7), the second fundamental forms of local isometric immersions of surfaces described by the solutions of a PS equation have the form

$$
II = \omega_1\omega_{13} + \omega_2\omega_{23} = a_{11}dx^2 + 2a_{12}dx \cdot dt + a_{22}dt^2,
$$

with

$$
\begin{cases}
a_{11} = a f_{11}^2 + 2b f_{11} f_{21} + c f_{21}^2, \\
a_{12} = a f_{11} f_{12} + b\,(f_{11} f_{22} + f_{21} f_{12}) + c f_{21} f_{22}, \\
a_{22} = a f_{12}^2 + 2b f_{12} f_{22} + c f_{22}^2,
\end{cases}
\tag{1.3.1}
$$

and a, b, c differentiable functions of x, t, z and derivatives of z with respect to x and t.

It follows that, using the total derivative operators D_x and D_t, the two Codazzi equations (1.1.10) and the Gauss equation (1.1.9) have the form

$$\begin{cases} f_{11}D_t a + f_{21}D_t b - f_{12}D_x a - f_{22}D_x b - 2b\,(f_{11}f_{32} - f_{31}f_{12}) \\ +(a-c)\,(f_{21}f_{32} - f_{31}f_{22}) = 0, \\ \\ f_{11}D_t b + f_{21}D_t c - f_{12}D_x b - f_{22}D_x c + (a-c)\,(f_{11}f_{32} - f_{31}f_{12}) \\ +2b\,(f_{21}f_{32} - f_{31}f_{22}) = 0, \end{cases} \tag{1.3.2}$$

and

$$ac - b^2 = -1, \tag{1.3.3}$$

respectively.

In view of Bonnet theorem, the local isometric immersion of the pseudospherical surfaces described by the space of solutions of a PS equation exist if and only if there exist a solution $\{a, b, c\}$ of (1.3.2-1.3.3). In this thesis we will restrict the problem of determining such a triple $\{a, b, c\}$ in the case of PS equations described by Theorem 2.2.1, and also Theorem 1.2.2, under the assumption that the triple $\{a, b, c\}$ is of *finite-order*, i.e., depends only on x, t, z and finitely many derivatives of z with respect to x and t. We will refer to local isometric immersions described by such a *finite-order* triple $\{a, b, c\}$, as *finite-order local isometric immersions.*

1.4 Symmetries of differential equations

Let $\pi : E \to M$ be a fiber bundle, with $dim\, M = n$ and $dim\, E = n + m$. For any $k \in \mathbb{N}$ we denote by $J^k(\pi)$ the manifold of k-th order jets of sections of π and by $\pi_k : J^k(\pi) \to M$ the *k-order jet bundle* of sections of π. By definition, a point θ of $J^k(\pi)$ is an equivalence class $\theta = [s]^k_a$ of smooth sections s of π whose graphs at $a \in M$ pass through the same point $s(a)$, where they have the same contact up to order k. Hence, any section s of π together with its derivatives up to order k determines a section $j_k(s)$ of π_k which is called the *k-th order prolongation of* s. For $h > k$ we denote by $\pi_{h,k} : J^h(\pi) \to J^k(\pi)$ the *natural projection* of $J^h(\pi)$ onto $J^k(\pi)$, given by $[s]^h_a \to [s]^k_a$.

Denoting by $\{x_1, ..., x_n\}$ local coordinates on M and by $\{z^1, ..., z^m\}$ local fiber coordinates of π, the induced *natural coordinates* on $J^k(\pi)$ will be $\{x_i, z^j_\sigma\}$, where $i \in \{1, ..., n\}$, $j \in \{1, ..., m\}$ and $\sigma = (\sigma_1, ..., \sigma_n)$ is a multi-index of order $|\sigma| = \sigma_1 + ... + \sigma_n$ such that $0 \le |\sigma| \le k$. By definition, if $\theta = [s]^k_a$, then $x_i(\theta) := x_i(a)$ and $z^j_\sigma(\theta) := \frac{\partial^{|\sigma|} s^j}{\partial x_1^{\sigma_1} ... \partial x_n^{\sigma_n}}(a)$. Throughout the thesis, $C^\infty(M)$ will denote the algebra of smooth functions on M and $\mathcal{F}_k(\pi)$ will denote the algebra of *smooth functions on* $J^k(\pi)$.

The k-th order jet space $J^k(\pi)$ is naturally equipped with the *Cartan distribution*

$\mathcal{C}^k(\pi)$, which is spanned by tangent planes to graphs of k-th order jet prolongations of sections of π. In coordinates $\mathcal{C}^k(\pi)$ can be described either as the annihilator of the Pfaffian system $\{\theta^j_\sigma = dz^j_\sigma - \sum_i z^j_{\sigma+1_i}dx_i \ : \ 0 \leq |\sigma| \leq k-1, \ j = 1,...,m\}$, or as the distribution generated by the vector fields $\{\partial_{z^j_{\sigma+1_i}}, \ D^{(k)}_i \ : \ 0 \leq |\sigma| \leq k-1, \ j = 1,...,m\}$, with $D^{(k)}_i$ denoting the k-th order *truncated total derivative operators*, i.e.,

$$D^{(k)}_i := \partial_{x_i} + \sum_{|\sigma| \leq k-1} z^j_{\sigma+1_i}\partial_{z^j_\sigma}.$$

A k-th order system of differential equations $\mathcal{E}$, for the sections of π, can be geometrically interpreted as a submanifold $\mathcal{E} \subset J^k(\pi)$ and its solutions are just sections s of π whose k-th prolongations $j_k(s)$ lie in $\mathcal{E}$. Hence, solutions of $\mathcal{E}$ are sections of π whose k-th prolongations are integral manifolds of the *induced Cartan distribution* $\mathcal{C}^k(\mathcal{E}) := \mathcal{C}^k(\pi) \cap T\mathcal{E}$ over $\mathcal{E}$.

Classical finite symmetries of a k-th order system $\mathcal{E} \subset J^k(\pi)$ are finite symmetries of $\mathcal{C}^k(\pi)$ which leave invariant the submanifold $\mathcal{E}$. *Classical infinitesimal symmetries* of $\mathcal{E}$ are vector fields on $J^k(\pi)$ which are infinitesimal symmetries of $\mathcal{C}^k(\pi)$ and are tangent to $\mathcal{E}$. Hence, by definition, the flow of an infinitesimal symmetry of $\mathcal{E}$ is a 1-parameter local group of local diffeomorphisms which are finite symmetries of $\mathcal{E}$. A finite symmetry f is called *projectable* if $f^*(C^\infty(M)) \subset C^\infty(M)$. Analogously, an infinitesimal symmetry X is called projectable if $X(C^\infty(M)) \subset C^\infty(M)$.

The explicit description of infinitesimal symmetries of $\mathcal{C}^k(\pi)$ is particularly simple. Indeed, if one writes $D^{(k)}_\sigma := \left(D^{(k)}_1\right)^{\sigma_1} \circ ... \circ \left(D^{(k)}_n\right)^{\sigma_n}$, one can show that [44, 63]:

1. When $m = 1$, infinitesimal symmetries of $\mathcal{C}^k(\pi)$ are of the form

$$Y^{(k)}_\varphi = -\sum_{i=1}^{n} \frac{\partial\varphi}{\partial z_i} D^{(k)}_i + \sum_{|\sigma|=0}^{k-1} D^{(k)}_\sigma(\varphi)\partial_{z_\sigma}, \tag{1.4.1}$$

   where $\varphi \in \mathcal{F}(\pi)$.

2. When $m > 1$, infinitesimal symmetries of $\mathcal{C}^k(\pi)$ are of the form

$$Y^{(k)}_\varphi = -\sum_{i} \frac{\partial\varphi^a}{\partial z^a_i} D^{(k)}_i + \sum_{|\sigma|=0}^{k-1}\sum_{j=1}^{m} D^{(k)}_\sigma(\varphi^j)\partial_{z^j_\sigma}, \tag{1.4.2}$$

   where a is any fixed integer in $\{1,...,m\}$ and $\varphi = (\varphi^1,...,\varphi^m)$ is any smooth vector function on $J^1(\pi)$ such that for any $j \in \{1,...,m\}$ one has $\varphi^j = \sum_{i=1}^{n}\xi^i(x,z)z^j_i + \eta^j(x,z)$, for some smooth functions ξ^i and η^j.

The function φ is called the *generating function* of the k-th order classical symmetry $Y_\varphi^{(k)}$, since the symmetry is completely determined by φ.

On the other hand, for any $h > k$, the symmetries $Y_\varphi^{(h)}$ and $Y_\varphi^{(k)}$ generated by φ project one to the other under the action of the pushforward $\pi_{h,k*}$. Hence it is natural to say that $Y_\varphi^{(h)}$ is the *prolongation* of $Y_\varphi^{(k)}$ to $J^h(\pi)$. In particular, any classical infinitesimal symmetry of $\mathcal{C}^k(\pi)$ is the prolongation of a first order symmetry. However, when $m > 1$ the generating function $\varphi = (\varphi^1, ..., \varphi^m)$ is always linear in the variables z_i^j 's, hence $Y_\varphi^{(k)}$ always projects to the vector field $Y_\varphi^{(0)} := \xi^i \partial_{x^i} + \eta^j \partial_{z^j}$ on $J^0(\pi)$. In such a case one may call $Y_\varphi^{(k)}$ the prolongation to $J^k(\pi)$ of the vector field $Y_\varphi^{(0)}$ on $J^0(\pi)$. Traditionally, classical infinitesimal symmetries which are prolongations of vector fields on $J^0(\pi)$ are called *infinitesimal point symmetries.* On the contrary, infinitesimal classical symmetries which are not point symmetries are traditionally called *infinitesimal contact symmetries.* Infinitesimal contact symmetries only exist when $m = 1$, since for $m = dim\, \pi > 1$ one only has infinitesimal point symmetries.

In practice, *computing infinitesimal classical symmetries* $\mathcal{E} \subset J^k(\pi)$ consists in the search of generating functions φ such that $Y_\varphi^{(k)}$ are tangent to $\mathcal{E}$. This tangency condition returns a linear system of PDEs for the function φ, which is usually overdetermined and hence can be algorithmically studied by taking into consideration the full set of compatibility conditions. The analysis of this kind of system is in general more feasible if one makes use of symbolic packages like those available in Maple.

The *infinite jet space* $J^\infty(\pi)$ is the inverse limit of the sequence of surjections $M \xleftarrow{\pi} J^0(\pi) \xleftarrow{\pi_{1,0}} ... \xleftarrow{\pi_{k,k-1}} J^k(\pi) \xleftarrow{\pi_{k+1,k}} ...$. By definition, any $\theta \in J^\infty(\pi)$ is a sequence $\theta = \{\theta_r\}$ of points $\theta_r \in J^r(\pi)$ such that $\pi_{h,k}(\theta_h) = \theta_k$, for all $h > k$. Of course $J^\infty(\pi)$ is not a finite dimensional manifold, nevertheless one may introduce a differential calculus on $J^\infty(\pi)$ by making use of standard constructions of differential calculus over commutative algebras [63]. Indeed, by defining the exterior algebra $\Lambda^*(\pi)$ of *differential forms on* $J^\infty(\pi)$ as the direct limit of the sequences of embeddings $\Lambda^*\left(J^k(\pi)\right) \xrightarrow{\pi^*_{k+1,k}} \Lambda^*\left(J^{k+1}(\pi)\right) \xrightarrow{\pi^*_{k+2,k+1}}$ .... , one also defines the commutative algebra of *smooth functions on* $J^\infty(\pi)$ as $\mathcal{F}(\pi) := \Lambda^0(\pi)$. Since $\Lambda^*\left(J^k(\pi)\right) \subset \Lambda^*\left(J^{k+1}(\pi)\right) \subset ...$, $\Lambda^*(\pi)$ is a filtered algebra and one may think of any h-form over $J^\infty(\pi)$ as an h-form on some finite order jet space; this is true in particular for $\mathcal{F}(\pi)$. It follows that the *exterior differential* d naturally extends to the exterior algebra $\Lambda^*(\pi)$ and defines the de Rham complex of $J^\infty(\pi)$.

Then, the $\mathcal{F}(\pi)$-module $\mathcal{D}(\pi)$ of *vector fields* on $J^\infty(\pi)$ is by definition the module of all derivations of $\mathcal{F}(\pi)$ which preserve the natural filtration $\mathcal{F}_k(\pi) \subset \mathcal{F}_h(\pi) \subset ...$ of $\mathcal{F}(\pi)$, i.e., any $Z \in \mathcal{D}(\pi)$ has an associated *filtration degree* $l \in \mathbb{N}$ such that $Z(\mathcal{F}_k(\pi)) \subseteq \mathcal{F}_{k+l}(\pi)$, $\forall k \in \mathbb{N}$. Hence, in coordinates, a vector field Z can be identified with a formal series $Z = \sum \alpha_i \partial_{x_i} + \sum \beta_\sigma^j \partial_{z_\sigma^j}$, with $\alpha_i, \beta_\sigma^j \in \mathcal{F}(\pi)$. It follows that vector fields on

$J^\infty(\pi)$, contrary to the finite dimensional case, do not have in general an associated flow. However, any Z with zero filtration degree is the inverse limit of an inverse sequence $\{Z^{(k)}\}$ of vector fields on finite order jet spaces, hence one may define the flow of Z as being the inverse limit of a sequence of flows on finite order jet spaces. Moreover, one can define *Lie derivative* of functions, vector fields or forms on $J^\infty(\pi)$ in a completely algebraic way. For instance, the Lie derivative of a function $G \in \mathcal{F}(\pi)$ along a vector field $Z \in \mathcal{D}(\pi)$ is $L_Z(G) := Z(G)$, and the Lie derivative of $Y \in \mathcal{D}(\pi)$ along Z is $L_Z Y := [Z, Y]$. Whereas, the Lie derivative of a form $\omega \in \Lambda^*(\pi)$ along Z is defined as $L_Z\omega := i_Z(d\omega) + d(i_Z\omega)$, where i_Z denotes the *inner product* operation $i_Z : \Lambda^h(\pi) \longrightarrow \Lambda^{h-1}(\pi)$.

Also $J^\infty(\pi)$ is naturally equipped with a Cartan distribution denoted by $\mathcal{C}(\pi)$ and defined as the inverse limit of the sequence of surjections $\mathcal{C}^1(\pi) \overset{\pi_{1,0*}}{\longleftarrow} \mathcal{C}^2(\pi) \overset{\pi_{2,1*}}{\longleftarrow} \ldots \overset{\pi_{k,k-1*}}{\longleftarrow} \mathcal{C}^k(\pi) \overset{\pi_{k+1,k*}}{\longleftarrow} \ldots$. In coordinates $\mathcal{C}(\pi)$ can be described either as the annihilator of the Pfaffian system $\{\theta^j_\sigma = dz^j_\sigma - \sum_i z^j_{\sigma+1_i} dx_i : \ |\sigma| \geq 0, \ j = 1, ..., m\}$, or as the distribution generated by the totality of vector fields $\{\partial_{z^j_{\sigma+1_i}}, \ D_i : \ |\sigma| \geq 0, \ j = 1, ..., m\}$, where

$$D_i := \partial_{x_i} + \sum_{|\rho|\geq 0} \sum_{j=1}^{m} z^j_{\rho+1_i} \partial_{z^j_\rho}$$

denote the *total derivative operators*.

Given a k-th order equation (or system) $\mathcal{E} = \{F = 0\} \subset J^k(\pi)$, under regularity conditions one may consider the *r-th order prolongation* $\mathcal{E}^{(r)} = \{D_\mu F = 0 : \ 0 \leq |\mu| \leq r\}$. Equation $\mathcal{E}$ will be called *formally integrable* if, and only if, $\mathcal{E}^{(r)}$ are submanifolds of $J^{k+r}(\pi)$ and the maps $\pi_{k+r+1,k+r} : \mathcal{E}^{(r+1)} \to \mathcal{E}^{(r)}$ are smooth fiber bundles, for any $r \geq 0$.

By definition the *infinite prolongation* $\mathcal{E}^{(\infty)} = \{D_\mu F = 0 : \ |\mu| \geq 0\}$ of a formally integrable equation $\mathcal{E}$ is the inverse limit of the sequence of fiber bundles $\pi_{k+r+1,k+r} : \mathcal{E}^{(r+1)} \to \mathcal{E}^{(r)}$. By restricting $\Lambda^*(\pi)$ to $\mathcal{E}^{(\infty)}$ one gets the exterior algebra $\Lambda^*(\mathcal{E})$ of *differential forms on* $\mathcal{E}^{(\infty)}$ and in particular the algebra $\mathcal{F}(\mathcal{E})$ of *smooth functions on* $\mathcal{E}^{(\infty)}$. Moreover, since any prolongation $\mathcal{E}^{(r)}$ is naturally equipped with an induced Cartan distribution $\mathcal{C}^{k+r}(\mathcal{E}^{(r)})$, also $\mathcal{E}^{(\infty)}$ is naturally equipped with an induced *Cartan distribution* denoted by $\mathcal{C}(\mathcal{E})$ and defined as the inverse limit of the sequence of surjections $\mathcal{C}^k(\mathcal{E}) \overset{\pi_{k+1,k*}}{\longleftarrow} \mathcal{C}^{k+1}(\mathcal{E}^{(1)}) \overset{\pi_{k+2,k+1*}}{\longleftarrow} \ldots$. Hence one may further extend the notion of symmetry for a system $\mathcal{E}$: a vector field $Z \in \mathcal{D}(\pi)$ is a symmetry of $\mathcal{E}^{(\infty)}$ if, and only if, Z is a symmetry of $\mathcal{C}(\pi)$ which is tangent to $\mathcal{E}^{(\infty)}$. These symmetries are called *generalized symmetries of* $\mathcal{E}$, and we will refer to the restriction $\bar{Z}$ of a generalized symmetry Z to $\mathcal{E}^{(\infty)}$ as *a restricted generalized symmetry.*

Among the generalized symmetries of $\mathcal{E}$ one has the *classical generalized symmetries*, which by definition are infinite prolongations of classical symmetries of $\mathcal{E}$: recall that, given a classical symmetry Y_φ of $\mathcal{C}^k(\pi)$, its *infinite prolongation* $Y_\varphi^{(\infty)}$ is

$$Y_\varphi^{(\infty)} = -\sum_{i=1}^{n} \frac{\partial \varphi^a}{\partial z_i^a} D_i + \sum_{|\sigma|\geq 0} \sum_{j=1}^{m} D_\sigma(\varphi^j)\partial_{z_\sigma^j}.$$

In this thesis we are mainly concerned with classical generalized symmetries, since these are the only generalized symmetries which always admit a flow. For further details on the theory of generalized symmetries see [44, 63].

1.5 Horizontal forms with values in a Lie algebra and ZCRs

Since $\mathcal{C}(\pi)$ is totally horizontal with respect to the mapping $\pi_\infty : J^\infty(\pi) \to M$, the tangent bundle $\mathcal{T}(\pi)$ on $J^\infty(\pi)$ decomposes as $\mathcal{T}(\pi) = \mathcal{V}(\pi) \oplus \mathcal{C}(\pi)$, where $\mathcal{V}(\pi) := Ker\,(\pi_\infty)_*$. Dually one has $\Lambda^1(\pi) = \Lambda^{(1,0)}(\pi) \oplus \Lambda^{(0,1)}(\pi)$, where $\Lambda^{(1,0)}(\pi) := Ann\,(\mathcal{V}(\pi))$ and $\Lambda^{(0,1)}(\pi) := Ann\,(\mathcal{C}(\pi))$ are the $\mathcal{F}(\pi)$-modules of horizontal and vertical 1-forms on $J^\infty(\pi)$ locally generated by $\{dx^i\}$ and the Cartan forms $\{\theta_\sigma^j\}$, respectively. More in general, by considering $\Lambda^{(p,q)}(\pi) = \left(\bigwedge^p \Lambda^{(1,0)}(\pi)\right) \bigwedge \left(\bigwedge^q \Lambda^{(0,1)}(\pi)\right)$, the $\mathcal{F}(\pi)$-module of r-forms on $J^\infty(\pi)$ decomposes as $\Lambda^r(\pi) = \bigoplus_{p+q=r} \Lambda^{(p,q)}(\pi)$. By definition we set $\mathcal{F}(\pi) = \Lambda^{(0,0)}(\pi)$. Accordingly, the exterior differential splits into the sum $d = d_H + d_V$ of the horizontal and vertical differentials $d_H : \Lambda^{(p,q)}(\pi) \to \Lambda^{(p+1,q)}(\pi)$ and $d_V : \Lambda^{(p,q)}(\pi) \to \Lambda^{(p,q+1)}(\pi)$, satisfying $d_H^2 = d_V^2 = 0$ and $d_H \circ d_V = -d_V \circ d_H$. In coordinates, horizontal and vertical differentials can be easily computed, since they act as graded derivations on $\Lambda^*(\pi)$ and for any function $f \in \mathcal{F}(\pi)$ they are such that $d_H f := D_i(f)dx^i$, $d_V f := \frac{\partial f}{\partial u_\sigma^j}\theta_\sigma^j$.

Analogously, given a formally integrable equation $\mathcal{E}$, since $\mathcal{C}(\mathcal{E})$ is totally horizontal with respect to the mapping $\bar{\pi}_\infty : \mathcal{E}^{(\infty)} \to M$, the tangent bundle $\mathcal{T}(\mathcal{E})$ on $\mathcal{E}^{(\infty)}$ decomposes as $\mathcal{T}(\mathcal{E}) = \mathcal{V}(\mathcal{E}) \oplus \mathcal{C}(\mathcal{E})$, where $\mathcal{V}(\mathcal{E}) := Ker\,(\bar{\pi}_{\infty *})$ is the vertical bundle on $\mathcal{E}^{(\infty)}$. Hence the $\mathcal{F}(\mathcal{E})$-modules $\Lambda^{(1,0)}(\mathcal{E})$ and $\Lambda^{(0,1)}(\mathcal{E})$ of horizontal and vertical 1-forms on $\mathcal{E}^{(\infty)}$, locally generated by $\{dx^i\}$ and the restricted Cartan forms $\{\bar{\theta}_\sigma^j := \theta_\sigma^j|_{\mathcal{E}^{(\infty)}}\}$, can be used to decompose the $\mathcal{F}(\mathcal{E})$-module of r-forms on $\mathcal{E}^{(\infty)}$ as $\Lambda^r(\mathcal{E}) = \bigoplus_{p+q=r} \Lambda^{(p,q)}(\mathcal{E})$. By definition we set $\mathcal{F}(\mathcal{E}) = \Lambda^{(0,0)}(\mathcal{E})$. Accordingly, the restriction $\bar{d} := d|_{\mathcal{E}^{(\infty)}}$ splits into the sum $\bar{d} = \bar{d}_H + \bar{d}_V$ of the horizontal and vertical differentials $\bar{d}_H : \Lambda^{(p,q)}(\mathcal{E}) \to \Lambda^{(p+1,q)}(\mathcal{E})$ and $\bar{d}_V : \Lambda^{(p,q)}(\mathcal{E}) \to \Lambda^{(p,q+1)}(\mathcal{E})$, which satisfy $\bar{d}_H^2 = \bar{d}_V^2 = 0$ and $\bar{d}_H \circ \bar{d}_V = -\bar{d}_V \circ \bar{d}_H$. In coordinates, $\bar{d}_H$ and $\bar{d}_V$ can be easily computed, since they act as graded derivations on $\Lambda^*(\mathcal{E})$ and for any function $f \in \mathcal{F}(\mathcal{E})$ they are such that $\bar{d}_H f := \bar{D}_i(f)dx^i$, $\bar{d}_V f := \frac{\partial f}{\partial u_\sigma^j}\bar{\theta}_\sigma^j$, with $\bar{D}_i$ denoting the total derivatives restricted to $\mathcal{E}^{(\infty)}$. For ease of notation, we will denote $\Lambda^{(p,0)}(\pi)$ and $\Lambda^{(p,0)}(\mathcal{E})$ by $\bar{\Lambda}^p(\pi)$ and $\bar{\Lambda}^p(\mathcal{E})$, respectively.

Given a Lie sub-algebra $\mathfrak{g}$ of $\mathfrak{gl}(n,\mathbb{R})$ (or $\mathfrak{gl}(n,\mathbb{C})$), one may consider the exterior algebras $\mathfrak{g}\otimes\Lambda^*(\pi)$ and $\mathfrak{g}\otimes\Lambda^*(\mathcal{E})$ of $\mathfrak{g}$-valued forms on $J^\infty(\pi)$ and $\mathcal{E}^{(\infty)}$, respectively. The graded algebra of $\mathfrak{g}$-valued horizontal forms on $J^\infty(\pi)$ and $\mathcal{E}^{(\infty)}$, will be denoted by $\mathfrak{g}\otimes\bar\Lambda^*(\pi)=\bigoplus_p\mathfrak{g}\otimes\bar\Lambda^p(\pi)$ and $\mathfrak{g}\otimes\bar\Lambda^*(\mathcal{E})=\bigoplus_p\mathfrak{g}\otimes\bar\Lambda^p(\mathcal{E})$, respectively. By definition, $\mathfrak{g}$-valued horizontal p-forms on $J^\infty(\pi)$ (resp., $\mathcal{E}^{(\infty)}$) are generated by $\mathfrak{g}$-valued p-forms $A\omega$, with A $\mathfrak{g}$-valued functions on $J^\infty(\pi)$ (resp., $\mathcal{E}^{(\infty)}$). Then, one may define a skew-symmetric product $[\ ,\ ]$ by linearly extending the product $[A_1\omega_1,A_2\omega_2]:=[A_1,A_2]\,\omega_1\wedge\omega_2$, between generators. One can check that $[\ ,\ ]$ satisfies the following properties:

$$[\rho,\sigma]=-(-1)^{rs}[\sigma,\rho], \tag{1.5.1}$$

$$(-1)^{rt}[\rho,[\sigma,\tau]]+(-1)^{sr}[\sigma,[\tau,\rho]]+(-1)^{ts}[\tau,[\rho,\sigma]]=0, \tag{1.5.2}$$

$$d_H[\rho,\sigma]=[d_H\rho,\sigma]+(-1)^r[\rho,d_H\sigma], \tag{1.5.3}$$

$$\bar d_H[\rho,\sigma]=[\bar d_H\rho,\sigma]+(-1)^r[\rho,\bar d_H\sigma], \tag{1.5.4}$$

where r,s and t are the degrees of the $\mathfrak{g}$-valued horizontal forms ρ,σ and τ, respectively, on $J^\infty(\pi)$ or $\mathcal{E}^{(\infty)}$.

Analogously, one may define an exterior product $\wedge$ on $\mathfrak{g}\otimes\bar\Lambda^*(\pi)$, or $\mathfrak{g}\otimes\bar\Lambda^*(\mathcal{E})$, by linearly extending the product $A_1\omega_1\wedge A_2\omega_2=A_1A_2\omega_1\wedge\omega_2$.

In the forthcoming sections, for any pair of natural numbers (a,b), the natural projections $\Lambda^*(\pi)\to\Lambda^{(a,b)}(\pi)$ and $\mathfrak{g}\otimes\Lambda^*(\pi)\to\mathfrak{g}\otimes\Lambda^{(a,b)}(\pi)$ will be both denoted by $\pi^{(a,b)}$. Analogously, the projections $\Lambda^*(\mathcal{E})\to\Lambda^{(a,b)}(\mathcal{E})$ and $\mathfrak{g}\otimes\Lambda^*(\mathcal{E})\to\mathfrak{g}\otimes\Lambda^{(a,b)}(\mathcal{E})$ will be both denoted by $\bar\pi^{(a,b)}$. Moreover, when an explicit reference to equation $\mathcal{E}$ is necessary, instead of $\bar\pi^{(a,b)}$ and $\bar d_H$, we will use $\bar\pi^{(a,b)}_{\mathcal{E}}$ and $\bar d_{H,\mathcal{E}}$.

We will use the following

Definition 1.5.1. Let $\mathcal{E}$ be a formally integrable equation. A $\mathfrak{g}$-valued *zero-curvature representation* (ZCR) of $\mathcal{E}$ is a non-vanishing 1-form $\alpha\in\mathfrak{g}\otimes\bar\Lambda^1(\mathcal{E})$ such that

$$\bar d_H\alpha-\frac{1}{2}[\alpha,\alpha]=0. \tag{1.5.5}$$

A 1-parameter family of $\mathfrak{g}$-valued ZCRs of $\mathcal{E}$ is a smooth map $\lambda\mapsto\alpha_\lambda$ defined on an open interval $I\subset\mathbb{R}$, such that $\alpha_\lambda\in\mathfrak{g}\otimes\bar\Lambda^1(\mathcal{E})$ is a $\mathfrak{g}$-valued ZCRs of $\mathcal{E}$ for any $\lambda\in I$.

Remark 1.5.2. We notice that (1.5.5) is written sometimes in the literature in the following equivalent form $\bar d_H\alpha-\alpha\wedge\alpha=0$. We also notice that, since $\mathcal{E}^{(\infty)}\subset J^\infty(\pi)$, any element of $\mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ can be identified with an element of $\mathfrak{g} \otimes \bar{\Lambda}^1(\pi)$. Hence (1.5.5) can also be rewritten as $d_H\alpha - \frac{1}{2}[\alpha, \alpha] = 0 \mod \mathcal{E}^{(\infty)}$. In particular, when $\mathcal{E} = \{F^j = 0, j = 1, ..., h\}$, under regularity assumptions (i.e., if any prolongation $\mathcal{E}^{(\infty)}$, $h \geq 0$, is totally non-degenerating [44]) (1.5.5) can also be rewritten in "characteristic" form $d_H\alpha - \frac{1}{2}[\alpha, \alpha] = \sum D_\sigma(F^j)\omega_j^\sigma$, where $\omega_j^\sigma \in \mathfrak{g} \otimes \bar{\Lambda}^2(\pi)$. This shows that in general (1.5.5) may involve differential consequences of $\mathcal{E}$ and, in such a case, (1.5.5) is not equivalent to $\{F^j = 0, j = 1, ..., h\}$.

1.6 Gauge transformations, horizontal gauge complex and removability of a parameter from a ZCR

Let $\mathfrak{g}$ be the Lie algebra of a matrix Lie group G, and α be a $\mathfrak{g}$-valued ZCR of a formally integrable equation $\mathcal{E}$. As already observed in the particular case $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$ of Section 1.2, one can check that also in general the 1-form $\alpha^S := \bar{d}_H S \cdot S^{-1} + S \cdot \alpha \cdot S^{-1}$ is another $\mathfrak{g}$-valued ZCR of $\mathcal{E}$, for any given G-valued smooth function S on $\mathcal{E}^{(\infty)}$. Hence a transformation $\alpha \mapsto \alpha^S$ will be still referred to as a *gauge transformation*, and α^S will be called gauge equivalent to α. It is easy to check that $\left(\alpha^{S_1}\right)^{S_2} = \alpha^{S_2 S_1}$, for any pair of G-valued smooth functions S_1, S_2 on $\mathcal{E}^{(\infty)}$.

Of course, given a $\mathfrak{g}$-valued ZCR α, one may always embed α into a 1-parameter family of $\mathfrak{g}$-valued ZCRs $\alpha_\lambda := \alpha^{M_\lambda}$, with M_λ any G-valued smooth function on $I \times \mathcal{E}^{(\infty)}$. However, in such a case, $\alpha = (\alpha_\lambda)^{M_\lambda^{-1}} = \left(\alpha^{M_\lambda}\right)^{M_\lambda^{-1}}$ and the parameter λ can always be removed by means of a gauge transformation. Also, since for any $\lambda_0 \in I$ one has that $\alpha_{\lambda_0} = (\alpha_\lambda)^{\left(M_{\lambda_0} M_\lambda^{-1}\right)}$, one may adopt the following

Definition 1.6.1. Let $\lambda \in]a, b[\subset \mathbb{R}$ and α_λ be a 1-parameter family of $\mathfrak{g}$-valued ZCRs of $\mathcal{E}$. The parameter λ is **removable** from α_λ if for any $\lambda_0 \in]a, b[$ there exists a G-valued smooth function S_λ such that $S_{\lambda_0} = \mathbb{I}$ (identity) and $\alpha_{\lambda_0} = \alpha_\lambda^{S_\lambda^{-1}}$. When λ is not removable, α_λ is called a **nontrivial** 1-parameter family of $\mathfrak{g}$-valued ZCRs of $\mathcal{E}$.

We will also use the following

Definition 1.6.2. Two 1-parameter families α_λ and β_η of $\mathfrak{g}$-valued ZCRs of $\mathcal{E}$ are called **equivalent** if there exists some reparametrization $\lambda = f(\eta)$, $f'(\eta) \neq 0$, such that $\alpha_{f(\eta)}$ is gauge equivalent to β_η.

In the paper [38], Marvan proved that the obstruction to removability of λ from a 1-parameter family of $\mathfrak{g}$-valued ZCR α_λ is the first cohomology group $\bar{H}^1_{\alpha_\lambda}(\mathcal{E}, \mathfrak{g})$ of the *horizontal gauge complex*:

$$0 \to \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E}) \xrightarrow{\bar{\partial}_\alpha} \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E}) \xrightarrow{\bar{\partial}_\alpha} \mathfrak{g} \otimes \bar{\Lambda}^2(\mathcal{E}) \longrightarrow \ldots \longrightarrow \mathfrak{g} \otimes \bar{\Lambda}^n(\mathcal{E}) \longrightarrow 0$$

where $\bar{\partial}_\alpha \omega = \bar{d}_H \omega - [\alpha, \omega]$.

Indeed, in view of (1.5.5), the horizontal gauge differential $\bar{\partial}_\alpha$ is such that $\bar{\partial}_\alpha^2 = 0$ and one has the following

Theorem 1.6.3. *(Marvan) If α_λ is a 1-parameter family of $\mathfrak{g}$-valued ZCRs for $\mathcal{E}$, with $\lambda \in]a, b[$, then $\dot{\alpha}_\lambda := \frac{d}{d\lambda}\alpha_\lambda$ is a 1-cocycle with respect to $\bar{\partial}_{\alpha_\lambda}$, i.e., $\bar{\partial}_{\alpha_\lambda}(\dot{\alpha}_\lambda) = 0$. In particular, the parameter λ is removable if, and only if, there exists a solution $K \in \mathfrak{g} \otimes \bar{\Lambda}^0(\mathcal{E})$ of the equation*

$$\dot{\alpha}_\lambda = \bar{\partial}_{\alpha_\lambda}(K) . \tag{1.6.1}$$

For any solution K of (1.6.1) and $\lambda_0 \in]a, b[$, the G-valued matrix S_λ such that $\alpha_{\lambda_0} = \alpha_\lambda^{S_\lambda^{-1}}$ is the solution of the Cauchy problem

$$\begin{cases} \dot{S}_\lambda = K\, S_\lambda, \\ S_{\lambda_0} = \mathbb{I}. \end{cases}$$

It is usually true that an integrable equation admits a $\mathfrak{g}$-valued ZCR, for some matrix Lie-algebra $\mathfrak{g}$. The cases when such a ZCR is embeddable into a 1-parameter family are considered the most important, since the presence of a parameter is crucial from many points of view. For instance, it is known [20, 56] that in the case of equations describing pseudospherical surfaces, the parameter may guarantee the existence of an infinite sequence of nontrivial (and possibly nonlocal) conservation laws, which is usually considered a remarkable attribute of integrable equations. Indeed, an equation (or system) $\mathcal{E}$ of order k in two independent variables (x_1, x_2) is said to describe pseudospherical surfaces if there exists an $\mathfrak{sl}(2, \mathbb{R})$-valued form

$$\beta = \frac{1}{2}\begin{pmatrix} f_{21} & f_{11} - f_{31} \\ f_{11} + f_{31} & -f_{21} \end{pmatrix} dx_1 + \frac{1}{2}\begin{pmatrix} f_{22} & f_{12} - f_{32} \\ f_{12} + f_{32} & -f_{22} \end{pmatrix} dx_2, \tag{1.6.2}$$

with functions f_{ij} satisfying the non-degeneracy condition

$$f_{11}f_{22} - f_{12}f_{21} \neq 0, \tag{1.6.3}$$

and such that the zero-curvature condition $d_H\beta - \frac{1}{2}[\beta, \beta] = 0 \mod \mathcal{E}^{(\infty)}$ is equivalent to $\mathcal{E}$. As a consequence, on any $\mathcal{E}$ describing pseudospherical surfaces, the following system for $\rho = \rho(x_1, x_2)$

$$\begin{cases} \rho_{x_1} = -f_{31} + f_{11} sin(\rho) - f_{21} cos(\rho) , \\ \rho_{x_2} = -f_{32} + f_{12} sin(\rho) - f_{22} cos(\rho) , \end{cases}$$

is compatible and the 1-form

$$\omega = (-f_{31} + f_{11} sin\,(\rho) - f_{21} cos\,(\rho))\, dx_1 + (-f_{32} + f_{12} sin\,(\rho) - f_{22} cos\,(\rho))\, dx_2$$

represents a nonlocal conservation law for $\mathcal{E}$. This entails, when f_{ij} are analytic functions of a parameter λ, that ω can be expanded in a power series $\omega = \sum \omega_i \lambda^i$, where ω_i are possibly nonlocal conservation laws for $\mathcal{E}$. However, this expansion guarantees the existence of an infinite sequence of nontrivial conservation laws only when λ is not removable. Indeed, as observed in [38], when λ is removable from (1.6.2), one could check that $\omega = \mu + \bar{d}_H(f_\lambda)$, with f_λ a function depending on λ and μ a 1-form independent of λ (the explicit formulas for f_λ and μ are too huge to be reported here).

More in general, the importance of nontrivial 1-parameter families of ZCRs is also clear from the fact that the non-removability of the parameter is crucial for the application of some integration techniques, such as the inverse scattering method. For instance, as observed in [20], the applications given by Ablowitz et al. in [1] of the inverse scattering method concern equations which describe pseudospherical surfaces with a family of ZCRs depending on a parameter λ such that

$$f_{21} = \lambda, \qquad \frac{df_{11}}{d\lambda} = \frac{df_{31}}{d\lambda} = 0. \tag{1.6.4}$$

In the paper [16] we used Theorem 1.6.3 to investigate the removability of a parameter λ, from the ZCR (1.6.2) of an equation describing pseudospherical surfaces, when conditions (1.6.4) are satisfied. Interestingly we found that in the case of evolutive equations, (1.6.4) guarantees the non-removability of the parameter.

Chapter 2

Second order evolution PS equations

In this chapter we give a complete and explicit classification of second order evolution PS equations of the form $z_t = A(x,t,z)z_2 + B(x,t,z,z_1)$, with $z = z(x,t)$ and $z_i = \frac{\partial^i z}{\partial x^i}$, under the assumptions that $f_{ij} = f_{ij}\,(x,t,z,z_1,z_2)$ and $f_{21} = \eta$. According to this classification, the considered PS equations are subdivided into three main classes (referred to as Types I-III) together with the corresponding system of 1-forms $\{\omega_1, \omega_2, \omega_3\}$. Some already known equations are found to belong to the considered class of PS equations, like Svinolupov-Sokolov equations admitting higher weakly nonlinear symmetries, Boltzmann equation and reaction-diffusion equations like Murray equation. Other explicit examples are presented, as well.

The chapter is organized as follows. In Section 2.1, we give a preliminary characterization which naturally led us to distinguish between the *generic* and *special* cases $f_{11,z} \neq 0$ and $f_{11,z} = 0$, respectively. In Section 2.2 we state the main result of the chapter, Theorem 2.2.1, which classifies the differential equations into Types I-III and we summarize the classification scheme followed in the subsequent sections. Moreover, we give some simple examples from the classes of equations described in the main theorem. Section 2.3 is devoted to the complete analysis of the generic case $f_{11,z} \neq 0$, which will lead to identify the following types: Type I (a); Type I (b); Type II (a); Type III (a); and Type III (b). Section 2.4 is devoted to the complete analysis of the special case $f_{11,z} = 0$, which will lead to identify the following remaining types: Type II (b); Type III (c). Finally, in Section 2.5 we provide additional examples with the aim of illustrating further aspects of the given classification.

2.1 A characterization of PS equations of the form $z_t = A(x,t,z)\, z_2 + B(x,t,z,z_1)$

Necessary and sufficient conditions for equation (1.2.8) to describe pseudospherical surfaces are given by the following

Theorem 2.1.1. *(See [53].) A differential equation of the form (1.2.8) is a PS equation with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$, *depending on* $(x, t, z, ..., z_k)$ *if, and only if, there exist functions* f_{ij} *satisfying the following conditions*

$$f_{11}f_{22} - f_{12}f_{21} \neq 0, \tag{2.1.1}$$

$$\Delta = (f_{11,z})^2 + (f_{21,z})^2 + (f_{31,z})^2 \neq 0, \tag{2.1.2}$$

$$f_{i1,z_j} = 0,\ f_{i2,z_k} = 0, \qquad 1 \leq i \leq 3,\ 1 \leq j \leq k, \tag{2.1.3}$$

and such that

$$\begin{cases} f_{21}f_{32} - f_{22}f_{31} - f_{11,z}F + f_{12,x} - f_{11,t} + \displaystyle\sum_{i=0}^{k-1} f_{12,z_i} z_{i+1} = 0, \\ f_{12}f_{31} - f_{11}f_{32} - f_{21,z}F + f_{22,x} - f_{21,t} + \displaystyle\sum_{i=0}^{k-1} f_{22,z_i} z_{i+1} = 0, \\ f_{12}f_{21} - f_{11}f_{22} - f_{31,z}F + f_{32,x} - f_{31,t} + \displaystyle\sum_{i=0}^{k-1} f_{32,z_i} z_{i+1} = 0. \end{cases} \tag{2.1.4}$$

In order to obtain classification results of PS equations of the form (1.2.8) one has to obtain F and the functions f_{ij} satisfying (2.1.1-2.1.4).

As a consequence of Theorem 2.1.1, the following theorem gives a characterization of the PS equations of the form

$$z_t = A(x,t,z)\, z_2 + B(x,t,z,z_1), \qquad A \neq 0. \tag{2.1.5}$$

Theorem 2.1.2. *A differential equation of the form (2.1.5) is a PS equation with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$, *depending on* (x, t, z, z_1, z_2) *if, and only if, the functions* f_{ij} *have the form*

$$f_{i1} = f_{i1}(x,t,z), \qquad i = 1,2,3 \tag{2.1.6}$$

$$f_{i2} = A\left(x,t,z\right) f_{i1,z} z_1 + \psi_{i2}\left(x,t,z\right), \qquad\qquad i = 1,2,3 \tag{2.1.7}$$

and in addition satisfy non-degeneracy conditions (2.1.1-2.1.2) and the system

$$\begin{cases} \left(Af_{11,z}\right)_{,z} z_1^2 + \left[\psi_{12,z} + \left(Af_{11,z}\right)_{,x} + A\left(f_{21}f_{31,z} - f_{21,z}f_{31}\right)\right] z_1 \\ +f_{21}\psi_{32} - f_{31}\psi_{22} - f_{11,z}B - f_{11,t} + \psi_{12,x} = 0, \\ \left(Af_{21,z}\right)_{,z} z_1^2 + \left[\psi_{22,z} + \left(Af_{21,z}\right)_{,x} + A\left(f_{31}f_{11,z} - f_{31,z}f_{11}\right)\right] z_1 \\ +f_{31}\psi_{12} - f_{11}\psi_{32} - f_{21,z}B - f_{21,t} + \psi_{22,x} = 0, \\ \left(Af_{31,z}\right)_{,z} z_1^2 + \left[\psi_{32,z} + \left(Af_{31,z}\right)_{,x} + A\left(f_{21}f_{11,z} - f_{21,z}f_{11}\right)\right] z_1 \\ +f_{21}\psi_{12} - f_{11}\psi_{22} - f_{31,z}B - f_{31,t} + \psi_{32,x} = 0, \end{cases} \tag{2.1.8}$$

where ψ_{i2}, $i = 1,2,3$, are differentiable functions.

Proof. In view of Theorem 2.1.1, a differential equation of the form (2.1.5) is a PS equation with 1-forms ω_i depending on (x,t,z,z_1,z_2) if, and only if, the functions f_{ij} satisfy (2.1.1-2.1.4), with $k = 2$. Thus, equations (2.1.3) are equivalent to (2.1.6) and $f_{i2} = f_{i2}(x,t,z,z_1)$, $i = 1,2,3$. On the other hand, under the condition $F = Az_2 + B$, equations (2.1.4) rewrite as

$$\begin{cases} \left(-Af_{11,z} + f_{12,z_1}\right) z_2 + R_1 = 0, \\ \left(-Af_{21,z} + f_{22,z_1}\right) z_2 + R_2 = 0, \\ \left(-Af_{31,z} + f_{32,z_1}\right) z_2 + R_3 = 0, \end{cases} \tag{2.1.9}$$

with

$$\begin{aligned} R_1 &= f_{12,z}z_1 - Bf_{11,z} + f_{21}f_{32} - f_{31}f_{22} + f_{12,x} - f_{11,t}, \\ R_2 &= f_{22,z}z_1 - Bf_{21,z} + f_{31}f_{12} - f_{11}f_{32} + f_{22,x} - f_{21,t}, \\ R_3 &= f_{32,z}z_1 - Bf_{31,z} + f_{21}f_{12} - f_{11}f_{22} + f_{32,x} - f_{31,t}. \end{aligned}$$

It follows that (2.1.9) is equivalent to the system formed by the equations $R_i = 0$, $i = 1,2,3$, together with the following three equations

$$-Af_{11,z} + f_{12,z_1} = 0, \qquad -Af_{21,z} + f_{22,z_1} = 0, \qquad -Af_{31,z} + f_{32,z_1} = 0. \tag{2.1.10}$$

Then conditions (2.1.7) readily follows by integrating (2.1.10) with respect to f_{12}, f_{22} and f_{32}, and by substituting (2.1.7) into the remaining three equations $R_i = 0$, $i = 1,2,3$, one finally gets equations (2.1.8). □

In view of Theorem 2.1.2 one has the following

Proposition 2.1.3. *Consider a second order PS equation of the form (2.1.5) with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$, *depending on* (x, t, z, z_1, z_2). *The function* f_{11} *satisfies the condition* $f_{11,z} \neq 0$ *if, and only if*

$$f_{21} = m_2(x,t) f_{11}(x,t,z) + h_2(x,t), \qquad f_{31} = m_3(x,t) f_{11}(x,t,z) + h_3(x,t), \tag{2.1.11}$$

with m_2, m_3 *and* h_2, h_3 *differentiable functions.*

Proof. If $f_{11,z} \neq 0$, the first equation of (2.1.8) gives

$$\begin{aligned} B = {} & \tfrac{1}{f_{11,z}} \left[A f_{11,zz} + A_{,z} f_{11,z} \right] z_1^2 \\ & + \tfrac{1}{f_{11,z}} \left[A \left(f_{21} f_{31,z} + f_{11,xz} - f_{31} f_{21,z} \right) + \psi_{12,z} + A_{,x} f_{11,z} \right] \\ & + \tfrac{1}{f_{11,z}} \left[f_{21} \psi_{32} + \psi_{12,x} - f_{11,t} - f_{31} \psi_{22} \right]. \end{aligned} \tag{2.1.12}$$

Then, by substituting (2.1.12) into the remaining two equations of (2.1.8), one gets

$$\begin{cases} A \left[f_{11,z} f_{21,zz} - f_{21,z} f_{11,zz} \right] z_1^2 + \big[f_{11,z} \psi_{22,z} - f_{21,z} \psi_{12,z} + A \big(f_{31} f_{21,z}^2 \\ -f_{21,z} f_{21} f_{31,z} - f_{21,z} f_{11,xz} - f_{11,z} f_{11} f_{31,z} + f_{11,z}^2 f_{31} + f_{11,z} f_{21,xz} \big) \big] z_1 \\ +\psi_{12} f_{31} f_{11,z} - f_{21,z} \psi_{12,x} - f_{11} \psi_{32} f_{11,z} - f_{21,t} f_{11,z} + \psi_{22,x} f_{11,z} \\ +f_{21,z} f_{31} \psi_{22} - f_{21,z} f_{21} \psi_{32} + f_{21,z} f_{11,t} = 0, \\ A \left[f_{11,z} f_{31,zz} - f_{31,z} f_{11,zz} \right] z_1^2 + \big[f_{11,z} \psi_{32,z} - f_{31,z} \psi_{12,z} + A \big(f_{31} f_{21,z} f_{31,z} \\ -f_{21} f_{31,z}^2 - f_{31,z} f_{11,xz} - f_{11,z} f_{11} f_{21,z} + f_{11,z}^2 f_{21} + f_{11,z} f_{31,xz} \big) \big] z_1 \\ +\psi_{12} f_{21} f_{11,z} - f_{31,z} \psi_{12,x} - f_{11} \psi_{22} f_{11,z} - f_{31,t} f_{11,z} + \psi_{32,x} f_{11,z} \\ +f_{31,z} f_{31} \psi_{22} - f_{31,z} f_{21} \psi_{32} + f_{31,z} f_{11,t} = 0. \end{cases} \tag{2.1.13}$$

Therefore, in view of the independence of f_{11}, f_{21}, f_{31}, ψ_{12}, ψ_{22}, ψ_{32} on z_1, by deriving (2.1.13) twice with respect to z_1, it is easily seen that

$$\left(\frac{f_{21,z}}{f_{11,z}} \right)_{,z} = 0, \qquad \left(\frac{f_{31,z}}{f_{11,z}} \right)_{,z} = 0,$$

and (2.1.11) readily follow by an integration of the last two equations.

Conversely, if (2.1.11) holds, then non-degeneracy condition (2.1.2) entails that $(1 + m_2^2 + m_3^2) f_{11,z} \neq 0$ and consequently that $f_{11,z} \neq 0$. □

On the other hand, by solving the third equation of (2.1.8) with respect to B and substituting in the remaining two equations, when $f_{31,z} \neq 0$ the following analogue of Proposition 2.1.3 can be readily proved.

Proposition 2.1.4. *Consider a second order PS equation of the form (2.1.5) with associated* 1*-forms* $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$, *depending on* (x, t, z, z_1, z_2)*. The function* f_{31} *satisfies the condition* $f_{31,z} \neq 0$ *if, and only if*

$$f_{11} = m_1(x,t)f_{31}(x,t,z) + h_1(x,t), \qquad f_{21} = m_2(x,t)f_{31}(x,t,z) + h_2(x,t), \tag{2.1.14}$$

with m_1, m_2 *and* h_1, h_2 *differentiable functions.*

Solving equations (2.1.8) in general is very complicated, however as proved in Sections 2.3 and 2.4 they can be explicitly solved under the assumption $f_{21} = \eta$. To this end, by taking advantage of the suitable form taken by f_{31}, in view of Proposition 2.1.3, we will distinguish between the following two cases:

(i) **Generic case** $f_{11,z} \neq 0$, where

$$f_{31} = mf_{11} + h, \tag{2.1.15}$$

in view of Proposition 2.1.3;

(ii) **Special case** $f_{11,z} = 0$.

The generic case will be treated in Section 2.3, whereas the special case will be treated in Section 2.4.

2.2 Main theorem and simple examples

In this section we present our main classification result, which is Theorem 2.2.1, and illustrate some of its concrete applications by means of simple examples.

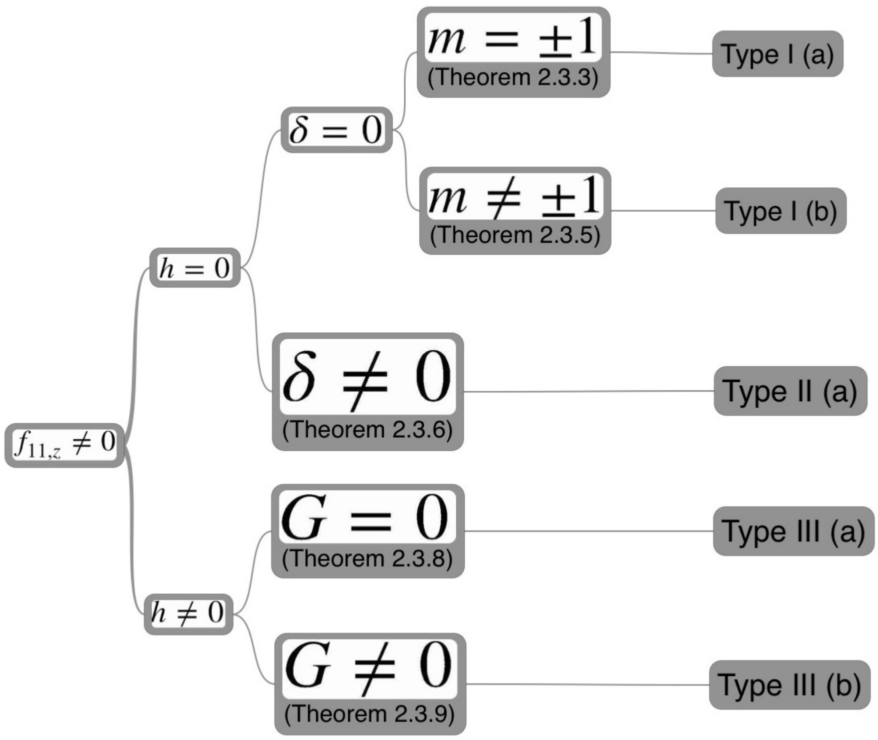
According to Theorem 2.2.1, a given second order differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x, t, z, z_1, z_2) and satisfying

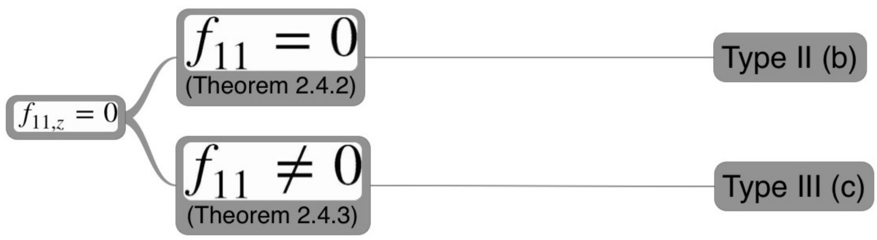
$$f_{21} = \eta, \qquad \eta \in \mathbb{R}, \tag{2.2.1}$$

iff it belongs to one of Types I-III classified by theorem. Once a given equation is recognized to belong to one of these types, Theorem 2.2.1 explicitly gives also the associated functions f_{ij}. It is noteworthy to observe the possibility to have, in some cases, multiple linear problems for the same given equation, which is an interesting feature of the given classification since it may provide pairs of non gauge-equivalent linear problems (see for instance Example 2.5.6).

The proof of Theorem 2.2.1 is based on the results of the subsequent Sections 2.3 and 2.4, which are graphically illustrated in the diagrams below where the branches occurring in the generic case $f_{11,z} \neq 0$ have been distinguished by means of the following functions of m and h (see (2.1.15)):

$$
\begin{aligned}
\alpha &= mh^3 + h\left(\delta_{,x} - \eta m \delta\right) - \delta h_{,x}, \\
\beta &= h^2\left(m^2 - 1\right) + \delta^2, \\
\delta &= m_{,x} + \eta\left(1 - m^2\right), \\
G &= \alpha + \beta f_{11}.
\end{aligned}
\tag{2.2.2}
$$





According to the diagrams, the analysis of generic case in Section 2.3 leads to equations of Types I-III, with linear problems (a) and (b) for Type I, linear problem (a) for Type II and linear problems (a) and (b) for Type III. Whereas by analyzing the special case $f_{11,z} = 0$, one gets linear problems (b) and (c) for Type II and III, respectively.

Theorem 2.2.1. *A PS equation*

$$z_t = A\left(x, t, z\right) z_2 + B\left(x, t, z, z_1\right), \qquad A \neq 0,$$

with associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$, $1 \leq i \leq 3$, *depending on* (x, t, z, z_1, z_2) *and satisfying* $f_{21} = \eta$, *belongs to one of the following types, where* δ *and* G *are as in (2.2.2).*

Type I

$$z_t = \frac{1}{f_{,z}}\left[\varphi z_2 + \varphi_{,z} z_1^2 + (\varphi_{,x} + \psi_{,z})\, z_1 + \psi_{,x} - f_{,t}\right], \tag{2.2.3}$$

where $\varphi = \varphi(x,t,z)$, $\psi = \psi(x,t,z)$, $f = f(x,t,z)$ *are arbitrary differentiable functions, with* $\varphi f_{,z} \neq 0$ *on a nonempty open set, and the following two alternatives occur with* $g = g(t)$ *an arbitrary differentiable function such that* $\eta^2 + (g')^2 \neq 0$*:*

a) *the functions* f_{ij} *are*

$$\begin{aligned}
f_{11} &= e^{-\epsilon(\eta x + g)} f, & f_{12} &= e^{-\epsilon(\eta x+g)}\left(\varphi z_1 + \psi\right),\\
f_{21} &= \eta, & f_{22} &= g',\\
f_{31} &= \epsilon f_{11}, & f_{32} &= \epsilon f_{12},
\end{aligned} \tag{2.2.4}$$

with $\epsilon = \pm 1$*;*

b) *the functions* f_{ij} *are*

$$\begin{aligned}
f_{11} &= cosh(\eta x + g) f, & f_{12} &= cosh(\eta x + g)\left(\varphi z_1 + \psi\right),\\
f_{21} &= \eta, & f_{22} &= g',\\
f_{31} &= -tanh\,(\eta x + g)\, f_{11}, & f_{32} &= -tanh\,(\eta x + g)\, f_{12}.
\end{aligned} \tag{2.2.5}$$

Type II

$$\begin{aligned}
z_t = {} & \epsilon_1\epsilon_2\, f^2 z_2 + \epsilon_1\epsilon_2\, \frac{f^2 f_{,zz}}{f_{,z}} z_1^2 + \frac{f^2}{g}\left(\epsilon_1\epsilon_2 \frac{g f_{,xz}}{f_{,z}} - \frac{\psi_{,z}}{f_{,z}} + \epsilon_1\epsilon_2 g_{,x} + \epsilon_1 \eta m g\right) z_1\\
& - \frac{f^2}{g f_{,z}}\left(\psi_{,x} + \eta m \psi + \epsilon_1\epsilon_2\, \eta f g \delta\right) - \frac{f_{,t}}{f_{,z}} + \frac{f g_{,t}}{g f_{,z}} + \epsilon_1 \frac{m f}{f_{,z}} \int g^2 \delta dx,
\end{aligned} \tag{2.2.6}$$

where $\epsilon_1, \epsilon_2 = \pm 1$, $g = g(x,t)$ *and* $f = f(x,t,z)$ *are arbitrary differentiable functions with* $g f_{,z} \neq 0$ *on a nonempty open set, and the following two alternatives occur:*

a) $\epsilon_2 = 1$ *and the functions* f_{ij} *are*

$$\begin{aligned}
f_{11} &= \frac{g}{f}, & f_{12} &= -\epsilon_1 g f_{,z} z_1 + \psi,\\
f_{21} &= \eta, & f_{22} &= \epsilon_1 \int g^2 \delta dx,\\
f_{31} &= \frac{mg}{f}, & f_{32} &= m\left(-\epsilon_1 g f_{,z} z_1 + \psi\right) + \epsilon_1 f g \delta.
\end{aligned} \tag{2.2.7}$$

with

$$\begin{aligned}
\psi = {} & -\epsilon_1 g f_{,x} + \epsilon_1 \frac{f}{\delta}\left(\eta m g \delta - g \delta_{,x} - g_{,x}\delta\right)\\
& + \epsilon_1 \frac{g}{f\delta}\left[\epsilon_1 m_{,t} + (1 - m^2) \int g^2 \delta dx\right],
\end{aligned} \tag{2.2.8}$$

and $m = m(x,t)$ *is an arbitrary differentiable function such that* $\delta = m_{,x} + \eta\left(1 - m^2\right) \neq 0$ *on a nonempty open set;*

b) $\epsilon_2 = -1$, $m = 0$ *(hence* $\delta = \eta$*) and the functions* f_{ij} *are*

$$\begin{aligned} f_{11} &= 0, & f_{12} &= -\epsilon_1 \eta g f, \\ f_{21} &= \eta, & f_{22} &= \epsilon_1 \textstyle\int \eta g^2 dx, \\ f_{31} &= \tfrac{g}{f}, & f_{32} &= \epsilon_1 g f_{,z} z_1 + \psi, \end{aligned} \tag{2.2.9}$$

with

$$\psi = \epsilon_1 g f_{,x} + \epsilon_1 f g_{,x} + \epsilon_1 \frac{g \int g^2 dx}{f}. \tag{2.2.10}$$

Type III

$$\begin{aligned} z_t = & -\epsilon_2 \tfrac{g_{,z}}{f_{,z}} z_2 - \epsilon_2 \tfrac{g_{,zz}}{f_{,z}} z_1^2 + \epsilon_2 \left[\tfrac{g_{,z}}{f_{,z}}\left(\tfrac{f\delta - h_{,x}}{h} - \eta m\right) - \tfrac{2g_{,xz}}{f_{,z}} + \tfrac{g\delta}{h} + \epsilon_2 \tfrac{q}{h}\right] z_1 \\ & + \tfrac{g}{f_{,z}} \left[\tfrac{(\delta_{,x} + \eta m \delta) f + \epsilon_2 f_{,x} \delta}{h} + \epsilon_2 \eta \delta - h^2 - mhf - \epsilon_2 \tfrac{f h_{,x} \delta}{h^2}\right. \\ & \left. + \epsilon_2 \left(\tfrac{h_{,x}}{h}\right)^2 - \epsilon_2 \tfrac{h_{,xx}}{h} - \eta m \tfrac{h_{,x}}{h}\right] + \epsilon_2 \tfrac{g_{,x}}{f_{,z}} \left(\tfrac{f\delta - h_{,x}}{h} - \epsilon_2 \eta m\right) - \epsilon_2 \tfrac{g_{,xx}}{f_{,z}} \\ & + \tfrac{f}{h f_{,z}} \left(q_{,x} + \eta m q\right) + \tfrac{q f_{,x}}{h f_{,z}} - \tfrac{q f h_{,x}}{f_{,z} h^2} + \tfrac{\eta q - f_{,t}}{f_{,z}}, \end{aligned} \tag{2.2.11}$$

where $\epsilon_2 = \pm 1$, $f = f(x,t,z)$ *is an arbitrary differentiable function with* $f_{,z} \neq 0$ *on a nonempty open set, and the following three alternatives occur:*

a) $\epsilon_2 = 1$, $m = m(x,t)$ *and* $g = g(x,t,z)$ *are arbitrary differentiable functions such that* $-1 < m < 1$, $\delta g_{,z} \neq 0$ *on a nonempty open set and the functions* f_{ij} *are*

$$\begin{aligned} f_{11} &= f, & f_{12} &= -g_{,z} z_1 - g_{,x} - g \tfrac{h_{,x}}{h} + \tfrac{qf}{h} + \epsilon_1 f g \sqrt{1 - m^2}, \\ f_{21} &= \eta, & f_{22} &= gh, \\ f_{31} &= mf + h, & f_{32} &= m f_{12} + \epsilon_1 g h \sqrt{1 - m^2} + q, \end{aligned} \tag{2.2.12}$$

with h *and* q *such that*

$$\left\{ \begin{array}{ll} h = \dfrac{\epsilon_1 \delta}{\sqrt{1 - m^2}}, & \epsilon_1 = \pm 1, \\ q = \frac{\epsilon_1 m_{,t}}{\sqrt{1 - m^2}}; & \end{array} \right. \tag{2.2.13}$$

b) $\epsilon_2 = 1$, $m = m(x,t)$, $q = q(x,t)$ *and* $h = h(x,t)$ *are arbitrary differentiable functions* *such that* $h\beta \neq 0$ *on a nonempty open set and the functions* f_{ij} *are*

$$\begin{aligned} f_{11} &= f, & f_{12} &= -g_{,z} z_1 + \tfrac{qf}{h} - g_{,x} - \tfrac{g h_{,x}}{h} + \tfrac{\delta f g}{h}, \\ f_{21} &= \eta, & f_{22} &= gh, \\ f_{31} &= mf + h, & f_{32} &= m f_{12} + \delta g + q, \end{aligned} \tag{2.2.14}$$

with

$$g = -\tfrac{1}{G}\left[h\left(q_{,x} - h_{,t} - \eta m q\right) + \left(q\delta - m_{,t} h\right) f\right] \; ; \tag{2.2.15}$$

c) $\epsilon_2 = -1$, $m = 0$ *(hence* $\delta = \eta$*),* $h = h(x,t)$, $q = q(x,t)$ *are arbitrary differentiable functions and the functions* f_{ij} *are*

$$\begin{aligned} f_{11} &= h, & f_{12} &= q - \eta g, \\ f_{21} &= \eta, & f_{22} &= gh, \\ f_{31} &= f & f_{32} &= g_{,z} z_1 + (q - \eta g)\tfrac{f}{h} + g_{,x} + \tfrac{g h_{,x}}{h}, \end{aligned} \tag{2.2.16}$$

with

$$g = -\tfrac{\left[h(q_{,x} - h_{,t}) + \eta q f\right]}{\eta h_{,x} - (\eta^2 + h^2) f}. \tag{2.2.17}$$

Here are some examples of equations described by the classification given in this chapter. Further examples will be found in Section 2.5.

Example 2.2.2. Burgers equation

$$z_t = z_2 + z z_1,$$

is a particular instance of Type I (a), for $f = z$, $\varphi = 1$, $\psi = \frac{z^2}{2}$. Hence, by using Theorem 2.2.1, one gets that the associated 1-forms $\omega_i = f_{i1} dx + f_{i2} dt$ are given by the functions

$$\begin{aligned} f_{11} &= e^{-\epsilon(\eta x + g)} z, & f_{12} &= e^{-\epsilon(\eta x + g)}\left(z_1 + \tfrac{z^2}{2}\right), \\ f_{21} &= \eta, & f_{22} &= g', \\ f_{31} &= \epsilon e^{-\epsilon(\eta x + g)} z, & f_{32} &= \epsilon e^{-\epsilon(\eta x + g)}\left(z_1 + \tfrac{z^2}{2}\right), \end{aligned}$$

where $g = g\,(t)$ is an arbitrary differentiable function, $\epsilon = \pm 1$ and $\eta^2 + (g')^2 \neq 0$.

Example 2.2.3. Potential Burgers equation

$$z_t = z_2 + z_1^2,$$

is a particular instance of Type I (b), for $f=\varphi=e^z$ and $\psi=0$. Hence, by using Theorem 2.2.1, one gets that the associated 1-forms $\omega_i=f_{i1}dx+f_{i2}dt$ are given by the functions

$$\begin{aligned}
f_{11} &= cosh\,(\eta x+g)\,e^z, & f_{12} &= cosh\,(\eta x+g)\,e^z z_1,\\
f_{21} &= \eta, & f_{22} &= g',\\
f_{31} &= -sinh\,(\eta x+g)\,e^z, & f_{32} &= -sinh\,(\eta x+g)\,e^z z_1,
\end{aligned}$$

where $g=g(t)$ is an arbitrary differentiable function and $\eta^2+(g')^2\neq 0$.

Example 2.2.4. Equation

$$z_t=z^2z_2+xz_1-z-\eta^2z^3,$$

is a particular instance of Type II (a), for $\epsilon_1=1$, $m=\mu\in\,]-1,1[$, $f=z$ and $g=\frac{1}{\sqrt{1-\mu^2}}$. Hence, by using Theorem 2.2.1, one gets that the associated 1-forms $\omega_i=f_{i1}dx+f_{i2}dt$ are given by the functions

$$\begin{aligned}
f_{11} &= \tfrac{1}{z\sqrt{1-\mu^2}}, & f_{12} &= -\tfrac{z_1}{\sqrt{1-\mu^2}}+\tfrac{\mu\eta z}{\sqrt{1-\mu^2}}+\tfrac{x}{z\sqrt{1-\mu^2}},\\
f_{21} &= \eta, & f_{22} &= \eta x,\\
f_{31} &= \tfrac{\mu}{z\sqrt{1-\mu^2}}, & f_{32} &= -\tfrac{\mu z_1}{\sqrt{1-\mu^2}}+\tfrac{\eta z}{\sqrt{1-\mu^2}}+\tfrac{\mu x}{z\sqrt{1-\mu^2}}.
\end{aligned}$$

Example 2.2.5. The equation

$$z_t=2zz_2+2z_1^2+3z^2z_1,$$

is a particular instance of Type III (a), for $m=0$, $f=z$, $g=-z^2$ and $\epsilon_1=-1$. Hence, by using Theorem 2.2.1, one gets that the associated 1-forms $\omega_i=f_{i1}dx+f_{i2}dt$ are given by the functions

$$\begin{aligned}
f_{11} &= z, & f_{12} &= 2zz_1+z^3,\\
f_{21} &= \eta, & f_{22} &= \eta z^2,\\
f_{31} &= -\eta, & f_{32} &= -\eta z^2,
\end{aligned}$$

where $\eta\neq 0$.

Example 2.2.6. Equation

$$z_t=\frac{z_2}{z^2}-\frac{2z_1^2}{z^3}+xz_1+z-\frac{1}{z},$$

is a particular instance of Type III (b), for $m = \mu \in \,]-1,1[$, $h = \sqrt{(\eta^2+1)(1-\mu^2)}$, $f = \frac{z}{\sqrt{1-\mu^2}} - \frac{\alpha}{\beta}$ and $q = \frac{\beta x}{\lambda(\mu^2-1)}$. Hence, by using Theorem 2.2.1, one gets that the associated 1-forms $\omega_i = f_{i1}dx + f_{i2}dt$ are given by the functions

$$
\begin{aligned}
f_{11} &= \tfrac{z}{\sqrt{1-\mu^2}} + \tfrac{\mu\sqrt{1+\eta^2}}{\sqrt{1-\mu^2}}, & f_{12} &= \tfrac{z_1}{z^2\sqrt{1-\mu^2}} + \tfrac{\mu\eta}{z\sqrt{1-\mu^2}} + \tfrac{xz}{\sqrt{1-\mu^2}} + \tfrac{\mu x\sqrt{1+\eta^2}}{\sqrt{1-\mu^2}},\\
f_{21} &= \eta, & f_{22} &= \eta x + \tfrac{\sqrt{1+\eta^2}}{z},\\
f_{31} &= \tfrac{\mu z}{\sqrt{1-\mu^2}} + \tfrac{\mu^2\sqrt{1+\eta^2}}{\sqrt{1-\mu^2}} + & f_{32} &= \tfrac{\mu z_1}{z^2\sqrt{1-\mu^2}} + \tfrac{\eta}{z\sqrt{1-\mu^2}} + \tfrac{\mu xz}{\sqrt{1-\mu^2}} + \tfrac{x\sqrt{1+\eta^2}}{\sqrt{1-\mu^2}}.\\
&\quad + \sqrt{(1+\eta^2)(1-\mu^2)}, & &
\end{aligned}
$$

2.3 Analysis of the generic case $f_{11,z} \neq 0$

The following theorem gives a characterization of PS equations of the form (2.1.5) with associated 1-forms (1.2.2) satisfying $f_{21} = \eta$ and $f_{11,z} \neq 0$.

Theorem 2.3.1. *A differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x,t,z,z_1,z_2) with $f_{21} = \eta$ and $f_{11,z} \neq 0$ if, and only if, B has the form*

$$B = \phi_1\, z_1^2 + \phi_2\, z_1 + \phi_3, \tag{2.3.1}$$

where

$$
\begin{cases}
\phi_1 = \frac{1}{f_{11,z}}\left[A f_{11,zz} + A_{,z} f_{11,z}\right],\\
\phi_2 = \frac{1}{f_{11,z}}\left[A f_{11,xz} + A_{,x} f_{11,z} + \eta m A f_{11,z} + \psi_{12,z}\right],\\
\phi_3 = \frac{1}{f_{11,z}}\left[\psi_{12,x} + \eta\psi_{32} - m f_{11} f_{22} - h f_{22} - f_{11,t}\right],
\end{cases} \tag{2.3.2}
$$

with $f_{11} = f_{11}(x,t,z)$, $f_{22} = f_{22}(x,t,z)$ and remaining f_{ij} satisfy the non-degeneracy conditions (2.1.1-2.1.2) and have the form

$$f_{31} = m f_{11} + h, \qquad f_{12} = A f_{11,z}\, z_1 + \psi_{12}, \qquad f_{32} = m A f_{11,z}\, z_1 + \psi_{32}, \tag{2.3.3}$$

with $m = m(x,t)$, $h = h(x,t)$, $\psi_{12} = \psi_{12}(x,t,z)$, $\psi_{32} = \psi_{32}(x,t,z)$ differentiable func-

tions satisfying the system

$$\begin{cases} f_{22,z} + hAf_{11,z} = 0, \\ f_{22,x} + mf_{11}\psi_{12} + h\psi_{12} - f_{11}\psi_{32} = 0, \\ \psi_{32,z} - m\psi_{12,z} + \delta Af_{11,z} = 0, \\ \left[\left(m^2 - 1\right) f_{11} + mh\right] f_{22} + \psi_{32,x} - m\psi_{12,x} \\ -\eta\left(m\psi_{32} - \psi_{12}\right) - m_{,t} f_{11} - h_{,t} = 0, \end{cases} \tag{2.3.4}$$

with δ given by (2.2.2).

Proof. The formulas (2.3.3) follow from (2.1.7) and second formula of (2.1.11). Hence, solving the first equation of (2.1.8) with respect to B, one gets (2.3.1) with ϕ_1, ϕ_2, ϕ_3 given by (2.3.2). Thus, the remaining two equations of (2.1.8) reduce to

$$\begin{cases} \left(f_{22,z} + hAf_{11,z}\right) z_1 + f_{22,x} + mf_{11}\psi_{12} + h\psi_{12} - f_{11}\psi_{32} = 0, \\ \left[\psi_{32,z} - m\psi_{12,z} + \delta Af_{11,z}\right] z_1 + \left[\left(m^2 - 1\right) f_{11} + mh\right] f_{22} \\ +\psi_{32,x} - m\psi_{12,x} - \eta\left(m\psi_{32} - \psi_{12}\right) - m_{,t} f_{11} - h_{,t} = 0. \end{cases} \tag{2.3.5}$$

Therefore in view of the independence of A, f_{11}, f_{31}, f_{22}, ψ_{12} and ψ_{32} on z_1, equations (2.3.4) follow from the derivation of (2.3.5) with respect to z_1. $\square$

The rest of this section is devoted to the characterization of PS equations of the form (2.1.5) under the assumption (2.2.1) and with $f_{11,z} \neq 0$. The analysis of this case naturally splits into two cases $h = 0$ and $h \neq 0$. In Subsection 2.3.1, we consider the case $h = 0$, whereas in Subsection 2.3.2 we treat the case $h \neq 0$.

2.3.1 Subcases with $h = 0$

According to the diagram of Section 2.2, the analysis of the case $\{f_{11,z} \neq 0,\ h = 0\}$ naturally splits into further subcases which finally lead to distinguish the following types of equations: Type I (a) with $\{\delta = 0, m = \pm 1\}$; Type I (b) with $\{\delta = 0, m \neq \pm 1\}$; Type II (a) with $\delta \neq 0$. In this subsection we aim at giving a detailed analysis of these three subcases.

We start with the following auxiliary

Lemma 2.3.2. *A differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x, t, z, z_1, z_2) with $f_{21} = \eta$ and $\{f_{11,z} \neq 0,\ h = 0\}$ if, and* *only if, B has the form (2.3.1) where*

$$\begin{aligned} \phi_1 &= \tfrac{1}{f_{11,z}}\left[Af_{11,zz} + A_{,z}f_{11,z}\right], \\ \phi_2 &= \tfrac{1}{f_{11,z}}\left[Af_{11,xz} + A_{,x}f_{11,z} + \eta m A f_{11,z} + \psi_{12,z}\right], \\ \phi_3 &= \tfrac{1}{f_{11,z}}\left[\psi_{12,x} + \eta m \psi_{12} + \eta\delta A f_{11} - m f_{11} f_{22} - f_{11,t}\right], \end{aligned} \tag{2.3.6}$$

with $f_{11} = f_{11}(x,t,z)$, $f_{22} = f_{22}(x,t)$ and remaining f_{ij} satisfy non-degeneracy conditions (2.1.1-2.1.2) and have the form

$$f_{31} = mf_{11}, \qquad f_{12} = Af_{11,z}z_1 + \psi_{12}, \qquad f_{32} = mAf_{11,z}z_1 + \psi_{32}, \tag{2.3.7}$$

with $m = m(x,t)$, $\psi_{12} = \psi_{12}(x,t,z)$, $\psi_{32} = \psi_{32}(x,t,z)$, differentiable functions satisfying the system

$$\begin{cases} \psi_{32} = m\psi_{12} + \delta A f_{11}, \\ f_{22,x} - \delta A f_{11}^2 = 0, \\ (m^2-1)\, f_{11}f_{22} + \delta\psi_{12} + (\delta A f_{11})_{,x} - (\eta m \delta A + m_{,t})\, f_{11} = 0. \end{cases} \tag{2.3.8}$$

Proof. Under the given assumptions, formulas (2.3.3) entail that (2.3.7) hold. Moreover, the first equation of (2.3.4) implies that $f_{22} = f_{22}(x,t)$. Thus, by deriving the second equation of (2.3.4) with respect to z, one obtains

$$f_{11,z}(m\psi_{12} - \psi_{32}) + f_{11}(m\psi_{12,z} - \psi_{32,z}) = 0, \tag{2.3.9}$$

and (2.3.8) readily follows from (2.3.4) and (2.3.9), as well as (2.3.6) follows from (2.3.2) and first equation of (2.3.8). □

In the next two subsections we will solve (2.3.8) under the assumption that $\delta = 0$, i.e., that $m = m(x,t)$ satisfies the Riccati type equation

$$m_{,x} + \eta\left(1 - m^2\right) = 0. \tag{2.3.10}$$

Therefore we will distinguish the degenerate cases $m = \pm 1$, from the general case where (2.3.10) has the solution $m = -tanh\,(\eta x + g(t))$.

2.3.1.1 Type I (a)

Using Lemma 2.3.2 one gets the following

Theorem 2.3.3. *In the case $\{f_{11,z} \neq 0,\ h = 0,\ \delta = 0,\ m = \pm 1\}$, a differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x,t,z,z_1,z_2)*

with $f_{21}=\eta$ if, and only if, the differential equation has the form

$$z_t = \tfrac{1}{f_{,z}}\left[\varphi z_2 + \varphi_{,z} z_1^2 + \left(\psi_{,z}+\varphi_{,x}\right) z_1 + \psi_{,x} - f_{,t}\right], \tag{2.3.11}$$

where $\varphi=\varphi\left(x,t,z\right)$, $\psi=\psi\left(x,t,z\right)$ and $f=f\left(x,t,z\right)$ are arbitrary functions, such that $\varphi f_{,z}\neq 0$ on a nonempty open set, and the functions f_{ij} are

$$\begin{aligned}
f_{11} &= e^{-\epsilon(\eta x+g)} f, & f_{12} &= e^{-\epsilon(\eta x+g)}\left(\varphi z_1+\psi\right),\\
f_{21} &= \eta, & f_{22} &= g',\\
f_{31} &= \epsilon f_{11}, & f_{32} &= \epsilon f_{12},
\end{aligned} \tag{2.3.12}$$

with $\epsilon=\pm 1$ and $g=g(t)$ arbitrary differentiable function such that $\eta^2+\left(g'\right)^2\neq 0$.

Proof. Under the given assumptions, formulas (2.3.7-2.3.8) entail that $f_{31}=\epsilon f_{11}$, $\psi_{32}=\epsilon\psi_{12}$ and $f_{22}=f_{22}\left(t\right)$. Thus, in view of (2.3.1) and (2.3.6), (2.1.5) takes the form

$$\begin{aligned}
z_t = Az_2 + \tfrac{1}{f_{11,z}}&\left[\left(Af_{11,z}\right)_{,z} z_1^2 + \left(\psi_{12,z}+\epsilon\eta A f_{11,z} + \left(Af_{11,z}\right)_{,x}\right) z_1\right.\\
&\left.+\epsilon\eta\psi_{12} - \epsilon f_{11} f_{22} + \psi_{12,x} - f_{11,t}\right],
\end{aligned} \tag{2.3.13}$$

and, by introducing the functions

$$g=\int f_{22}\,dt, \qquad f=e^{\epsilon(\eta x+g)} f_{11}, \qquad \varphi = Af_{,z}, \qquad \psi = e^{\epsilon(\eta x+g)}\psi_{12},$$

equation (2.3.13) reduces to (2.3.11). Moreover, in view of (2.3.7), the functions f_{ij} are given by (2.3.12).

Notice that, the non-degeneracy condition (2.1.1) holds in view of the fact that $\eta^2+\left(g'\right)^2\neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. □

Observe that equation (2.3.11) coincides with (2.2.3) and it is referred to as of Type I in our main classification result, Theorem 2.2.1.

Remark 2.3.4. It is noteworthy to remark that equation (2.3.11) can be written in the form

$$D_t\left(f\right) = D_x\left(\varphi z_1+\psi\right),$$

and by means of the point transformation $\{x=x,\ t=t,\ \bar z = f(x,t,z)\}$, it reduces to

$$\bar z_t = D_x\left(\bar\varphi D_x(\sigma)+\bar\psi\right)$$

where $z=\sigma\left(x,t,\bar z\right)$ is inverse of $\bar z=f(x,t,z)$ and $\bar\varphi=\bar\varphi\left(x,t,\sigma\left(x,t,\bar z\right)\right)$, $\bar\psi=\bar\psi\left(x,t,\sigma\left(x,t,\bar z\right)\right)$.

2.3.1.2 Type I (b)

Using Lemma 2.3.2 one gets the following

Theorem 2.3.5. *In the case* $\{f_{11,z} \neq 0,\ h = 0,\ \delta = 0,\ m \neq \pm 1\}$*, a differential equation of the form (2.1.5) is a PS equation with associated* 1*-forms (1.2.2) depending on* (x, t, z, z_1, z_2) *with* $f_{21} = \eta$ *if, and only if, the differential equation has the form*

$$z_t = \tfrac{1}{f_{,z}} \left[\varphi z_2 + \varphi_{,z} z_1^2 + (\psi_{,z} + \varphi_{,x}) z_1 + \psi_{,x} - f_{,t}\right], \tag{2.3.14}$$

where $\varphi = \varphi(x, t, z)$, $\psi = \psi(x, t, z)$ *and* $f = f(x, t, z)$ *are arbitrary functions, such that* $\varphi f_{,z} \neq 0$ *on a nonempty open set, and the functions* f_{ij} *are*

$$\begin{aligned} f_{11} &= cosh(\eta x + g) f, & f_{12} &= cosh(\eta x + g)(\varphi z_1 + \psi_{12}), \\ f_{21} &= \eta, & f_{22} &= g', \\ f_{31} &= -tanh(\eta x + g) f_{11}, & f_{32} &= -tanh(\eta x + g) f_{12}. \end{aligned} \tag{2.3.15}$$

with $g = g(t)$ *arbitrary differentiable function and* $\eta^2 + (g')^2 \neq 0$.

Proof. Under the given assumptions, from (2.3.10) and (2.3.7-2.3.8) one gets

$$f_{22} = f_{22}(t), \qquad f_{31} = m f_{11}, \qquad \psi_{32} = m\psi_{12},$$

with $m = -tanh(\eta x + g(t))$ and in addition $f_{22} = g'$, since $f_{11,z} \neq 0$. Thus, in view of (2.3.1) and (2.3.6), (2.1.5) takes the form

$$\begin{aligned} z_t = \tfrac{1}{f_{11,z}} \{& A f_{11,z} z_2 + (A f_{11,z})_{,z} z_1^2 + [\psi_{12,z} + (A f_{11,z})_{,x} - \eta\, tanh(\eta x + g) A f_{11,z}] z_1 \\ & + \psi_{12,x} - \eta\, tanh(\eta x + g) \psi_{12} + tanh(\eta x + g) f_{11} g' - f_{11,t}\}, \end{aligned}$$

which in its turn reduces to (2.3.14), by introducing the functions

$$f = f_{11}/\cosh(\eta x + g), \quad \varphi = A f_{,z}, \quad \psi = \psi_{12}/\cosh(\eta x + g).$$

Then, by taking into consideration (2.3.7), the functions f_{ij} reduce to (2.3.15).

Notice that, the non-degeneracy condition (2.1.1) holds in view of the fact that $\eta^2 + (g')^2 \neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. □

Observe that equation (2.3.14) coincides with (2.2.3) and it is referred to as of Type I in our main classification result, Theorem 2.2.1. In particular Remark 2.3.4 still applies to equation (2.3.14).

2.3.1.3 Type II (a)

Using Lemma 2.3.2 one gets the following

Theorem 2.3.6. *In the case* $\{f_{11,z} \neq 0,\ h = 0,\ \delta \neq 0\}$*, a differential equation of the form (2.1.5) is a PS equation with associated* 1*-forms (1.2.2) depending on* (x, t, z, z_1, z_2) *with* $f_{21} = \eta$ *if, and only if, the differential equation has the form*

$$\begin{aligned} z_t = {} & \epsilon_1 f^2 z_2 + \epsilon_1 \tfrac{f^2 f_{,zz}}{f_{,z}} z_1^2 + \tfrac{f^2}{g}\left(\epsilon_1 \tfrac{g f_{,xz}}{f_{,z}} - \tfrac{\psi_{,z}}{f_{,z}} + \epsilon_1 g_{,x} + \epsilon_1 \eta m g\right) z_1 \\ & - \tfrac{f^2}{g f_{,z}}\left(\psi_{,x} + \eta m \psi + \epsilon_1 \eta f g \delta\right) - \tfrac{f_{,t}}{f_{,z}} + \tfrac{f g_{,t}}{g f_{,z}} + \epsilon_1 \tfrac{m f}{f_{,z}} \textstyle\int g^2 \delta dx, \end{aligned} \tag{2.3.16}$$

where $\epsilon_1 = \pm 1$*,*

$$\psi = -\epsilon_1 g f_{,x} + \epsilon_1 \frac{f}{\delta}\left(\eta m g \delta - g \delta_{,x} - g_{,x}\delta\right) + \epsilon_1 \frac{g}{f\delta}\left[\epsilon_1 m_{,t} + \left(1 - m^2\right)\int g^2 \delta dx\right], \tag{2.3.17}$$

and $g = g(x,t)$*,* $m = m(x,t)$*,* $f = f(x,t,z)$ *are arbitrary differentiable functions, such that* $g f_{,z} \neq 0$ *on a nonempty open set, and the functions* f_{ij} *are*

$$\begin{aligned} & f_{11} = \tfrac{g}{f}, && f_{12} = -\epsilon_1 g f_{,z} z_1 + \psi, \\ & f_{21} = \eta, && f_{22} = \epsilon_1 \textstyle\int g^2 \delta dx, \\ & f_{31} = \tfrac{mg}{f}, && f_{32} = m\left(-\epsilon_1 g f_{,z} z_1 + \psi\right) + \epsilon_1 f g \delta. \end{aligned} \tag{2.3.18}$$

Proof. Under the given assumptions, from the second equation of (2.3.8) one gets $A = \frac{f_{22,x}}{f_{11}^2 \delta}$. On the other hand, in view of (2.3.7) and the first equation of (2.3.8), the functions f_{ij} are such that

$$f_{31} = m f_{11}, \quad f_{12} = \tfrac{f_{22,x} f_{11,z}}{f_{11}^2 \delta} z_1 + \psi_{12}, \quad f_{32} = m\left(\tfrac{f_{22,x} f_{11,z}}{f_{11}^2 \delta} z_1 + \psi_{12}\right) + \tfrac{f_{22,x}}{f_{11}}\ ,$$

where $f_{11} = f_{11}(x,t,z)$, $f_{22} = f_{22}(x,t)$, $f_{11} \neq 0$ on a nonempty open set, $m = m(x,t)$ and

$$\psi_{12} = \frac{1}{\delta}\left[\left(1 - m^2\right) f_{11} f_{22} + m_{,t} f_{11} + \frac{f_{11,x} f_{22,x}}{f_{11}^2} + \frac{\eta m f_{22,x}}{f_{11}} - \frac{f_{22,xx}}{f_{11}}\right],$$

in view of the third equation of (2.3.8). Thus, by (2.3.1) and (2.3.6), (2.1.5) takes the form

$$\begin{aligned} z_t = {} & \tfrac{f_{22,x}}{f_{11}^2 \delta} z_2 + \tfrac{f_{22,x}}{f_{11}^2 \delta}\left[\tfrac{f_{11,zz}}{f_{11,z}} - \tfrac{2 f_{11,z}}{f_{11}}\right] z_1^2 \\ & + \left\{\tfrac{1}{f_{11}^2 \delta^2}\left[\left(\eta m \delta - \delta_{,x}\right) f_{22,x} + f_{22,xx}\delta\right] + \tfrac{f_{22,x}}{f_{11}^2 \delta}\left[\tfrac{f_{11,xz}}{f_{11,z}} - \tfrac{2 f_{11,x}}{f_{11}}\right] + \tfrac{\psi_{12,z}}{f_{11,z}}\right\} z_1 \\ & + \tfrac{1}{f_{11,z}}\left[\psi_{12,x} - f_{11,t} + m\left(\eta \psi_{12} - f_{11} f_{22}\right) + \tfrac{\eta f_{22,x}}{f_{11}}\right], \end{aligned}$$

and one gets (2.3.16) and (2.3.17-2.3.18), after introducing the differentiable functions

$\psi(x,t,z)$, $f(x,t,z)$ and $g(x,t)$ such that

$$\psi = \psi_{12}, \qquad \epsilon_1 \frac{f_{22,x}}{\delta} = g^2, \qquad f = \frac{g}{f_{11}},$$

with $\epsilon_1 = \operatorname{sgn}\left(\frac{f_{22,x}}{\delta}\right)$.

Notice that, the non-degeneracy condition (2.1.1) holds in view of the fact that $gf_{,z} \neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. $\square$

Observe that equation (2.3.16) coincides with (2.2.6), where $\epsilon_2 = 1$ and ψ satisfies (2.2.8) and it is referred to as of Type II in our main classification result, Theorem 2.2.1.

2.3.2 Subcases with $h \neq 0$

When $h \neq 0$, in view of the system (2.3.4), one is naturally lead to distinguish the cases $G = 0$ and $G \neq 0$, where $G = \alpha + \beta f_{11}$ and α, β are given by (2.2.2). Hence, as illustrated in diagram of Section 2.2, the analysis of the case $\{f_{11,z} \neq 0,\ h \neq 0\}$ naturally leads to Type III (a) and Type III (b) equations, which correspond to $G = 0$ and $G \neq 0$, respectively. In this subsection we aim at giving a detailed analysis of these two subcases.

We start with the following auxiliary

Lemma 2.3.7. *A differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x,t,z,z_1,z_2) with $f_{21} = \eta$, $f_{11,z} \neq 0$ and $h \neq 0$ if, and only if, f_{ij} satisfy non-degeneracy conditions (2.1.1-2.1.2), equations (2.3.3) hold with*

$$\begin{aligned} A &= -\tfrac{f_{22,z}}{hf_{11,z}}, \\ \psi_{12} &= \tfrac{f_{11}f_{22}\delta}{h^2} + \tfrac{qf_{11}}{h} - \tfrac{f_{22,x}}{h}, \\ \psi_{32} &= m\left(\tfrac{f_{11}f_{22}\delta}{h^2} + \tfrac{qf_{11}}{h} - \tfrac{f_{22,x}}{h}\right) + \tfrac{f_{22}\delta}{h} + q, \end{aligned} \tag{2.3.19}$$

and $m = m(x,t)$, $q = q(x,t)$, $h = h(x,t)$, $f_{22} = f_{22}(x,t,z)$, $f_{11} = f_{11}(x,t,z)$, moreover B has the form (2.3.1) with ϕ_i given by (2.3.2) and in addition the following differential equation is satisfied

$$Gf_{22} + h\left(q\delta - m_{,t}h\right)f_{11} + h^2\left(q_{,x} - h_{,t} - \eta mq\right) = 0, \tag{2.3.20}$$

where $G = \alpha + \beta f_{11}$ and α, β are given by (2.2.2).

Proof. Under the given assumptions, equations (2.3.19) are equivalent to the first three equations of (2.3.4). Equation (2.3.20) readily follows from substituting (2.3.19) into the last equation of (2.3.4). □

It is noteworthy to remark here that, if $h \neq 0$, then the condition $\beta = 0$ implies that $\alpha = 0$. Indeed, $\beta = 0$ entails that $h = \epsilon_1 \delta/\sqrt{1-m^2}$, with $\epsilon_1 = \pm 1$ and $-1 < m < 1$. Then, using the obtained expression for h, one can easily check that $\beta_x = 2\delta\alpha/h$ and hence that $\alpha = 0$, because of $\beta_{,x} = 0$ and $\delta \neq 0$.

2.3.2.1 Type III (a)

Using Lemma 2.3.7 one gets the following

Theorem 2.3.8. *In the case $\{f_{11,z} \neq 0,\ h \neq 0,\ G = 0\}$, a differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x, t, z, z_1, z_2) with $f_{21} = \eta$ if, and only if, the differential equation has the form*

$$\begin{aligned} z_t = &-\tfrac{g_{,z}}{f_{,z}} z_2 - \tfrac{g_{,zz}}{f_{,z}} z_1^2 + \left[\tfrac{g_{,z}}{f_{,z}}\left(\tfrac{f\delta - h_{,x}}{h} - \eta m\right) - \tfrac{2g_{,xz}}{f_{,z}} + \tfrac{g\delta}{h} + \tfrac{q}{h}\right] z_1 \\ &+\tfrac{g}{f_{,z}}\left[\tfrac{(\delta_{,x}+\eta m\delta)f + f_{,x}\delta}{h} + \eta\delta - h^2 - mhf - \tfrac{fh_{,x}\delta}{h^2} + \left(\tfrac{h_{,x}}{h}\right)^2 - \tfrac{h_{,xx}}{h} - \eta m \tfrac{h_{,x}}{h}\right] \\ &+\tfrac{g_{,x}}{f_{,z}}\left(\tfrac{f\delta - h_{,x}}{h} - \eta m\right) - \tfrac{g_{,xx}}{f_{,z}} + \tfrac{f}{hf_{,z}}\left(q_{,x} + \eta m q\right) + \tfrac{qf_{,x}}{hf_{,z}} - \tfrac{qfh_{,x}}{f_{,z}h^2} + \tfrac{\eta q - f_{,t}}{f_{,z}}, \end{aligned} \tag{2.3.21}$$

where $f = f(x,t,z)$, $g = g(x,t,z)$, $m = m(x,t)$ are arbitrary differentiable functions with $\delta f_{,z} g_{,z} \neq 0$ on a nonempty open set, h and q are given by

$$\begin{cases} h = \dfrac{\epsilon_1 \delta}{\sqrt{1-m^2}}, \qquad -1 < m < 1, \qquad \epsilon_1 = \pm 1, \\ q = \dfrac{\epsilon_1 m_{,t}}{\sqrt{1-m^2}}, \end{cases} \tag{2.3.22}$$

and the functions f_{ij} have the form

$$\begin{aligned} &f_{11} = f, && f_{12} = -g_{,z} z_1 - g_{,x} - g\tfrac{h_{,x}}{h} + \tfrac{qf}{h} + \epsilon_1 f g \sqrt{1-m^2}, \\ &f_{21} = \eta, && f_{22} = gh, \\ &f_{31} = mf + h, && f_{32} = mf_{12} + \epsilon_1 g h \sqrt{1-m^2} + q. \end{aligned} \tag{2.3.23}$$

Proof. Under the given assumptions, one has that $G = 0$ entails that $\alpha = 0$, $\beta = 0$. Hence in view of (2.3.20) one has that

$$\begin{aligned}
&mh^3 + h(\delta_{,x} - \eta m\delta) - \delta h_{,x} = 0,\\
&h^2(m^2-1) + \delta^2 = 0,\\
&q\delta - m_{,t}h = 0,\\
&q_{,x} - h_{,t} - \eta mq = 0.
\end{aligned} \tag{2.3.24}$$

Then by solving the second equation of (2.3.24) with respect to h one obtains that

$$h = \frac{\epsilon_1\delta}{\sqrt{1-m^2}}, \qquad \epsilon_1 = \pm 1, \tag{2.3.25}$$

with m taking its values in the open interval $]-1,1[$. Thus, by substituting (2.3.25) into the third equation of (2.3.24) one concludes that

$$q = \frac{\epsilon_1 m_{,t}}{\sqrt{1-m^2}},$$

and the first and fourth equations of (2.3.24) are automatically satisfied.

Now in view of (2.3.19), one gets

$$\begin{aligned}
&A = -\tfrac{f_{22,z}}{hf_{11,z}},\\
&\psi_{12} = \tfrac{\epsilon\sqrt{1-m^2}f_{11}f_{22}}{h} + \tfrac{\epsilon m_{,t}f_{11}}{h\sqrt{1-m^2}} - \tfrac{f_{22,x}}{h},\\
&\psi_{32} = m\left(\tfrac{\epsilon\sqrt{1-m^2}f_{11}f_{22}}{h} + \tfrac{\epsilon m_{,t}f_{11}}{h\sqrt{1-m^2}} - \tfrac{f_{22,x}}{h}\right) + \epsilon\sqrt{1-m^2}f_{22} + \tfrac{\epsilon m_{,t}}{\sqrt{1-m^2}},
\end{aligned} \tag{2.3.26}$$

and using (2.3.1-2.3.2), (2.1.5) takes the form

$$\begin{aligned}
z_t = &-\tfrac{f_{22,z}}{hf_{11,z}}z_2 - \tfrac{f_{22,zz}}{hf_{11,z}}z_1^2 + \left[\tfrac{f_{22,z}}{f_{11,z}}\left(\tfrac{h_{,x}+f_{11}\delta}{h^2} - \tfrac{\eta m}{h}\right) - \tfrac{2f_{22,xz}}{hf_{11,z}} + \tfrac{f_{22}\delta}{h^2} + \tfrac{q}{h}\right]z_1\\
&+\tfrac{f_{22}}{f_{11,z}}\left[\tfrac{(\delta_{,x}+\eta m\delta)f_{11}+f_{11,x}\delta}{h^2} + \tfrac{\eta\delta}{h} - h - mf_{11} - \tfrac{2h_{,x}f_{11}\delta}{h^3}\right] + \tfrac{qf_{11,x}}{hf_{11,z}}\\
&+\tfrac{f_{22,x}}{f_{11,z}}\left(\tfrac{h_{,x}+f_{11}\delta}{h^2} - \tfrac{\eta m}{h}\right) + \tfrac{\eta q - f_{11,t}}{f_{11,z}} + \tfrac{f_{11}}{hf_{11,z}}(q_{,x} + \eta mq)\\
&-\tfrac{f_{22,xx}}{hf_{11,z}} - \tfrac{qf_{11}h_{,x}}{f_{11,z}h^2}.
\end{aligned}$$

where $f_{11} = f_{11}(x,t,z)$ and $f_{22} = f_{22}(x,t,z)$ are arbitrary differentiable functions. Moreover one has that

$$\begin{aligned}
&f_{31} = mf_{11} + h,\\
&f_{12} = \tfrac{1}{h}\left[-f_{22,z}z_1 + qf_{11} - f_{22,x} + \tfrac{f_{11}f_{22}\delta}{h}\right],\\
&f_{32} = mf_{12} + \tfrac{f_{22}\delta}{h} + q,
\end{aligned}$$

and by introducing the differentiable functions $f = f(x,t,z)$, $g = g(x,t,z)$ such that

$$f_{11} = f, \qquad f_{22} = gh,$$

one finally gets (2.3.21) and (2.3.23).

Notice that the non-degeneracy condition (2.1.1) holds in view of the fact that $f_{,z}g_{,z} \neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. □

Observe that equation (2.3.21) coincides with (2.2.11), where $\epsilon_2 = 1$ and h, m and q satisfy (2.2.13), and it is referred to as of Type III in our main classification result, Theorem 2.2.1.

2.3.2.2 Type III (b)

Using Lemma 2.3.7 one gets the following

Theorem 2.3.9. *In the case $\{f_{11,z} \neq 0,\ h \neq 0,\ G \neq 0\}$, a differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x,t,z,z_1,z_2) with $f_{21} = \eta$ if, and only if, the differential equation has the form*

$$\begin{aligned} z_t = &-\tfrac{g_{,z}}{f_{,z}} z_2 - \tfrac{g_{,zz}}{f_{,z}} z_1^2 + \left[\tfrac{g_{,z}}{f_{,z}} \left(\tfrac{f\delta - h_{,x}}{h} - \eta m\right) - \tfrac{2g_{,xz}}{f_{,z}} + \tfrac{g\delta}{h} + \tfrac{q}{h}\right] z_1 \\ &+ \tfrac{g}{f_{,z}} \left[\tfrac{(\delta_{,x} + \eta m\delta) f + f_{,x}\delta}{h} + \eta\delta - h^2 - mhf - \tfrac{f h_{,x}\delta}{h^2} + \left(\tfrac{h_{,x}}{h}\right)^2 - \tfrac{h_{,xx}}{h} - \eta m \tfrac{h_{,x}}{h}\right] \\ &+ \tfrac{g_{,x}}{f_{,z}} \left(\tfrac{f\delta - h_{,x}}{h} - \eta m\right) - \tfrac{g_{,xx}}{f_{,z}} + \tfrac{f}{hf_{,z}} (q_{,x} + \eta m q) + \tfrac{q f_{,x}}{h f_{,z}} - \tfrac{q f h_{,x}}{f_{,z} h^2} + \tfrac{\eta q - f_{,t}}{f_{,z}}, \end{aligned} \tag{2.3.27}$$

where

$$g = -\tfrac{1}{G} \left[h\left(q_{,x} - h_{,t} - \eta m q\right) + \left(q\delta - m_{,t} h\right) f\right],$$

$f = f(x,t,z)$, $m = m(x,t)$, $h = h(x,t)$ and $q = q(x,t)$ are arbitrary differentiable functions, with $hf_{,z} \neq 0$ on a nonempty open set, and the functions f_{ij} have the form

$$\begin{aligned} f_{11} &= f, & f_{12} &= -g_{,z} z_1 + \tfrac{qf}{h} - g_{,x} - \tfrac{g h_{,x}}{h} + \tfrac{fg\delta}{h}, \\ f_{21} &= \eta, & f_{22} &= gh, \\ f_{31} &= mf + h, & f_{32} &= m f_{12} + g\delta + q. \end{aligned} \tag{2.3.28}$$

Proof. Under the given assumption, by rewriting (2.3.20) as

$$f_{22} = -\frac{1}{G} \left[h^2\left(q_{,x} - h_{,t} - \eta m q\right) + \left(qh\delta - m_{,t} h^2\right) f_{11}\right],$$

and using the expression of A, ψ_{12} and ψ_{32} provided by (2.3.19) in the formulas (2.3.1-2.3.2) and (2.1.7), one gets that (2.1.5) takes the form

$$
\begin{aligned}
z_t = & -\tfrac{f_{22,z}}{hf_{11,z}} z_2 - \tfrac{f_{22,zz}}{hf_{11,z}} z_1^2 + \left[\tfrac{f_{22,z}}{f_{11,z}} \left(\tfrac{h_{,x}+f_{11}\delta}{h^2} - \tfrac{\eta m}{h}\right) - \tfrac{2f_{22,xz}}{hf_{11,z}} + \tfrac{f_{22}\delta}{h^2} + \tfrac{q}{h}\right] z_1 \\
& + \tfrac{f_{22}}{f_{11,z}} \left[\tfrac{(\delta_{,x}+\eta m\delta)f_{11}+f_{11,x}\delta}{h^2} + \tfrac{\eta\delta}{h} - h - mf_{11} - \tfrac{2h_{,x}f_{11}\delta}{h^3}\right] + \tfrac{qf_{11,x}}{hf_{11,z}} \\
& + \tfrac{f_{22,x}}{f_{11,z}} \left(\tfrac{h_{,x}+f_{11}\delta}{h^2} - \tfrac{\eta m}{h}\right) + \tfrac{\eta q - f_{11,t}}{f_{11,z}} + \tfrac{f_{11}}{hf_{11,z}} \left(q_{,x} + \eta m q\right) \\
& - \tfrac{f_{22,xx}}{hf_{11,z}} - \tfrac{qf_{11}h_{,x}}{f_{11,z}h^2},
\end{aligned}
\tag{2.3.29}
$$

where $f_{11} = f_{11}(x,t,z)$, $m = m(x,t)$, $h = h(x,t)$, $q = q(x,t)$ are arbitrary differentiable functions and the remaining f_{ij} are such that

$$
\begin{aligned}
f_{31} &= mf_{11} + h, \\
f_{12} &= \tfrac{1}{h}\left[-f_{22,z}z_1 + qf_{11} - f_{22,x} + \tfrac{f_{11}f_{22}\delta}{h}\right], \\
f_{32} &= mf_{12} + \tfrac{f_{22}\delta}{h} + q.
\end{aligned}
\tag{2.3.30}
$$

Finally (2.3.27) and (2.3.28) are easily obtained from (2.3.29) and (2.3.30) by introducing the new functions f and g such that

$$f_{11} = f, \qquad f_{22} = gh.$$

Notice that the non-degeneracy condition (2.1.1) holds in view of the fact that $f_{,z}g_{,z} \neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. $\square$

Observe that equation (2.3.27) coincides with (2.2.11), where $\epsilon_2 = 1$ and g satisfies (2.2.15), and it is referred to as of Type III in our main classification result, Theorem 2.2.1.

2.4 Analysis of the special case $f_{11,z} = 0$

The following theorem gives a characterization of PS equations of the form (2.1.5) with associated 1-forms (1.2.2) satisfying (2.2.1) and $f_{11,z} = 0$.

Lemma 2.4.1. *A differential equation of the form (2.1.5) is a PS equation with associated 1-forms (1.2.2) depending on (x,t,z,z_1,z_2) with $f_{21} = \eta$ and $f_{11,z} = 0$ if, and only if, B has the form*

$$B = \phi_1 z_1^2 + \phi_2 z_1 + \phi_3, \tag{2.4.1}$$

where

$$\begin{cases} \phi_1 = \frac{1}{f_{31,z}}\left[Af_{31,zz} + A_{,z}f_{31,z}\right], \\ \phi_2 = \frac{1}{f_{31,z}}\left[Af_{31,xz} + A_{,x}f_{31,z} + \psi_{32,z}\right], \\ \phi_3 = \frac{1}{f_{31,z}}\left[\psi_{32,x} + \eta f_{12} - f_{11}f_{22} - f_{31,t}\right], \end{cases} \tag{2.4.2}$$

with $f_{11} = f_{11}(x,t)$, $f_{31} = f_{31}(x,t,z)$, $f_{12} = f_{12}(x,t,z)$, $f_{22} = f_{22}(x,t,z)$, $f_{32} = Af_{31,z}z_1 + \psi_{32}(x,t,z)$ *differentiable functions such that*

$$f_{11}f_{22} - \eta f_{12} \neq 0, \qquad \Delta = f_{31,z} \neq 0, \tag{2.4.3}$$

and satisfying the system

$$\begin{cases} f_{12,z} + \eta A f_{31,z} = 0, \\ f_{12,x} - f_{11,t} + \eta\psi_{32} - f_{31}f_{22} = 0, \\ f_{22,z} - Af_{11}f_{31,z} = 0, \\ f_{22,x} + f_{31}f_{12} - f_{11}\psi_{32} = 0. \end{cases} \tag{2.4.4}$$

Proof. Equations (2.4.1) and (2.4.2) follow by solving the third equation of (2.1.8) with respect to B. On the other hand the remaining two equations of (2.1.8) reduce to

$$\begin{cases} (f_{12,z} + \eta A f_{31,z})\, z_1 + f_{12,x} - f_{11,t} + \eta\psi_{32} - f_{31}f_{22} = 0, \\ (f_{22,z} - Af_{11}f_{31,z})\, z_1 + f_{22,x} + f_{31}f_{12} - f_{11}\psi_{32} = 0, \end{cases} \tag{2.4.5}$$

and, in view of the independence of f_{11}, f_{31}, f_{12}, f_{22}, ψ_{32} with respect to z_1, one readily gets equations (2.4.4). Finally, non-degeneracy conditions (2.4.3) are direct consequences of (2.1.1-2.1.2). $\square$

The following two subsections are devoted to the classification of PS equations of the form (2.1.5) under the assumption (2.2.1) with $f_{11,z} = 0$. This is a special case, where the analysis noteworthy simplifies and one only finds the two further types of equations II (b) and III (c), as illustrated in diagram of Section 2.2.

2.4.1 Type II (b)

Using Lemma 2.4.1 one gets the following

Theorem 2.4.2. *In the case* $f_{11} = 0$*, a differential equation of the form (2.1.5) is a PS equation with associated* 1*-forms (1.2.2) depending on* (x,t,z,z_1,z_2) *with* $f_{21} = \eta$ *if, and* *only if, the differential equation has the form*

$$\begin{aligned} z_t = & -\epsilon_1 f^2 z_2 - \epsilon_1 \tfrac{f^2 f_{,zz}}{f_{,z}} z_1^2 - \tfrac{f^2}{g}\left(\epsilon_1 \tfrac{g f_{,xz}}{f_{,z}} + \tfrac{\psi_{,z}}{f_{,z}} + \epsilon_1 g_{,x}\right) z_1 \\ & -\tfrac{f^2}{g f_{,z}}\left(\psi_{,x} - \epsilon_1 \eta^2 g f\right) - \tfrac{f_{,t}}{f_{,z}} + \tfrac{f g_{,t}}{g f_{,z}}, \end{aligned} \tag{2.4.6}$$

where $\epsilon_1 = \pm 1$,

$$\psi = \epsilon_1 g f_{,x} + \epsilon_1 f g_{,x} + \epsilon_1 \frac{g \int g^2 dx}{f}$$

and $g = g(x,t)$, $f = f(x,t,z)$ *are arbitrary differentiable functions, such that* $g f_{,z} \neq 0$ *on a nonempty open set, and the functions* f_{ij} *have the form*

$$\begin{aligned} f_{11} &= 0, & f_{12} &= -\epsilon_1 \eta g f, \\ f_{21} &= \eta, & f_{22} &= \epsilon_1 \textstyle\int \eta g^2 dx, \\ f_{31} &= \tfrac{g}{f}, & f_{32} &= \epsilon_1 g f_{,z} z_1 + \psi. \end{aligned} \tag{2.4.7}$$

Proof. Under the given assumptions the system (2.4.4) reduces to

$$\begin{aligned} & f_{12,z} + \eta A f_{31,z} = 0, \\ & f_{12,x} + \eta \psi_{32} - f_{31} f_{22} = 0, \\ & f_{22} = f_{22}(x,t), \\ & f_{22,x} + f_{31} f_{12} = 0, \end{aligned} \tag{2.4.8}$$

whereas conditions (2.4.3) become $\eta f_{12,x} \neq 0$ and $f_{31,z} \neq 0$, respectively. Thus, (2.4.8) provide

$$A = -\frac{f_{12,z}}{\eta f_{31,z}}, \qquad \psi_{32} = \frac{f_{31} f_{22} - f_{12,x}}{\eta}, \qquad f_{12} = -\frac{f_{22,x}}{f_{31}},$$

and hence $f_{32} = \frac{1}{\eta}\left(-f_{12,z} z_1 + f_{31} f_{22} - f_{12,x}\right)$. Then in view of (2.4.1-2.4.2), (2.1.5) takes the form

$$\begin{aligned} z_t = & -\tfrac{f_{12,z}}{\eta f_{31,z}} z_2 - \tfrac{f_{12,zz}}{\eta f_{31,z}} z_1^2 - \left(\tfrac{2 f_{12,xz}}{\eta f_{31,z}} - \tfrac{f_{22}}{\eta}\right) z_1 - \tfrac{f_{31,t} - \eta f_{12}}{f_{31,z}} \\ & + \tfrac{f_{22,x} f_{31}}{\eta f_{31,z}} + \tfrac{f_{31,x} f_{22} - f_{12,xx}}{\eta f_{31,z}}, \end{aligned}$$

and by introducing the differentiable functions $f = f(x,t,z)$ and $g = g(x,t)$ such that

$$f_{31} = \frac{g}{f}, \qquad \epsilon_1 \frac{f_{22,x}}{\eta} = g^2, \qquad \epsilon_1 = \operatorname{sgn}\left(\frac{f_{22,x}}{\eta}\right),$$

one finally gets (2.4.6) and (2.4.7).

Notice that, the functions f_{ij} satisfy the non-degeneracy condition (2.1.1) in view of the fact that $g f_{,z} \neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. □

Observe that equation (2.4.6) coincides with (2.2.6), where $\epsilon_2 = -1$, $m = 0$, $\delta = \eta$ and ψ satisfies (2.2.10) and it is referred to as of Type II in our main classification result, Theorem 2.2.1.

2.4.2 Type III (c)

Using Lemma 2.4.1 one gets the following

Theorem 2.4.3. *In the case* $\{f_{11,z} = 0,\ f_{11} \neq 0\}$, *a differential equation of the form (2.1.5) is a PS equation with associated* 1*-forms (1.2.2) depending on* (x, t, z, z_1, z_2) *with* $f_{21} = \eta$ *if, and only if, the differential equation has the form*

$$\begin{aligned} z_t = {} & \tfrac{g_{,z}}{f_{,z}} z_2 + \tfrac{g_{,zz}}{f_{,z}} z_1^2 - \left[\tfrac{g_{,z}}{f_{,z}}\left(\tfrac{f\eta - h_{,x}}{h}\right) - \tfrac{2g_{,xz}}{f_{,z}} + \tfrac{g\eta}{h} - \tfrac{q}{h}\right] z_1 \\ & + \tfrac{g}{f_{,z}}\left[-\tfrac{f_{,x}\eta}{h} - \eta^2 - h^2 + \tfrac{f h_{,x}\eta}{h^2} - \left(\tfrac{h_{,x}}{h}\right)^2 + \tfrac{h_{,xx}}{h}\right] \\ & - \tfrac{g_{,x}}{f_{,z}}\left(\tfrac{f\eta - h_{,x}}{h}\right) + \tfrac{g_{,xx}}{f_{,z}} + \tfrac{f q_{,x}}{h f_{,z}} + \tfrac{q f_{,x}}{h f_{,z}} - \tfrac{q f h_{,x}}{f_{,z} h^2} + \tfrac{\eta q - f_{,t}}{f_{,z}}, \end{aligned} \tag{2.4.9}$$

where

$$g = -\tfrac{[h(q_{,x} - h_{,t}) + \eta q f]}{\eta h_{,x} - (\eta^2 + h^2) f}.$$

and $h = h(x,t)$, $q = q(x,t)$, $f = f(x,t,z)$ *are arbitrary differentiable functions, such that* $f_{,z} \neq 0$ *on a nonempty open set, and the functions* f_{ij} *have the form*

$$\begin{aligned} & f_{11} = h, && f_{12} = q - \eta g, \\ & f_{21} = \eta, && f_{22} = gh, \\ & f_{31} = f && f_{32} = g_{,z} z_1 + (q - \eta g)\tfrac{f}{h} + g_{,x} + \tfrac{g h_{,x}}{h}. \end{aligned} \tag{2.4.10}$$

Proof. Under the given assumptions, the first and the two last equations of (2.4.4) provide

$$f_{12} = q - \frac{\eta f_{22}}{f_{11}}, \qquad A = \frac{f_{22,z}}{f_{11} f_{31,z}}, \qquad \psi_{32} = \frac{f_{22,x} + f_{31} f_{12}}{f_{11}}, \tag{2.4.11}$$

where $q = q(x,t)$ is an arbitrary differentiable function. Then in view of Lemma 2.4.1 one gets

$$f_{32} = \frac{f_{22,z}}{f_{11}} z_1 + \left(q - \frac{\eta f_{22}}{f_{11}}\right)\frac{f_{31}}{f_{11}} + \frac{f_{22,x}}{f_{11}},$$

and by substituting (2.4.11) into the second equation of (2.4.4) one gets

$$f_{22} = \frac{\eta q f_{11} f_{31} + (q_{,x} - f_{11,t}) f_{11}^2}{f_{31}(\eta^2 + f_{11}^2) - \eta f_{11,x}}.$$

Hence, in view of (2.4.1-2.4.2), (2.1.5) takes the form

$$\begin{aligned}
z_t = {} & \tfrac{f_{22,z}}{f_{11}f_{31,z}} z_2 + \tfrac{f_{22,zz}}{f_{11}f_{31,z}} z_1^2 - \left[\tfrac{1}{f_{31,z}}\left(\tfrac{f_{22,z}(f_{11,x}+\eta f_{31})}{f_{11}^2} - \tfrac{2f_{22,xz}}{f_{11}}\right) - \tfrac{q}{f_{11}} + \tfrac{\eta f_{22}}{f_{11}^2}\right] z_1 \\
& + \tfrac{f_{31}}{f_{31,z}}\left[\tfrac{2\eta f_{11,x} f_{22}}{f_{11}^3} + \tfrac{q_{,x}}{f_{11}} - \tfrac{q f_{11,x} + \eta f_{22,x}}{f_{11}^2}\right] - \tfrac{f_{22}}{f_{31,z}}\left[f_{11} + \tfrac{\eta^2}{f_{11}} + \tfrac{\eta f_{31,x}}{f_{11}^2}\right] \\
& + \eta q - f_{31,t} + \tfrac{q f_{31,x} + f_{22,xx}}{f_{11}} - \tfrac{f_{22,x} f_{11,x}}{f_{11}^2}.
\end{aligned}$$

Thus, by introducing the differentiable functions $f = f(x,t,z)$ and $g = g(x,t,z)$ such that

$$f_{11} = h, \qquad f_{31} = f, \qquad f_{22} = gh,$$

one gets (2.4.9) and (2.4.10).

Notice that, the functions f_{ij} satisfy the non-degeneracy condition (2.1.1), since

$$f_{11}f_{22} - f_{21}f_{12} = g\left(h^2 + \eta\right) - \eta q,$$

is nonzero in view of the fact that $g_{,z} \neq 0$ on a nonempty open set.

The converse of the theorem is a straightforward computation. □

Observe that equation (2.4.9) coincides with (2.2.11), where $\epsilon_2 = -1$, $m = 0$, $\delta = \eta$ and g satisfies (2.2.17), and it is referred to as of Type III in our main classification result, Theorem 2.2.1.

2.5 Additional examples

Here are some additional examples of equations obtained from the given classification.

Example 2.5.1. Equations classified by Theorem 2.2.1 include the nonlinear second order evolution equations admitting "higher weakly nonlinear symmetries", which have been classified by Svinolupov and Sokolov [43] and up to contact transformations can be written in one of the following forms:

$$\begin{aligned}
z_t &= z_2 + 2zz_1 + k(x), \\
z_t &= z^2 z_2 - \lambda x z_1 + \lambda z, \\
z_t &= z^2 z_2 + \lambda z^2, \\
z_t &= z^2 z_2 - \lambda x^2 z_1 + 3\lambda x z,
\end{aligned} \tag{2.5.1}$$

with $\lambda \in \mathbb{R}$ and $k(x)$ an arbitrary differentiable function.

Indeed one can readily check what follows:

(i) The first equation of (2.5.1) is an example of the Type I (a), as well as of Type I (b), with $f = z$, $\varphi = 1$, $\psi = z^2 + \int k\,dx$. For instance, using formulas (2.3.12), one easily gets the corresponding 1-forms

$$\begin{aligned}
\omega_1 &= e^{-\epsilon(\eta x+g)}\left[z\,dx + \left(z_1 + z^2 + \int k\,dx\right)\,dt\right],\\
\omega_2 &= \eta\,dx + g'\,dt,\\
\omega_3 &= \epsilon\omega_1,
\end{aligned}$$

with $g = g(t)$ an arbitrary differentiable function, $\epsilon = \pm 1$ and $\eta^2 + (g')^2 \neq 0$. It follows that this equation is the integrability condition of a triangular linear problem given by (1.2.6).

The first equation of (2.5.1) is also an example of the Type III (a) with $f = z - p$, $g = -z - p$, $m = 0$, $\epsilon_1 = -1$. Using formulas (2.3.23), one easily gets the corresponding 1-forms

$$\begin{aligned}
\omega_1 &= (z-p)\,dx + \left(z_1 + z^2 + p' - p^2\right)\,dt,\\
\omega_2 &= \eta\,dx + \eta\,(z+p)\,dt,\\
\omega_3 &= -\eta\,dx - \eta\,(z+p)\,dt,
\end{aligned} \tag{2.5.2}$$

where $p = p(x)$ satisfies $k = p'' - 2pp'$.

(ii) The second equation of (2.5.1) is an example of the Type I (a), and Type I (b), with $f = z^{-1}$, $\varphi = -1$, $\psi = -\lambda x z^{-1}$. Using formulas (2.3.12), one easily gets the corresponding 1-forms

$$\begin{aligned}
\omega_1 &= e^{-\epsilon(\eta x+g)}\left[z^{-1}\,dx + \left(-z_1 - \lambda x z^{-1}\right)\,dt\right],\\
\omega_2 &= \eta\,dx + g'\,dt,\\
\omega_3 &= \epsilon\omega_1,
\end{aligned}$$

with $g = g(t)$ an arbitrary differentiable function, $\epsilon = \pm 1$ and $\eta^2 + (g')^2 \neq 0$. It follows that this equation is the integrability condition of a triangular linear problem given by (1.2.6).

(iii) The third equation of (2.5.1) is an example of the Type I (a), and Type I (b), with $f = xz^{-1}$, $\varphi = -x$, $\psi = z - \frac{\lambda x^2}{2}$. Using formulas (2.3.12), one easily gets the corresponding 1-forms

$$\begin{aligned}
\omega_1 &= e^{-\epsilon(\eta x+g)}\left[xz^{-1}\,dx + \left(-xz_1 + z - \tfrac{\lambda x^2}{2}\right)\,dt\right],\\
\omega_2 &= \eta\,dx + g'\,dt,\\
\omega_3 &= \epsilon\omega_1,
\end{aligned}$$

with $g = g(t)$ an arbitrary differentiable function, $\epsilon = \pm 1$ and $\eta^2 + (g')^2 \neq 0$. It follows that this equation is the integrability condition of a triangular linear problem given by (1.2.6).

(iv) The fourth equation of (2.5.1) is an example of the Type I (a), and Type I (b), with $f = xz^{-1}$, $\varphi = -x$, $\psi = z - \frac{\lambda x^3}{z}$. Using formulas (2.3.12), one easily gets the corresponding 1-forms

$$\omega_1 = e^{-\epsilon(\eta x + g)} \left[xz^{-1}\, dx + \left(-xz_1 + z - \tfrac{\lambda x^3}{z} \right) dt \right],$$
$$\omega_2 = \eta\, dx + g'\, dt,$$
$$\omega_3 = \epsilon \omega_1,$$

with $g = g(t)$ an arbitrary differentiable function, $\epsilon = \pm 1$ and $\eta^2 + (g')^2 \neq 0$. It follows that this equation is the integrability condition of a triangular linear problem given by (1.2.6).

Similar results have been obtained by Reyes in [49].

Remark 2.5.2. It is noteworthy to note that for all the linear problems of previous example the parameter η is always removable. Indeed η is removable from the linear problems described by Theorems 2.3.3 and 2.3.5 by means of the gauge transformation defined by

$$S = \begin{pmatrix} e^{\frac{\eta x}{2}} & 0 \\ 0 & e^{-\frac{\eta x}{2}} \end{pmatrix}.$$

In fact, under such a transformation, the 1-forms ω_i of Theorem 2.3.3 are transformed to

$$\begin{cases} \omega_1^S = e^{-\epsilon g} \left(f\, dx + (\varphi z_1 + \psi)\, dt \right), \\ \omega_2^S = g'\, dt, \\ \omega_3^S = \epsilon \omega_1^S, \end{cases} \tag{2.5.3}$$

whereas the 1-forms ω_i of Theorem 2.3.5 are transformed to

$$\begin{cases} \omega_1^S = \cosh(g) \left(f\, dx + (\varphi z_1 + \psi)\, dt \right), \\ \omega_2^S = g'\, dt, \\ \omega_3^S = -\tanh(g) \omega_1^S. \end{cases} \tag{2.5.4}$$

Notice that, despite the analogy between (2.5.3) and (2.5.4), it may be checked that the corresponding zero-curvature representations are not gauge equivalent.

On the other hand one can easily check that by means of the gauge transformation defined by

$$S = \begin{pmatrix} \frac{1}{2\sqrt{\eta}} & -\sqrt{\eta} \\ \frac{1}{2\sqrt{\eta}} & \sqrt{\eta} \end{pmatrix}, \tag{2.5.5}$$

η is removable also from the linear problem given by (2.5.2).

Example 2.5.3. Murray equation

$$z_t = z_2 + \lambda_1 z z_1 + \lambda_2 z - \lambda_3 z^2,$$

is another example of Type I (a), and Type I (b), corresponding to the choice:

$$\begin{aligned} f &= e^{-\frac{1}{\lambda_1^2}\left[\left(\lambda_1^2\lambda_2 + 4\lambda_3^2\right)t + 2\lambda_1\lambda_3 x\right]} z, \quad \varphi = e^{-\frac{1}{\lambda_1^2}\left[\left(\lambda_1^2\lambda_2 + 4\lambda_3^2\right)t + 2\lambda_1\lambda_3 x\right]}, \\ \psi &= e^{-\frac{1}{\lambda_1^2}\left[\left(\lambda_1^2\lambda_2 + 4\lambda_3^2\right)t + 2\lambda_1\lambda_3 x\right]} \left(\frac{\lambda_1 z^2}{2} + \frac{2\lambda_3 z}{\lambda_1}\right). \end{aligned}$$

For instance, by using formulas (2.3.12), one easily gets the corresponding 1-forms

$$\begin{aligned} \omega_1 &= e^{-\epsilon(\eta x + g)} e^{-\frac{1}{\lambda_1^2}\left[\left(\lambda_1^2\lambda_2 + 4\lambda_3^2\right)t + 2\lambda_1\lambda_3 x\right]} \left[z\,dx + \left(z_1 + \frac{\lambda_1 z^2}{2} + \frac{2\lambda_3 z}{\lambda_1}\right) dt\right], \\ \omega_2 &= \eta\,dx + g'\,dt, \\ \omega_3 &= \epsilon\omega_1, \end{aligned}$$

with $g = g(t)$ an arbitrary differentiable function, $\epsilon = \pm 1$ and $\eta^2 + (g')^2 \neq 0$. It follows that this equation is the integrability condition of a triangular linear problem given by (1.2.6).

Example 2.5.4. Boltzman equation

$$z_t = z z_2 + z_1^2,$$

is another example of Type I (a), and Type I (b), corresponding to the choice:

$$f = \varphi = z, \quad \psi = 0, \quad g = g(t).$$

Using formulas (2.3.15), one easily gets the corresponding 1-forms

$$\begin{aligned} \omega_1 &= cosh\,(\eta x + g)\,(z\,dx + z z_1\,dt), \\ \omega_2 &= \eta\,dx + g'\,dt, \\ \omega_3 &= -sinh\,(\eta x + g)\,(z\,dx + z z_1\,dt), \end{aligned}$$

with $g = g(t)$ an arbitrary differentiable function and $\eta^2 + (g')^2 \neq 0$. It follows that this equation is the integrability condition of the linear problem given by (1.2.6).

Example 2.5.5. The equation

$$z_t = z^2 z_2 + 4x z_1 - 4z^3 - 4z, \tag{2.5.6}$$

is the "simplest" member, up to contact transformations, of the class of second order evolution equations described by Michal Marvan in [39]. It is another example of Type II (a) corresponding to the choice

$$\epsilon_1 = 1, \quad m = 0, \quad f = z, \quad g = \eta = 2, \tag{2.5.7}$$

which, in view of (2.3.18), also gives the corresponding 1-forms

$$\begin{aligned} \omega_1 &= \tfrac{2}{z}\, dx + \left(-2z_1 + \tfrac{8x}{z}\right) dt, \\ \omega_2 &= 2\, dx + 8x\, dt, \\ \omega_3 &= 4z\, dt. \end{aligned} \tag{2.5.8}$$

One can check that in view of Theorem 6 of [39], (2.5.6) is up to contact transformations the unique equation described by Theorem 2.2.1 which admits an irreducible zero-curvature representation. In particular, we notice that the irreducible zero-curvature representation obtained in [39] for (2.5.6) coincides with the one obtained from (1.2.6) and (2.5.8) by passing to new 1-forms $\omega_1 \mapsto \omega_2$, $\omega_2 \mapsto \omega_1$ and $\omega_3 \mapsto -\omega_3$.

Example 2.5.6. Burgers equation

$$z_t = z_2 + z z_1, \tag{2.5.9}$$

can be embedded in Type I (a-b) with $f = z$, $\varphi = 1$, $\psi = z^2/2$, as well as in Type III (a) with $f = z/2$, $g = -z/2$, $m = 0$, $\epsilon_1 = -1$. In particular the linear problem corresponding to Type III (a) coincides with the one already given by Chern and Tenenblat in [20].

These two linear problems of (2.5.9) provide of an example of a pair of linear problems which are non gauge equivalent. Indeed only the second linear problem admits non gauge-like symmetries, as one can check by using the method discussed in Chapter 4. Hence, the two linear problems must be considered as being structurally different. In particular, as shown in Chapter 4, after removing the parameter η with the gauge transformation defined by (2.5.5), one can insert a non-removable parameter in the second linear problem by using the flow A_λ of the non gauge-like symmetry generated by $t\partial_x - \partial_z$. Indeed, in this way one get the following family of ZCRs of (2.5.9)

$$\alpha_\lambda = \begin{pmatrix} \frac{z}{4} - \frac{\lambda}{4} & 0 \\ -\frac{1}{2} & \frac{\lambda}{4} - \frac{z}{4} \end{pmatrix} dx + \begin{pmatrix} \frac{z_1}{4} + \frac{(z-\lambda)^2}{8} + \frac{\lambda(z-\lambda)}{4} & 0 \\ -\frac{\lambda}{4} - \frac{z}{4} & -\frac{z_1}{4} - \frac{(z-\lambda)^2}{8} - \frac{\lambda(z-\lambda)}{4} \end{pmatrix} dt,$$

which depends on a non-removable parameter λ.

Example 2.5.7. Equation

$$z_t = xz_2 + 2\,(xz+1)\,z_1 + z^2, \tag{2.5.10}$$

is another example of Type III (a) corresponding to the choice

$$f = z, \qquad g = -xz \qquad m = 0, \qquad \epsilon_1 = -1.$$

Using formulas (2.3.23), one easily gets the corresponding 1-forms

$$\begin{aligned} \omega_1 &= z\,dx + \left(xz_1 + xz^2 + z\right)\,dt, \\ \omega_2 &= \eta\,dx + \eta xz\,dt, \\ \omega_3 &= -\omega_2, \end{aligned} \tag{2.5.11}$$

with $\eta \neq 0$. Hence equation (2.5.10) is the integrability condition of a linear depending on a parameter η.

This parameter is removable by means of the gauge transformation defined by (2.5.5). However, by using the method discussed in Chapter 4, one can easily construct a linear problem depending on a non-removable parameter λ.

Indeed, the symmetry generated by $xt\partial_x + \frac{t^2}{2}\partial_t - \left(\frac{1}{2} + tz\right)\partial_z$ is non gauge-like for the given zero-curvature representation and its flow A_λ can be used to insert a non-removable parameter λ into it. Indeed, by preliminarily removing the parameter η through the gauge transformation defined by (2.5.5), one gets the following family of ZCRs of (2.5.10)

$$\begin{aligned} \alpha_\lambda = &\begin{pmatrix} \frac{\lambda+\lambda tz-2z}{2(\lambda t-2)} & 0 \\ -\frac{2}{(\lambda t-2)^2} & -\frac{\lambda+\lambda tz-2z}{2(\lambda t-2)} \end{pmatrix} dx \\ &+ \begin{pmatrix} \frac{1}{2}\left[xz_1 + xz^2 + z + \frac{\lambda(\lambda t-\lambda x-2)}{(\lambda t-2)^2}\right] & 0 \\ -\frac{2xz}{(\lambda t-2)^2} + \frac{2\lambda x}{(\lambda t-2)^3} & -\frac{1}{2}\left[xz_1 + xz^2 + z + \frac{\lambda(\lambda t-\lambda x-2)}{(\lambda t-2)^2}\right] \end{pmatrix} dt, \end{aligned}$$

which depends on a non-removable parameter λ.

Remark 2.5.8. In the context of equations describing pseudospherical surfaces the occurrence of differential substitutions that, like the celebrated Cole-Hopf transformation, map solutions of an already known integrable equation (e.g. a linear equation) to solutions of a new equation, is quite natural. Indeed, as already observed by Reyes (see for instance [53]) these substitutions can be often obtained from the following Riccati first order system for an auxiliary function $\Gamma = \Gamma(x,t)$

$$\begin{cases} \Gamma_{,x} = -\frac{1}{2}\left(f_{31} - f_{21}\right)\Gamma^2 + f_{11}\Gamma - \frac{1}{2}\left(f_{31} + f_{21}\right), \\ \Gamma_{,t} = -\frac{1}{2}\left(f_{32} - f_{22}\right)\Gamma^2 + f_{12}\Gamma - \frac{1}{2}\left(f_{32} + f_{22}\right), \end{cases} \tag{2.5.12}$$

which is naturally defined by the 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$ of an equation describing pseudospherical surfaces. This fact is due to the remarkable property that the integrability of (2.5.12) is equivalent to the structure equations (1.2.1), and will be illustrated below by means of next three examples.

Example 2.5.9. Here we will use the Riccati first order system (2.5.12) to identify a differential substitution which "linearizes" the first equation of (2.5.1) (generalized Burgers equation). To this end we observe that by using the linear problem of Type III (a) given in the Example 2.5.1, (2.5.12) takes the form

$$\begin{cases} \Gamma_{,x} = \eta\Gamma^2 + (z - p)\,\Gamma, \\ \Gamma_{,t} = \eta\,(z + p)\,\Gamma^2 + \left(z_1 + z^2 + p_{,x} - p^2\right)\Gamma. \end{cases}$$

Hence, by assuming that $\Gamma \neq 0$, (2.5.12) can be rewritten as

$$\begin{cases} z = \frac{\Gamma_{,x}}{\Gamma} - \eta\Gamma + p, \\ \Gamma_{,t} = \Gamma_{,xx} - 2\eta\Gamma\Gamma_{,x} + 2p\Gamma_{,x} + 2p_{,x}\Gamma. \end{cases} \tag{2.5.13}$$

We notice that (2.5.13) is well defined whatever is the value of η and in particular that it is defined also for $\eta = 0$.

When $\eta = 0$, if $p = p(x)$ is such that $p'' - 2pp' = k(x)$ then (2.5.13) provides the differential substitution

$$z = \frac{\Gamma_{,x}}{\Gamma} + p,$$

which transforms the nonzero solutions of the linear equation

$$\Gamma_{,t} = \Gamma_{,xx} + 2p\Gamma_{,x} + 2p'\Gamma,$$

to solutions of the generalized Burgers equation

$$z_t = z_2 + 2zz_1 + k.$$

When $p = 0$ above transformation reduces to the celebrated Cole-Hopf transformation.

Example 2.5.10. Here we will use the Riccati first order system (2.5.12) to identify a differential substitution which "linearizes" equation (2.5.10). To this end we observe that by using the linear problem (2.5.11), (2.5.12) takes the following form

$$\begin{cases} \Gamma_{,x} = \eta\Gamma^2 - z\Gamma, \\ \Gamma_{,t} = \eta xz\Gamma^2 + \left(xz_1 + z + xz^2\right)\Gamma. \end{cases}$$

Hence, by assuming that $\Gamma \neq 0$, (2.5.12) can be rewritten as

$$\begin{cases} z = \frac{\Gamma_{,x}}{\Gamma} - \eta\Gamma, \\ \Gamma_{,t} = x\Gamma_{,xx} - 2\eta x\Gamma\Gamma_{,x} + \Gamma_{,x} - \eta\Gamma^2. \end{cases} \tag{2.5.14}$$

Also in this case, the Riccati system is well defined also for the particular value $\eta = 0$ by which it provides the differential substitution

$$z = \frac{\Gamma_{,x}}{\Gamma},$$

transforming the nonzero solutions of the linear equation

$$\Gamma_{,t} = x\Gamma_{,xx} + \Gamma_{,x},$$

to solutions of (2.5.10), i.e.,

$$z_t = xz_2 + 2\left(xz + 1\right)z_1 + z^2.$$

Example 2.5.11. Another application of the method illustrated in the last two examples can be given by considering the following class of equations

$$z_t = A_2 z_2 + \left(A_1 + 2A_2 z + A_{2,x}\right)z_1 + A_{2,x}z^2 + A_{1,x}z + A_{1,xx} - A_{2,xxx}, \tag{2.5.15}$$

where $A_1 = A_1(x,t)$ and $A_2 = A_2(x,t)$ are arbitrary differentiable functions, with $A_2 \neq 0$. Equations (2.5.15) are of Type III (a), as one can check by taking

$$f = z, \qquad g = -A_2 z + A_{2,x} - A_1, \qquad \epsilon_1 = -1, \qquad m = 0, \tag{2.5.16}$$

in the equation (2.2.11). As in the previous example, we will use the Riccati system (2.5.12) to show that the whole class (2.5.15) can be "linearized" by using a differential substitution.

Indeed, in view of (2.5.16), by using the corresponding f_{ij}, the Riccati system (2.5.12) takes the following form

$$\begin{cases} \Gamma_{,x} = \eta\Gamma^2 - z\Gamma, \\ \Gamma_{,t} = \eta\left(A_2 z - A_{2,x} + A_1\right)\Gamma^2 + \left(A_2 z_1 + A_2 z^2 + A_1 z + A_{1,x} - A_{2,xx}\right)\Gamma, \end{cases}$$

and assuming that $\Gamma \neq 0$ it can be rewritten as

$$\begin{cases} z = \frac{\Gamma_{,x}}{\Gamma} - \eta\Gamma, \\ \Gamma_t = A_2\Gamma_{,xx} - 2\eta A_2\Gamma\Gamma_{,x} + A_1\Gamma_{,x} - \eta A_{2,x}\Gamma^2 + \left(A_{1,x} - A_{2,xx}\right)\Gamma. \end{cases} \tag{2.5.17}$$

Thus, since for $\eta = 0$ the system (2.5.17) reduces to

$$\begin{cases} z = \frac{\Gamma_{,x}}{\Gamma}, \\ \Gamma_t = A_2\Gamma_{,xx} + A_1\Gamma_{,x} + \left(A_{1,x} - A_{2,xx}\right)\Gamma, \end{cases}$$

it follows that the non-vanishing solutions of the linear equation

$$\Gamma_t = A_2\Gamma_{,xx} + A_1\Gamma_{,x} + \left(A_{1,x} - A_{2,xx}\right)\Gamma,$$

are transformed by means of $z = \frac{\Gamma_{,x}}{\Gamma}$ to solutions of the nonlinear equations (2.5.15).

More in general, it is not difficult to prove that the class of equations

$$z_t = A_2 z_2 + \left(A_1 + 2A_2 z + A_{2,x}\right) z_1 + A_{2,x} z^2 + A_{1,x} z + A_{0,x}, \tag{2.5.18}$$

where $A_0 = A_0(x,t)$, $A_1 = A_1(x,t)$ and $A_2 = A_2(x,t) \neq 0$ are arbitrary differentiable functions, is the most general class of equations of the form $z_t = F(x,t,z,z_1,z_2)$ which can be "linearized", in the above sense, to

$$\Gamma_{,t} = A_2\Gamma_{,xx} + A_1\Gamma_{,x} + A_0\Gamma,$$

by means of the "Cole-Hopf" differential substitution

$$z = \frac{\Gamma_{,x}}{\Gamma}.$$

Equations (2.5.15) are just obtained from (2.5.18) by choosing $A_0 = -A_{2,xx} + A_{1,x}$.

Remark 2.5.12. According to the convention introduced by Calogero in [10], equations like Burgers and those considered in the Examples 2.5.9, 2.5.10 and 2.5.11 are called C-integrable. On the other hand, the type of equations considered by Svinolupov and Sokolov in [43] is sometimes referred to as symmetry-integrable. It is noteworthy to remark here that these two notions of integrability are not coincident. Indeed one has examples of equations, like Burgers equation, which are both C-integrable and symmetry-integrable. However it is easy to find examples of equations which are integrable in a sense but not in the other. For instance, equation (2.5.10) is C-integrable but it is neither linearizable by contact transformations (Indeed Cole-Hopf transformation is not a contact transformation) nor equivalent to one of the four equations (2.5.1). Indeed, the algebra of classical symmetries of (2.5.10) is 3-dimensional and hence (2.5.10) cannot be contact equivalent to a linear equation. On the other hand, it can also be shown that none of the four equations (2.5.1) is contact equivalent to (2.5.10). Indeed

$$\bar{x} = f(x,t), \qquad \bar{t} = g(t), \qquad \bar{z} = h(x,t,z). \tag{2.5.19}$$

is the most general contact transformation which leave invariant the class of evolution equations

$$\bar{z}_{\bar{t}} = a_3(\bar{x},\bar{t},\bar{z})\bar{z}_{\bar{x}\bar{x}} + a_2(\bar{x},\bar{t},\bar{z})\bar{z}_{\bar{x}}^2 + a_1(\bar{x},\bar{t},\bar{z})\bar{z}_{\bar{x}} + a_0(\bar{x},\bar{t},\bar{z}), \tag{2.5.20}$$

where $\bar{z}_{\bar{t}}$, $\bar{z}_{\bar{x}}$ $\bar{z}_{\bar{x}\bar{x}}$ are partial derivatives of $\bar{z} = \bar{z}(\bar{x},\bar{t})$ and a_i are arbitrary differentiable functions such that $\bar{a}_3 \neq 0$. Hence, under transformations (2.5.19), any equation of the form (2.5.20) is mapped to

$$\begin{aligned} z_t = {} & \frac{g' a_3}{f_{,x}^2} z_2 + \frac{g'\, f_{,x} \left[h_{,z}^2 a_2 + h_{,zz}\, a_3\right]}{f_{,x}^3 h_{,z}} z_1^2 \\ & + \frac{\left[h_{,z}\, f_{,x}^2 g'\, a_1 + 2g'\, h_{,x}\, h_{,z}\, f_{,x}\, a_2 + (2g'\, h_{,xz}\, f_{,x} - g'\, h_{,z}\, f_{,xx})\, a_3 + h_{,z}\, f_{,t}\, f_{,x}^2\right]}{f_{,x}^3 h_{,z}} z_1 \\ & - \frac{\left[g'\, f_{,x}^3 - g'\, h_{,x}\, f_{,x}^2 a_1 - g'\, h_{,x}^2 f_{,x}\, a_2 + (g'\, h_{,x}\, f_{,xx} - g'\, h_{,xx}\, f_{,x})\, a_3 + h_{,t}\, f_x^3 - h_{,x}\, f_{,t}\, f_{,x}^2\right]}{f_{,x}^3 h_{,z}}. \end{aligned} \tag{2.5.21}$$

In view of (2.5.21) it is not difficult to check that, by means of a contact transformation, none of the four equations (2.5.1) can be transformed to (2.5.10).

Chapter 3

Finite-order local isometric immersions of pseudospherical surfaces described by second order evolution PS equations and generalizations

In this chapter we consider the problem of existence of local isometric immersions, into the 3-dimensional Euclidean space $\mathbf{E}^3$, for the families of pseudospherical surfaces described by PS equations classified in the Chapter 2. We will show that only Type I equations admit such a kind of immersion and, on the base of this result, we also provide an extension of the results to the case of k-th order evolution equations in the conservation law form $D_t\left(f(x,t,z)\right) = D_x\left(\Omega(x,t,z,z_1,\ldots,z_k)\right)$. The examples discussed in the end of this chapter include second order equations as Boltzmann, Murray and Svinolupov-Sokolov equations, as well as higher order equations like Kuramoto-Sivashinsky, Sawada-Kotera and Kaup-Kupershmidt equations, and also full hierarchies of integrable equations like Burgers, mKdV and KdV, which were not covered by the results of previous papers [32, 33].

The chapter is organized as follows. In Section 3.1 we state the Theorem 3.1.1 and Theorem 3.1.2, which are the main results of the chapter, and in Section 3.2 we give detailed proofs of these theorems. Finally, in Section 3.3 we illustrate these results by means of some examples.

3.1 Main results

The chapter is mainly concerned with the following question:

Do finite-order local isometric immersions exist for the family of pseudospherical surfaces described by the evolution second order PS equations of Theorem 2.2.1?

The answer to this question is provided by Theorem 3.1.1, which is the main result of the present chapter and is stated below. According to this theorem such an immersion only exists for equations of Type I.

Theorem 3.1.1. *For second order PS equations classified by Theorem 2.2.1, there exists no finite-order local isometric immersions for the families of pseudospherical surfaces described by Types II and III, whereas for those described by Type I such an immersion exists if, and only if, there are constants γ, $\zeta \in \mathbb{R}$, $\gamma \neq 0$, $\zeta > 0$, $\zeta^2 - 4\gamma^2 > 0$ such that:*

(i) *for Type I (a) the generic solutions z and associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$, given by (2.2.4), are defined on a strip of $\mathbb{R}^2$ of the form*

$$log\sqrt{\frac{\zeta - \sqrt{\zeta^2 - 4\gamma^2}}{2\gamma^2}} < \epsilon\,(\eta x + g) < log\sqrt{\frac{\zeta + \sqrt{\zeta^2 - 4\gamma^2}}{2\gamma^2}}, \tag{3.1.1}$$

and the functions a, b, c appearing in (1.3.1) are given by

$$\begin{aligned} a &= \nu\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1}, \\ b &= \gamma e^{2\epsilon(\eta x+g)}, \\ c &= \tfrac{b^2-1}{a} = \nu\tfrac{\gamma^2 e^{4\epsilon(\eta x+g)}-1}{\sqrt{\zeta e^{2\epsilon(\eta x+g)}-\gamma^2 e^{4\epsilon(\eta x+g)}-1}}, \end{aligned} \tag{3.1.2}$$

with $\nu = \pm 1$;

(ii) *for Type I (b) the generic solutions z and associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$, given by (2.2.5), are defined on a strip of $\mathbb{R}^2$ of the form*

$$arccosh\left(\sqrt{\frac{\zeta - \sqrt{\zeta^2 - 4\gamma^2}}{2}}\right) < \eta x + g < arccosh\left(\sqrt{\frac{\zeta + \sqrt{\zeta^2 - 4\gamma^2}}{2}}\right), \tag{3.1.3}$$

and the functions a, b, c appearing in (1.3.1) are given by

$$\begin{aligned} a &= \nu\tfrac{\sqrt{\zeta\, cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}}{cosh^2(\eta x+g)}, \\ b &= \tfrac{\gamma}{cosh^2(\eta x+g)}, \\ c &= \tfrac{b^2-1}{a} = \nu\tfrac{\gamma^2 - cosh^4(\eta x+g)}{cosh^2(\eta x+g)\sqrt{\zeta\, cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}}, \end{aligned} \tag{3.1.4}$$

with $\nu = \pm 1$.

On the other hand since Type I equations can be written in conservation law form, like the k-th order equations described by Theorem 1.2.2, the answer provided by Theorem 3.1.1 naturally led us to the following second question.

Do finite-order local isometric immersions exist for the family of pseudospherical surfaces described by the evolution k-th order PS equations of Theorem 1.2.2?

The answer to this second question is provided by the following

Theorem 3.1.2. *Finite-order local isometric immersions for the families of pseudospherical surfaces described by k-th order PS equations of Theorem 1.2.2 exist if, and only if, there are constants* γ, $\zeta \in \mathbb{R}$, $\gamma \neq 0$, $\zeta > 0$, $\zeta^2 - 4\gamma^2 > 0$ *such that:*

(i) *for type (a) the generic solutions z and associated 1-forms* $\omega_i = f_{i1}\, dx + f_{i2}\, dt$, *given by (1.2.10), are defined on a strip of* $\mathbb{R}^2$ *of the form*

$$log\sqrt{\frac{\zeta - \sqrt{\zeta^2 - 4\gamma^2}}{2\gamma^2}} < \epsilon\,(\eta x + g) < log\sqrt{\frac{\zeta + \sqrt{\zeta^2 - 4\gamma^2}}{2\gamma^2}}, \tag{3.1.5}$$

and the functions a, b, c appearing in (1.3.1) are given by

$$\begin{aligned} a &= \nu\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1},\\ b &= \gamma e^{2\epsilon(\eta x+g)},\\ c &= \tfrac{b^2-1}{a} = \nu\tfrac{\gamma^2 e^{4\epsilon(\eta x+g)}-1}{\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1}}, \end{aligned} \tag{3.1.6}$$

with $\nu = \pm 1$*;*

(ii) *for type (b) the generic solutions z and associated 1-forms* $\omega_i = f_{i1}\, dx + f_{i2}\, dt$, *given by (1.2.11), are defined on a strip of* $\mathbb{R}^2$ *of the form*

$$arccosh\left(\sqrt{\frac{\zeta - \sqrt{\zeta^2 - 4\gamma^2}}{2}}\right) < \eta x + g < arccosh\left(\sqrt{\frac{\zeta + \sqrt{\zeta^2 - 4\gamma^2}}{2}}\right), \tag{3.1.7}$$

and the functions a, b, c appearing in (1.3.1) are given by

$$\begin{aligned} a &= \nu\tfrac{\sqrt{\zeta\, cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}}{cosh^2(\eta x+g)},\\ b &= \tfrac{\gamma}{cosh^2(\eta x+g)},\\ c &= \tfrac{b^2-1}{a} = \nu\tfrac{\gamma^2 - cosh^4(\eta x+g)}{cosh^2(\eta x+g)\sqrt{\zeta\, cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}}, \end{aligned} \tag{3.1.8}$$

with $\nu = \pm 1$.

The proofs of Theorem 3.1.1 and the Theorem 3.1.2 are presented in Section 3.2.2 and Section 3.2.3, respectively.

3.2 Proofs of the main results

3.2.1 Auxiliary lemmas

We begin with the following

Lemma 3.2.1. *If $z_t = F(x,t,z,z_1,\dots,z_k)$ is a k-th order PS equation with associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$, $1 \le i \le 3$, depending on $(x,t,z,z_1,\dots,z_k)$ then*

$$f_{i1} = f_{i1}(x,t,z), \qquad f_{i2} = f_{i2}(x,t,z,z_1,\dots,z_{k-1}). \tag{3.2.1}$$

In particular if $f_{11} = f_{11}(x,t)$ and $f_{21} = \eta$, then

$$f_{12,z_{k-1}} = f_{22,z_{k-1}} = 0, \tag{3.2.2}$$

$$f_{31,z} \neq 0, \tag{3.2.3}$$

$$f_{32,z_{k-1}} \neq 0. \tag{3.2.4}$$

Proof. In view of Theorem 2.1.1 one has (3.2.1). On the other hand, by assuming that $f_{11} = f_{11}(x,t)$ and $f_{21} = \eta$, one can rewrite structure equations (1.2.1) as

$$\begin{aligned}
&f_{12,x} + f_{12,z}z_1 + \cdots + f_{12,z_{k-1}}z_k + \eta f_{32} - f_{31}f_{22} - f_{11,t} = 0,\\
&f_{22,x} + f_{22,z}z_1 + \cdots + f_{22,z_{k-1}}z_k + f_{12}f_{31} - f_{11}f_{32} = 0,\\
&f_{32,x} + f_{32,z}z_1 + \cdots + f_{32,z_{k-1}}z_k + \eta f_{12} - f_{11}f_{22} - f_{31,t} - f_{31,z}F = 0.
\end{aligned} \tag{3.2.5}$$

Then (3.2.2) follows by deriving first two equations of (3.2.5) with respect to z_k. On the other hand, equations (3.2.3) and (3.2.4) easily follow deriving the third equation of (3.2.5) with respect to z_k and by the non-degeneracy condition (2.1.2). □

The following lemma is an analogue of the main result of the paper [32] and will facilitate the proofs of Theorem 3.1.1 and Theorem 3.1.2, which are provided in Subsections 3.2.2 and 3.2.3, respectively.

Lemma 3.2.2. *Let $z_t = F(x,t,z,z_1,\dots,z_k)$ be a k-th order PS equation with $k \ge 2$ and associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$, $1 \le i \le 3$, depending on $(x,t,z,z_1,\dots,z_k)$ and satisfying $f_{21} = \eta$. If there exists a finite-order local isometric immersion for the pseudospherical surfaces described by solutions $z = z(x,t)$ of this PS equation, then the functions a, b and c, defined in (1.1.7), depend only on x and t.*

Proof. If the coefficients of the second fundamental form (1.3.1) depend on finite-order jet prolongations of solutions z, and the functions f_{ij} only depend on $(x, t, z, z_1, \dots, z_k)$, then a, b and c may depend only on x, t, z, z_1, ..., z_l, where l is a fixed positive integer. Hence, (1.3.2) rewrites as

$$\begin{cases} f_{11}a_{,t} + \eta b_{,t} - f_{12}a_{,x} - f_{22}b_{,x} - 2b\left(f_{11}f_{32} - f_{31}f_{12}\right) + (a-c)\left(\eta f_{32} - f_{31}f_{22}\right) \\ -\sum_{i=0}^{l}\left(f_{12}a_{,z_i} + f_{22}b_{,z_i}\right)z_{i+1} + \sum_{i=0}^{l}\left(f_{11}a_{,z_i} + \eta b_{,z_i}\right)z_{i,t} = 0, \\ \\ f_{11}b_{,t} + \eta c_{,t} - f_{12}b_{,x} - f_{22}c_{,x} + (a-c)\left(f_{11}f_{32} - f_{31}f_{12}\right) + 2b\left(\eta f_{32} - f_{31}f_{22}\right) \\ -\sum_{i=0}^{l}\left(f_{12}b_{,z_i} + f_{22}c_{,z_i}\right)z_{i+1} + \sum_{i=0}^{l}\left(f_{11}b_{,z_i} + \eta c_{,z_i}\right)z_{i,t} = 0. \end{cases} \tag{3.2.6}$$

We will prove the lemma by distinguishing the two cases: $\eta = 0$ and $\eta \neq 0$.

Case $\eta = 0$. In this case, the non-degeneracy condition $\omega_1 \wedge \omega_2 \neq 0$ rewrites as $f_{11}f_{22} \neq 0$. Hence, since $z_t = F$ is a k-th order equation, by deriving both equations of (3.2.6) with respect to z_{l+k}, one obtains

$$a_{,z_l} = 0, \qquad b_{,z_l} = 0, \tag{3.2.7}$$

and in view of Gauss equation (1.3.3) one has that

$$ac_{,z_l} = 0. \tag{3.2.8}$$

Thus when $a \neq 0$, in view of (3.2.8), $c_{,z_l} = 0$ and by successive differentiating equations (3.2.6) with respect to z_{i+k}, for $i = 0, ..., l-1$, one has that $a_{,z_i} = b_{,z_i} = c_{,z_i} = 0$.

On the contrary, when $a = 0$, then Gauss equation leads to $b = \epsilon = \pm 1$ and (3.2.6) becomes

$$\begin{cases} f_{31}f_{22}c - 2\epsilon\left(f_{11}f_{32} - f_{31}f_{12}\right) = 0, \\ f_{22}c_{,x} + c\left(f_{11}f_{32} - f_{31}f_{12}\right) + 2\epsilon f_{31}f_{22} + \sum_{i=0}^{l} f_{22}c_{,z_i}z_{i+1} = 0. \end{cases} \tag{3.2.9}$$

Hence, in (3.2.9), by substituting the expression of $f_{11}f_{32} - f_{31}f_{12}$ obtained from the first equation into the second equation one gets

$$c_{,x} + \sum_{i=0}^{l} c_{,z_i}z_{i+1} + \frac{\epsilon c^2 f_{31}}{2} + 2\epsilon f_{31} = 0,$$

and in view of Lemma 3.2.1, by means of successive differentiations with respect to z_{i+1}, for $i = 0, ..., l$, one gets that $c_{,z_i} = 0$.

Hence when $\eta = 0$, one has

$$a_{,z_i} = b_{,z_i} = c_{,z_i} = 0, \qquad i = 0, 1, \ldots, l.$$

Case $\eta \neq 0$. In view of (1.2.8), by deriving both equations of (3.2.6) with respect to z_{l+k}, one obtains

$$b_{,z_l} = -\frac{f_{11}}{\eta} a_{,z_l}, \qquad c_{,z_l} = \frac{f_{11}^2}{\eta^2} a_{,z_l}, \tag{3.2.10}$$

and the derivative of the Gauss equation (1.3.3) with respect to z_l returns

$$\left(c + \frac{a f_{11}^2}{\eta^2} + \frac{2 b f_{11}}{\eta} \right) a_{,z_l} = 0. \tag{3.2.11}$$

Now we will proceed by further distinguishing the two subcases:

(i) $a_{,z_l} = 0$, $c + \frac{a f_{11}^2}{\eta^2} + \frac{2 b f_{11}}{\eta} \neq 0$;

(ii) $c + \frac{a f_{11}^2}{\eta^2} + \frac{2 b f_{11}}{\eta} = 0$.

Subcase (i). In view of (3.2.10),

$$a_{,z_l} = b_{,z_l} = c_{,z_l} = 0, \tag{3.2.12}$$

and by substituting (3.2.12) in (3.2.6) one readily gets the following analogous of (3.2.10) and (3.2.11):

$$b_{,z_{l-1}} = -\frac{f_{11}}{\eta} a_{,z_{l-1}}, \qquad c_{,z_{l-1}} = \frac{f_{11}^2}{\eta^2} a_{,z_{l-1}}, \tag{3.2.13}$$

and

$$\left(c + \frac{a f_{11}^2}{\eta^2} + \frac{2 b f_{11}}{\eta} \right) a_{,z_{l-1}} = 0. \tag{3.2.14}$$

Hence in view of $c + \frac{a f_{11}^2}{\eta^2} + \frac{2 b f_{11}}{\eta} \neq 0$ one also obtains that

$$a_{,z_{l-1}} = b_{,z_{l-1}} = c_{,z_{l-1}} = 0. \tag{3.2.15}$$

Thus the desired result easily follows by observing that iterating above procedure one would get that

$$a_{,z_i} = b_{,z_i} = c_{,z_i} = 0, \qquad i = 0, 1, \ldots, l.$$

Subcase (ii). If $c + \frac{a f_{11}^2}{\eta^2} + \frac{2 b f_{11}}{\eta} = 0$, then by substituting $c = -\frac{a f_{11}^2}{\eta^2} - \frac{2 b f_{11}}{\eta}$ into the Gauss equation, one gets

$$b = \nu - \frac{a f_{11}}{\eta}, \tag{3.2.16}$$

where $\nu = \pm 1$. Hence
$$c = \frac{af_{11}^2}{\eta^2} - \frac{2\nu f_{11}}{\eta}, \tag{3.2.17}$$
and the following identities hold,
$$\begin{array}{ll} D_t b = -\frac{f_{11}D_t a}{\eta} - \frac{aD_t(f_{11})}{\eta}, & D_t c = \frac{f_{11}^2 D_t a}{\eta^2} + \frac{2}{\eta}\left(\frac{af_{11}}{\eta} - \nu\right) D_t\left(f_{11}\right), \\ D_x b = -\frac{f_{11}D_x a}{\eta} - \frac{aD_x(f_{11})}{\eta}, & D_x c = \frac{f_{11}^2 D_x a}{\eta^2} + \frac{2}{\eta}\left(\frac{af_{11}}{\eta} - \nu\right) D_x\left(f_{11}\right), \end{array} \tag{3.2.18}$$
where D_x and D_t are the total derivative operators. Then, by using (3.2.18), equations (1.3.2) rewrite as
$$\begin{cases} -af_{11,t} - af_{11,z}F + \frac{\Delta_{12}a_{,x}}{\eta} + \frac{\Delta_{12}}{\eta}\sum_{i=0}^{l} a_{,z_i}z_{i+1} + \dfrac{af_{22}D_x\left(f_{11}\right)}{\eta} \\ -2b\left(f_{11}f_{32} - f_{31}f_{12}\right) + (a-c)\left(\eta f_{32} - f_{31}f_{22}\right) = 0, \\ \\ \left(\frac{af_{11}}{\eta} - 2\nu\right) f_{11,t} + \left(\frac{af_{11}}{\eta} - 2\nu\right) f_{11,z}F - \frac{f_{11}}{\eta}\frac{\Delta_{12}a_{,x}}{\eta} - \frac{f_{11}}{\eta}\frac{\Delta_{12}}{\eta}\sum_{i=0}^{l} a_{,z_i}z_{i+1} \\ -\left[\frac{\Delta_{12}a}{\eta^2} + \frac{f_{22}}{\eta}\left(\frac{af_{11}}{\eta} - 2\nu\right)\right] D_x\left(f_{11}\right) + (a-c)\left(f_{11}f_{32} - f_{31}f_{12}\right) \\ +2b\left(\eta f_{32} - f_{31}f_{22}\right) = 0, \end{cases} \tag{3.2.19}$$
where $\Delta_{12} := f_{11}f_{22} - \eta f_{12} \neq 0$ in view of the non-degeneracy condition $\omega_1 \wedge \omega_2 \neq 0$.

Now to prove that in the current subcase a, b, c do not depend on $(z, z_1, \ldots, z_l)$ we analyze separately the cases $l \geq k$, $l = k-1$ and $l \leq k-2$.

When $l \geq k$, by deriving (3.2.19) with respect to z_{i+1}, $i = k, \ldots, l$, one gets that $\Delta_{12}a_{,z_i} = 0$. Therefore an argument similar to that used in the analysis of subcase (i), shows that
$$a_{,z_i} = b_{,z_i} = c_{,z_i} = 0, \qquad \forall\, i = k, \ldots, l. \tag{3.2.20}$$

When $l = k-1$, by deriving (3.2.19) with respect to z_k, one gets that
$$\begin{cases} -af_{11,z}F_{,z_k} + \frac{\Delta_{12}}{\eta}a_{,z_{k-1}} = 0, \\ \left(\frac{af_{11}}{\eta} - 2\nu\right) f_{11,z}F_{,z_k} - \frac{f_{11}}{\eta}\frac{\Delta_{12}}{\eta}a_{,z_{k-1}} = 0, \end{cases} \tag{3.2.21}$$
which easily leads to $a_{,z_{k-1}} = 0$ and hence, in view of (3.2.10), to
$$a_{,z_{k-1}} = b_{,z_{k-1}} = c_{,z_{k-1}} = 0. \tag{3.2.22}$$

Hence in view of (3.2.20) and (3.2.22) the jet-order of a, b and c cannot exceed $k-1$. However we will prove now that a, b, c may only depend on (x, t). Indeed, when $l \leq k-2$, by deriving (3.2.19) with respect to z_k, one gets that

$$\begin{cases} -af_{11,z}F_{,z_k}=0, \\ \left(\frac{af_{11}}{\eta}-2\nu\right)f_{11,z}F_{,z_k}=0, \end{cases}$$

which easily leads to $\nu f_{11,z}F_{,z_k}=0$ and hence to $f_{11}=f_{11}(x,t)$, since $z_t=F$ is by assumption a k-th order equation. Therefore in such a case equations (3.2.19) reduce to

$$\begin{cases} -af_{11,t}+\frac{\Delta_{12}}{\eta}a_{,x}+\frac{\Delta_{12}}{\eta}\sum_{i=0}^{l}a_{,z_i}z_{i+1}+\frac{af_{11,x}f_{22}}{\eta}-2b\left(f_{11}f_{32}-f_{31}f_{12}\right) \\ +(a-c)\left(\eta f_{32}-f_{31}f_{22}\right)=0, \\ \left(\frac{af_{11}}{\eta}-2\nu\right)f_{11,t}-\frac{f_{11}\Delta_{12}}{\eta^2}a_{,x}-\frac{f_{11}\Delta_{12}}{\eta^2}\sum_{i=0}^{l}a_{,z_i}z_{i+1} \\ +\left[\frac{\Delta_{12}a}{\eta^2}+\frac{f_{22}}{\eta}\left(\frac{af_{11}}{\eta}-2\nu\right)\right]f_{11,x}+(a-c)\left(f_{11}f_{32}-f_{31}f_{12}\right) \\ +2b\left(\eta f_{32}-f_{31}f_{22}\right)=0, \end{cases} \tag{3.2.23}$$

and in view of Lemma 3.2.1, conditions (3.2.2), (3.2.3) and (3.2.4) must be satisfied.

In particular, if $l=k-2$, by deriving (3.2.23) with respect to z_{k-1} one has

$$\begin{cases} \frac{\Delta_{12}a_{,z_{k-2}}}{\eta}-2bf_{11}f_{32,z_{k-1}}+(a-c)\,\eta f_{32,z_{k-1}}=0, \\ -\frac{f_{11}}{\eta}\left[\frac{\Delta_{12}a_{,z_{k-2}}}{\eta}-\eta(a-c)\,f_{32,z_{k-1}}-\frac{2\eta^2bf_{32,z_{k-1}}}{f_{11}}\right]=0, \end{cases} \tag{3.2.24}$$

where $f_{11}\neq 0$ in view of (3.2.4). In particular, by comparing first and second equation of (3.2.24), one gets

$$\eta f_{11}(a-c)=\left(f_{11}^2-\eta^2\right)b, \tag{3.2.25}$$

where $f_{11}^2-\eta^2\neq 0$, since otherwise $a-c=0$ and by (3.2.17) one would get $f_{11}=0$. Then, by substituting (3.2.16-3.2.17) into (3.2.25) one obtains $f_{11}^2+\eta^2=0$ which contradicts $\eta\neq 0$ and $f_{11}\neq 0$.

On the other hand, if $l<k-2$ then by deriving (3.2.23) with respect to z_{k-1} and using (3.2.4), one gets the system

$$\begin{pmatrix} -f_{11} & \eta \\ -\eta & f_{11} \end{pmatrix}\begin{pmatrix} 2b \\ a-c \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which in view of $f_{11}^2+\eta^2\neq 0$ immediately leads to $b=0$ and $a=c$, which contradicts the Gauss equation (1.3.3). $\square$

3.2.2 Proof of Theorem 3.1.1

In the proof of Theorem 3.1.1 we will analyze separately equations of Type I and equations of Types II-III.

3.2.2.1 Existence of finite-order local isometric immersions for Type I equations

To prove that equations of Type I admit finite-order local isometric immersions, we will distinguish between Type I (a) and Type I (b).

Type I (a). In view of Lemma 3.2.2 and (2.2.4), equations (1.3.2) reduce to

$$\begin{cases} -\varphi e^{-\epsilon(\eta x+g)}\left[a_{,x}-\epsilon\left(a-c\right)\eta\right]z_1-\psi e^{-\epsilon(\eta x+g)}\left[a_{,x}-\epsilon\left(a-c\right)\eta\right] \\ +\eta b_{,t}-g'b_{,x}+e^{-\epsilon(\eta x+g)}f\left[a_{,t}-\epsilon g'\left(a-c\right)\right]=0, \\ -\varphi e^{-\epsilon(\eta x+g)}\left[b_{,x}-2\epsilon\eta b\right]z_1-\psi e^{-\epsilon(\eta x+g)}\left[b_{,x}-2\epsilon\eta b\right] \\ +\eta c_{,t}-g'c_{,x}+e^{-\epsilon(\eta x+g)}f\left(b_{,t}-2\epsilon g'b\right)=0. \end{cases} \tag{3.2.26}$$

Hence, in view of the independence of φ, ψ, f, g and a, b, c on z_1, (3.2.26) splits into the following two systems

$$\begin{cases} a_{,x}-\epsilon\left(a-c\right)\eta=0, \\ b_{,x}-2\epsilon\eta b=0, \end{cases} \tag{3.2.27}$$

and

$$\begin{cases} e^{-\epsilon(\eta x+g)}f\left[a_{,t}-\epsilon g'\left(a-c\right)\right]+\eta b_{,t}-g'b_{,x}=0, \\ e^{-\epsilon(\eta x+g)}f\left(b_{,t}-2\epsilon g'b\right)+\eta c_{,t}-g'c_{,x}=0. \end{cases} \tag{3.2.28}$$

In its turn, in view of $f_{,z}\neq 0$ and the independence of g, a, b, c on z, the system (3.2.28) splits into the following system

$$\begin{cases} a_{,t}-\epsilon g'\left(a-c\right)=0, \\ b_{,t}-2\epsilon g'b=0, \\ \eta b_{,t}-g'b_{,x}=0, \\ \eta c_{,t}-g'c_{,x}=0. \end{cases} \tag{3.2.29}$$

Then, from the second equation of (3.2.27) and second equation of (3.2.29), one gets the expression of b given by (3.1.2). In particular the third equation of (3.2.29) is automatically satisfied.

On the other hand, in view of $\eta^2+\left(g'\right)^2\neq 0$, from the first equations of (3.2.27) and (3.2.29) one has that $a \neq 0$ and from the Gauss equation one gets

$$c = \frac{b^2 - 1}{a}. \tag{3.2.30}$$

Then in view of (3.2.30), first equations of (3.2.27) and (3.2.29) rewrite as

$$\begin{cases} aa_{,x} - \epsilon\eta\left[a^2 - \gamma^2 e^{4\epsilon(\eta x+g)} + 1\right] = 0, \\ aa_{,t} - \epsilon g'\left[a^2 - \gamma^2 e^{4\epsilon(\eta x+g)} + 1\right] = 0, \end{cases} \tag{3.2.31}$$

and can be readily integrated in the form

$$a^2 = \zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1, \quad \zeta \in \mathbb{R}, \tag{3.2.32}$$

by using the integrating factor $e^{-2\epsilon\eta x}$. Thus (3.2.32) entails that

$$a = \nu\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1},$$

which is defined whenever $\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1 > 0$. Therefore $\zeta > 0$ and

$$\frac{\zeta - \sqrt{\zeta^2 - 4\gamma^2}}{2\gamma^2} < e^{2\epsilon(\eta x+g)} < \frac{\zeta + \sqrt{\zeta^2 - 4\gamma^2}}{2\gamma^2},$$

i.e., a is defined on the strip described by (3.1.1). Finally, by substituting above results in (3.2.30) one gets the expression of c given in (3.1.2), and one can readily check that also the fourth equation of (3.2.29) is satisfied.

A straightforward computation shows that also the converse of the theorem holds for current type.

Type I (b). The proof is similar to that of Type I (a). In this case instead of (3.2.26) one has the following system

$$\begin{cases} -\varphi\left[cosh\left(\eta x+g\right)a_{,x} + \eta\, sinh\left(\eta x+g\right)(a-c)\right]z_1 + \eta b_{,t} - g' b_{,x} \\ -\psi\left[cosh\left(\eta x+g\right)a_{,x} + \eta\, sinh\left(\eta x+g\right)(a-c)\right] \\ +\left[cosh\left(\eta x+g\right)a_{,t} + sinh\left(\eta x+g\right)g'\left(a-c\right)\right]f = 0, \\ \\ -\varphi\left[cosh\left(\eta x+g\right)b_{,x} + 2\eta\, sinh\left(\eta x+g\right)\right]z_1 + \eta c_{,t} - g' c_{,x} \\ -\psi\left[cosh\left(\eta x+g\right)b_{,x} + 2\eta\, sinh\left(\eta x+g\right)\right] \\ +\left[cosh\left(\eta x+g\right)b_{,t} + 2\, sinh\left(\eta x+g\right)g' b\right]f = 0, \end{cases} \tag{3.2.33}$$

by which one obtains, instead of (3.2.27) and (3.2.28), the following two systems

$$\begin{cases} cosh\,(\eta x+g)\,a_{,x}+\eta\,sinh\,(\eta x+g)\,(a-c)=0, \\ cosh\,(\eta x+g)\,b_{,x}+2\eta\,sinh\,(\eta x+g)=0, \end{cases} \tag{3.2.34}$$

and

$$\begin{cases} \left[cosh\,(\eta x+g)\,a_{,t}+sinh\,(\eta x+g)\,g'\,(a-c)\right]f+\eta b_{,t}-g'b_{,x}=0, \\ \left[cosh\,(\eta x+g)\,b_{,t}+2\,sinh\,(\eta x+g)\,g'b\right]f+\eta c_{,t}-g'c_{,x}=0. \end{cases} \tag{3.2.35}$$

Hence, instead of (3.2.29), in this case one gets the system

$$\begin{cases} cosh\,(\eta x+g)\,a_{,t}+sinh\,(\eta x+g)\,g'\,(a-c)=0, \\ \eta b_{,t}-g'b_{,x}=0, \\ cosh\,(\eta x+g)\,b_{,t}+2\,sinh\,(\eta x+g)\,g'b=0, \\ \eta c_{,t}-g'c_{,x}=0, \end{cases} \tag{3.2.36}$$

and the proof runs as that of Type I (a). In particular one gets the expression of b given in (3.1.4), as well as that $c=\frac{b^2-1}{a}$. Moreover by integrating the following analogue of (3.2.31)

$$\begin{cases} aa_{,x}+\eta\,tanh\,(\eta x+g)\,a^2-\eta\,tanh\,(\eta x+g)\left[\frac{\gamma^2-cosh^4(\eta x+g)}{cosh^4(\eta x+g)}\right]=0, \\ aa_{,t}+g'tanh\,(\eta x+g)\,a^2-g'tanh\,(\eta x+g)\left[\frac{\gamma^2-cosh^4(\eta x+g)}{cosh^4(\eta x+g)}\right]=0, \end{cases} \tag{3.2.37}$$

by means of the integrating factor $e^{\int 2\eta\,tanh(\eta x+g)\,dx}=-cosh^2\,(\eta x+g)$, one gets

$$a^2=\frac{\zeta\,(t)\,cosh^2\,(\eta x+g)-cosh^4\,(\eta x+g)-\gamma^2}{cosh^4\,(\eta x+g)}, \qquad \zeta\in\mathbb{R}, \tag{3.2.38}$$

which is exactly the expression of a given (3.1.4) and is defined whenever

$$\zeta\,cosh^2\,(\eta x+g)-cosh^4\,(\eta x+g)-\gamma^2>0.$$

Therefore one has that $\zeta>0$ and

$$\frac{\zeta-\sqrt{\zeta^2-4\gamma^2}}{2}<cosh^2\,(\eta x+g)<\frac{\zeta+\sqrt{\zeta^2-4\gamma^2}}{2},$$

i.e., a is defined on the strip described by (3.1.3). By substituting above results for a and b into $c=\frac{b^2-1}{a}$ one obtains the expression of c given in (3.1.3) and one can readily prove that all equations of (3.2.34) and (3.2.36) are satisfied.

A straightforward computation shows that also the converse of the theorem holds for the current type.

3.2.2.2 Non-existence of finite-order local isometric immersions for Type II and III equations

To prove that equations of Type II and Type III do not admit finite-order local isometric immersions, we will separately analyze Type II (a), Type II (b) and Type III (c), whereas Type III (a) and Type III (b) will be analyzed almost simultaneously.

Type II (a). In view of Lemma 3.2.2 and (2.2.7), equations (1.3.2) reduce to

$$\begin{cases} [a_{,x} - \eta m\,(a-c)]\,\epsilon_1 g f_{,z} z_1 + \epsilon_1 \eta g \delta\,(a-c)\,f - 2\epsilon_1 g^2 b\delta - \epsilon_1 b_{,x} \int g^2 \delta\,dx \\ +\eta\,[b_{,t} + m\psi\,(a-c)] - a_{,x}\psi + \frac{g}{f}\left[a_{,t} - \epsilon_1 m\,(a-c) \int g^2\delta\,dx\right] = 0, \\ [b_{,x} - 2\eta m b]\,\epsilon_1 g f_{,z} z_1 + 2\epsilon_1 \eta g \delta b f + \epsilon_1 g^2 \delta\,(a-c) - \epsilon_1 c_{,x} \int g^2\delta\,dx \\ +\eta\,[c_{,t} + 2m\psi b] - b_{,x}\psi + \frac{g}{f}\left[b_{,t} - 2\epsilon_1 m b \int g^2\delta\,dx\right] = 0. \end{cases} \tag{3.2.39}$$

Hence, in view of the independence of δ, m, f, g, ψ and a, b, c on z_1, (3.2.39) splits into the following two systems

$$\begin{cases} a_{,x} - \eta m\,(a-c) = 0, \\ b_{,x} - 2\eta m b = 0, \end{cases} \tag{3.2.40}$$

and

$$\begin{cases} \epsilon_1 \eta g \delta\,(a-c)\,f^2 + \left(\eta b_{,t} - 2\epsilon_1 g^2 b\delta - \epsilon_1 b_{,x} \int g^2\delta\,dx\right) f \\ +g\left[a_{,t} - \epsilon_1 m\,(a-c) \int g^2\delta\,dx\right] = 0, \\ 2\epsilon_1 \eta g \delta b f^2 + \left[\epsilon_1 g^2 \delta\,(a-c) - \epsilon_1 c_{,x} \int g^2\delta\,dx + \eta c_{,t}\right] f \\ +g\left[b_{,t} - 2\epsilon_1 m b \int g^2\delta\,dx\right] = 0. \end{cases} \tag{3.2.41}$$

Then if $\eta \neq 0$, by deriving (3.2.41) twice with respect to z, one gets

$$b = 0, \qquad a = c,$$

which contradicts the Gauss equation (1.3.3).

On the other hand, if $\eta = 0$, then from (3.2.40) one gets $a_{,x} = b_{,x} = 0$ and (3.2.41) reduces to

$$\begin{cases} -2\epsilon_1 g^2 b\delta f + g\left[a_{,t} - \epsilon_1 m\,(a-c) \int g^2\delta\,dx\right] = 0, \\ \left[\epsilon_1 g^2 \delta\,(a-c) - \epsilon_1 c_{,x} \int g^2\delta\,dx\right] f + g\left[b_{,t} - 2\epsilon_1 m b \int g^2\delta\,dx\right] = 0. \end{cases} \tag{3.2.42}$$

Hence, by deriving the first equation of (3.2.42) with respect to z, one has that $b = 0$ and $a \neq 0$, because of Gauss equation (1.3.3). Also, by deriving the Gauss equation with respect to x, one gets $c_{,x} = 0$ and hence, in view of $f_{,z} \neq 0$, from the second equation of (3.2.42) one concludes that $a = c$, which contradicts the Gauss equation.

Type II (b). In view of Lemma 3.2.2 and (2.2.9), equations (1.3.2) reduce to

$$\begin{cases} \eta\epsilon_1 g f_{,z}(a-c)z_1 + \epsilon_1\eta g a_{,x} f - \epsilon_1\eta\left(2g^2 b + b_{,x}\int g^2\,dx\right) \\ +\eta\left[b_{,t} + \psi(a-c)\right] - \frac{\epsilon_1\eta g\int g^2\,dx}{f}(a-c) = 0, \\ \\ 2\eta\epsilon_1 g b f_{,z} z_1 + \epsilon_1\eta g b_{,x} f + \epsilon_1\eta g^2(a-c) - \epsilon_1\eta c_{,x}\int g^2\,dx \\ +\eta\left(c_{,t} + 2b\psi\right) - \frac{2\eta\epsilon_1 g b\int g^2\,dx}{f} = 0. \end{cases} \tag{3.2.43}$$

Hence, since $f_{11} = 0$, it follows that $\omega_1 \wedge \omega_2 \neq 0$ is equivalent to $\epsilon_1\eta^2 g f \neq 0$ and by deriving (3.2.43) with respect to z_1, one concludes that

$$b = 0, \qquad a = c,$$

which contradicts the Gauss equation (1.3.3).

Type III (a-b). In view of Lemma 3.2.2 and (2.2.12-2.2.14), equations (1.3.2) reduce to

$$\begin{cases} \left[a_{,x} - 2bh - \eta m(a-c)\right] g_{,z} z_1 + \left[a_{,x} - 2bh - \eta m(a-c)\right] g_{,x} \\ -\left[mh(a-c) - \frac{\delta\eta m(a-c) - \delta a_{,x}}{h}\right] fg + \left[a_{,t} + \frac{q\eta m(a-c) - q a_{,x}}{h}\right] f \\ +\left[\frac{h_{,x} a_{,x} - \eta m h_{,x}(a-c)}{h} - (a-c)h^2 - h b_{,x} - 2b h_{,x} + \eta\delta(a-c)\right] g \\ +\eta q(a-c) + \eta b_{,t} = 0, \\ \\ \left[b_{,x} - 2\eta m b + h(a-c)\right] g_{,z} z_1 + \left[b_{,x} - 2\eta m b + h(a-c)\right] g_{,x} \\ -\left[2mhb - \frac{2\eta m\delta h - \delta b_{,x}}{h}\right] fg + \left[b_{,t} + \frac{2\eta m q b - q b_{,x}}{h}\right] f \\ +\left[\frac{h_{,x} b_{,x} - 2\eta m b h_{,x}}{h} - 2bh^2 - c_{,x} h + h_{,x}(a-c) + 2\eta\delta b\right] g \\ +2\eta q b + \eta c_{,t} = 0. \end{cases} \tag{3.2.44}$$

Hence, in view of the independence of h, m, q, f, g and a, b, c on z_1, (3.2.44) splits into the following two systems

$$\begin{cases} a_{,x} = 2bh + \eta m(a-c), \\ b_{,x} = 2\eta m b - h(a-c), \end{cases} \tag{3.2.45}$$

and

$$\begin{cases} f a_{,t} + \eta b_{,t} + \left[mh\left(\frac{b^2-1}{a} - a\right) - 2\delta b\right] gf - 2qbf \\ + \left[\eta\delta\left(a + \frac{1-b^2}{a}\right) - 2\eta mhb\right] g + \eta q\left(a + \frac{1-b^2}{a}\right) = 0, \\ \frac{\eta\left(1-b^2\right)}{a^2} a_{,t} + \left(f + \frac{2\eta b}{a}\right) b_{,t} + \left[\delta\left(a + \frac{1-b^2}{a}\right) - 2mhb\right] gf \\ +q\left(a + \frac{1-b^2}{a}\right) f + \left[2\eta\delta b + \frac{\eta mh}{a}\left(\frac{2b^2-1-b^4}{a^2} - 1 - 3b^2\right)\right] g + 2b\eta q = 0, \end{cases} \tag{3.2.46}$$

where we have used the identity $c = \frac{b^2-1}{a}$. Indeed $a \neq 0$, since otherwise (3.2.45) reduces to

$$\begin{pmatrix} h & -\eta m \\ \eta m & h \end{pmatrix} \begin{pmatrix} 2b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $det \begin{pmatrix} h & -\eta m \\ \eta m & h \end{pmatrix} \neq 0$, and $b = c = a = 0$ contradicts Gauss equation. Therefore, by rewriting (3.2.46) as

$$\begin{cases} a_{,t} = \left[\frac{mh}{a}\left(a^2 - b^2 + 1\right) + 2\delta b\right] g + \frac{2qb}{a}, \\ b_{,t} = \left[2mhb - \frac{\delta}{a}\left(a^2 - b^2 + 1\right)\right] g - \frac{q}{a}\left(a^2 - b^2 + 1\right), \end{cases} \tag{3.2.47}$$

and deriving (3.2.47) with respect to z one gets

$$\begin{pmatrix} \delta & mh \\ mh & -\delta \end{pmatrix} \begin{pmatrix} 2ab \\ a^2 - b^2 + 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.2.48}$$

and

$$a_{,t} = \frac{2qb}{a}, \qquad b_{,t} = -\frac{q\left(a^2-b^2+1\right)}{a}. \tag{3.2.49}$$

In the case of Type III (a) one has that

$$-\left(\delta^2 + m^2h^2\right) = -\left[h^2\left(1 - m^2\right) + m^2h^2\right] = -h^2 \neq 0,$$

and hence (3.2.48) entails that $b = 0$ and $a^2 + 1 = 0$, which is a contradiction.

On the other hand, in the case of Type III (b) either $m^2 + \delta^2 \neq 0$ or $m = \delta = 0$, however in both cases equations (3.2.48) and (3.2.49) lead to a contradiction. Indeed, when $m^2 + \delta^2 \neq 0$, (3.2.48) entails that $b = a^2 + 1 = 0$. On the other hand, in view of (2.2.2), when $m = \delta = 0$ one also has $\eta = 0$ and hence from the compatibility of (3.2.45) and (3.2.49) one obtains

$$b\left(h_{,t} - q_{,x}\right) = 0, \tag{3.2.50}$$

where $h_{,t} - q_{,x} \neq 0$, otherwise by (2.2.15) one would get $g = 0$ and hence (2.2.11) would degenerate to a first-order equation. Thus, from (3.2.50) one has $b = 0$ and in view of $h \neq 0$, (3.2.49) and (3.2.45) one easily that $q = a - c = 0$, which contradicts the Gauss equation (1.3.3).

Type III (c). In view of Lemma 3.2.2 and (2.2.16), equations (1.3.2) reduce to

$$\begin{cases} [\eta (a-c) - 2hb]\, g_{,z} z_1 + [\eta (a-c) - 2hb]\, g_{,x} - gf \left[h (a-c) + \frac{\eta^2}{h} (a-c)\right] \\ +g \left[\eta a_{,x} - h b_{,x} - 2b h_{,x} + \frac{\eta h_{,x}}{h} (a-c)\right] \\ +\frac{\eta q f}{h} (a-c) + h a_{,t} + \eta b_{,t} - q a_{,x} = 0, \\ [h (a-c) + 2\eta b]\, g_{,z} z_1 + [h (a-c) + 2\eta b]\, g_{,x} - gf \left[2hb + 2\frac{\eta^2}{h} b\right] \\ +g \left[\eta b_{,x} - h c_{,x} + (a-c) h_{,x} + \frac{2\eta h_{,x}}{h} b\right] + \frac{2\eta q f}{h} b + h b_{,t} + \eta c_{,t} - q b_{,x} = 0. \end{cases} \tag{3.2.51}$$

Hence, in view of the independence of h, m, q, f and g on z_1, one readily gets that

$$\begin{pmatrix} -h & \eta \\ \eta & h \end{pmatrix} \begin{pmatrix} 2b \\ a-c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $det \begin{pmatrix} -h & \eta \\ \eta & h \end{pmatrix} \neq 0$, and hence that $b = a - c = 0$ which contradicts the Gauss equation (1.3.3).

3.2.3 Proof of Theorem 3.1.2

In the following proof of Theorem 3.1.2, we distinguish between the linear problems (a) and (b) provided by (1.2.10) and (1.2.11).

(a) In view of Lemma 3.2.2 and (1.2.10), equations (1.3.2) reduce to

$$\begin{cases} -\Omega e^{-\epsilon(\eta x + g)} \left[a_{,x} - \epsilon (a-c) \eta\right] + e^{-\epsilon(\eta x + g)} f \left[a_{,t} - \epsilon g' (a-c)\right] \\ +\eta b_{,t} - g' b_{,x} = 0, \\ -\Omega e^{-\epsilon(\eta x + g)} \left[b_{,x} - 2\epsilon \eta b\right] + e^{-\epsilon(\eta x + g)} f \left(b_{,t} - 2\epsilon g' b\right) + \eta c_{,t} - g' c_{,x} = 0, \end{cases} \tag{3.2.52}$$

where $\Omega_{,z_{k-1}} \neq 0$. Hence, in view of the independence of f, g and a, b, c on z_{k-1}, (3.2.52) splits into the following systems

$$\begin{cases} a_{,x} - \epsilon (a-c) \eta = 0, \\ b_{,x} - 2\epsilon \eta b = 0, \end{cases}$$

and

$$\begin{cases} e^{-\epsilon(\eta x+g)} f\left[a_{,t} - \epsilon g'\left(a-c\right)\right] + \eta b_{,t} - g' b_{,x} = 0, \\ e^{-\epsilon(\eta x+g)} f\left(b_{,t} - 2\epsilon g' b\right) + \eta c_{,t} - g' c_{,x} = 0. \end{cases}$$

The rest of the proof runs as that of Theorem 3.1.1, in the case of Type I (a).

(b) In view of Lemma 3.2.2 and (1.2.11), equations (1.3.2) reduce to

$$\begin{cases} -\Omega\left[cosh\left(\eta x+g\right) a_{,x} + \eta\, sinh\left(\eta x+g\right)\left(a-c\right)\right] + cosh\left(\eta x+g\right) a_{,t} \\ +\eta b_{,t} - g' b_{,x} + sinh\left(\eta x+g\right) g' f\left(a-c\right) = 0, \\ -\Omega\left[cosh\left(\eta x+g\right) b_{,x} + 2\eta\, sinh\left(\eta x+g\right)\right] + cosh\left(\eta x+g\right) b_{,t} \\ +\eta c_{,t} - g' c_{,x} + 2\, sinh\left(\eta x+g\right) g' f b = 0, \end{cases} \tag{3.2.53}$$

where $\Omega_{,z_{k-1}} \neq 0$. Hence, in view of the independence of f, g and a, b, c on z_{k-1}, (3.2.53) splits into the following systems

$$\begin{cases} cosh\left(\eta x+g\right) a_{,x} + \eta\, sinh\left(\eta x+g\right)\left(a-c\right) = 0, \\ cosh\left(\eta x+g\right) b_{,x} + 2\eta\, sinh\left(\eta x+g\right) = 0, \end{cases}$$

and

$$\begin{cases} cosh\left(\eta x+g\right) a_{,t} + \eta b_{,t} - g' b_{,x} + sinh\left(\eta x+g\right) g' f\left(a-c\right) = 0, \\ cosh\left(\eta x+g\right) b_{,t} + \eta c_{,t} - g' c_{,x} + 2\, sinh\left(\eta x+g\right) g' f b = 0. \end{cases}$$

The rest of the proof runs as that of Theorem 3.1.1, in the case of Type I (b).

3.3 Examples

Example 3.3.1. Boltzman equation

$$z_t = z z_2 + z_1^2, \tag{3.3.1}$$

is an example of Type I (a) and Type I (b).

For instance, by choosing 1-forms

$$\omega_1 = e^{-\epsilon(\eta x+g)}\left(z\, dx + z z_1\, dt\right), \quad \omega_2 = \eta\, dx + g'\, dt, \quad \omega_3 = \epsilon \omega_1,$$

equation (3.3.1) can be seen as a particular instance of Type I (a), described by (2.2.3), with $\epsilon = \pm 1$, $f = \varphi = z$, $\psi = 0$ and $g = g\left(t\right)$ an arbitrary differentiable function. In this case, equation (3.3.1) describes a family of pseudospherical surfaces with first fundamental form

$$I = \left[e^{-2\epsilon(\eta x+g)}z^2 + \eta^2\right]\,dx^2 + 2\left[e^{-2\epsilon(\eta x+g)}z^2 z_1 + \eta g'\right]\,dx\,dt + \left[e^{-2\epsilon(\eta x+g)}z^2 z_1^2 + (g')^2\right]\,dt^2,$$

and in view of Theorem 3.1.1, whenever the associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$ and the generic solutions z of (3.3.1) are defined on a strip of the form (3.1.1), such a family of pseudospherical surfaces admits a finite-order local isometric immersion with second fundamental form given by

$$II = a_{11}\,dx^2 + 2a_{12}\,dx\,dt + a_{22}dt^2,$$

where

$$\begin{aligned}
a_{11} &= \nu e^{-2\epsilon(\eta x+g)}z^2\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1} \\
&\quad + 2\eta\gamma e^{\epsilon(\eta x+g)}z + \nu\eta^2\frac{\gamma^2 e^{4\epsilon(\eta x+g)}-1}{\sqrt{\zeta e^{2\epsilon(\eta x+g)}-\gamma^2 e^{4\epsilon(\eta x+g)}-1}}, \\
a_{12} &= \nu e^{-2\epsilon(\eta x+g)}z^2 z_1\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1} \\
&\quad + \gamma e^{\epsilon(\eta x+g)}\left(g'z + \eta z z_1\right) + \nu\eta g'\frac{\gamma^2 e^{4\epsilon(\eta x+g)}-1}{\sqrt{\zeta e^{2\epsilon(\eta x+g)}-\gamma^2 e^{4\epsilon(\eta x+g)}-1}}, \\
a_{22} &= \nu e^{-2\epsilon(\eta x+g)}z^2 z_1^2\sqrt{\zeta e^{2\epsilon(\eta x+g)} - \gamma^2 e^{4\epsilon(\eta x+g)} - 1} \\
&\quad + 2\gamma e^{\epsilon(\eta x+g)}g' z z_1 + \left(g'\right)^2\nu\frac{\gamma^2 e^{4\epsilon(\eta x+g)}-1}{\sqrt{\zeta e^{2\epsilon(\eta x+g)}-\gamma^2 e^{4\epsilon(\eta x+g)}-1}}.
\end{aligned}$$

On the other hand, by choosing 1-forms

$$\omega_1 = cosh\,(\eta x + g)\,(z\,dx + z z_1\,dt)\,, \quad \omega_2 = \eta\,dx + g'\,dt, \quad \omega_3 = -tanh\,(\eta x + g)\,\omega_1,$$

equation (3.3.1) can be seen as a particular instance of Type I (b), described by (2.2.3), with $f = \varphi = z$, $\psi = 0$ and $g = g\,(t)$ an arbitrary differentiable function. In this case, equation (3.3.1) describes a family of pseudospherical surfaces with first fundamental form

$$\begin{aligned}
I &= \left[cosh^2\,(\eta x + g)\,z^2 + \eta^2\right]\,dx^2 + 2\left[cosh^2\,(\eta x + g)\,z^2 z_1 + \eta g'\right]\,dx\,dt \\
&\quad + \left[cosh^2\,(\eta x + g)\,z^2 z_1^2 + (g')^2\right]\,dt^2,
\end{aligned}$$

and in view of Theorem 3.1.1, whenever the associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$ and the generic solutions z are defined on a strip of the form (3.1.3), such a family of pseudospherical surfaces admits a finite-order local isometric immersion with second fundamental form given by

$$II = a_{11}\,dx^2 + 2a_{12}\,dx\,dt + a_{22}dt^2,$$

where

$$a_{11} = \nu z^2 \sqrt{\zeta cosh^2(\eta x + g) - cosh^4(\eta x + g) - \gamma^2}$$
$$+ \frac{2\gamma\eta z}{cosh(\eta x+g)} + \nu\eta^2 \frac{\gamma^2 - cosh^4(\eta x+g)}{cosh^2(\eta x+g)\sqrt{\zeta cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}},$$

$$a_{12} = \nu z^2 z_1 \sqrt{\zeta cosh^2(\eta x + g) - cosh^4(\eta x + g) - \gamma^2}$$
$$+ \frac{\gamma(g' z + \eta z z_1)}{cosh(\eta x+g)} + \eta g' \nu \frac{\gamma^2 - cosh^4(\eta x+g)}{cosh^2(\eta x+g)\sqrt{\zeta cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}},$$

$$a_{22} = \nu z z_1 \sqrt{\zeta cosh^2(\eta x + g) - cosh^4(\eta x + g) - \gamma^2}$$
$$+ \frac{2g'\gamma z z_1}{cosh(\eta x+g)} + (g')^2 \nu \frac{\gamma^2 - cosh^4(\eta x+g)}{cosh^2(\eta x+g)\sqrt{\zeta cosh^2(\eta x+g) - cosh^4(\eta x+g) - \gamma^2}}.$$

Example 3.3.2. Equation

$$z_t = x z_2 + 2(xz + 1) z_1 + z^2, \tag{3.3.2}$$

is an equation of Type I (a) and Type I (b), as well as of Type III (a).

For instance, if in Type I one chooses $f = z$, $\varphi = x$, $\psi = xz^2 + z$, one can interpret (3.3.2) as a particular instance of Type I (a), with associated 1-forms

$$\begin{aligned} \omega_1 &= e^{-\epsilon(\eta x+g)} \left[z\,dx + (xz_1 + xz^2 + z)\,dt\right], \\ \omega_2 &= \eta\,dx + g'\,dt, \\ \omega_3 &= \epsilon\omega_1, \end{aligned} \tag{3.3.3}$$

where $\epsilon = \pm 1$ and $g = g(t)$ is an arbitrary differentiable function. In this case, equation (3.3.2) describes a family of pseudospherical surfaces with first fundamental form $I = \omega_1^2 + \omega_2^2$ given by (3.3.3), and in view of Theorem 3.1.1, whenever the associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$ and the generic solutions z of (3.3.2) are defined on a strip of the form (3.1.1), such a family of pseudospherical surfaces admits a finite-order local isometric immersion. In such a case, the coefficients a_{ij} of the second fundamental form are given by (1.3.1) where a, b, c are given by (3.1.6).

On the other hand, if in Type III (a) one chooses $f = z$, $g = -xz$, $m = 0$, $h = -\eta$ and $\epsilon_1 = -1$, one can interpret (3.3.2) as a particular instance of that type with associated 1-forms

$$\begin{aligned} \omega_1 &= z\,dx + \left[x z_2 + 2(xz + 1) z_1 + z^2\right] dt, \\ \omega_2 &= \eta\,dx + \eta x z\,dt, \\ \omega_3 &= -\omega_2, \end{aligned} \tag{3.3.4}$$

where $\eta \neq 0$. In this case, equation (3.3.2) describes pseudospherical surfaces with first fundamental form $I = \omega_1^2 + \omega_2^2$ given by (3.3.4), however in view of Theorem 3.1.1, such a family of pseudospherical surfaces does not admit any finite-order local isometric immersion.

This proves that the existence of finite-order local isometric immersions depends on the particular choice of the associated linear problem.

Example 3.3.3. In view of Theorem 1.2.2, any evolution equation written in conservation law form is a PS equation. Hence, in view of Theorem 3.1.2, whenever the associated 1-forms $\omega_i = f_{i1}\,dx + f_{i2}\,dt$ and the generic solutions z are defined on a strip of the form (3.1.5) or (3.1.7), there exists a local finite-order isometric immersion in $\mathbf{E}^3$ of the corresponding family of pseudospherical surfaces described by such a PS equation. In such a case, the coefficients a_{ij} of the second fundamental form are given by (1.3.1), where a, b, c are given by (3.1.6) or (3.1.8).

Examples of this type are provided by many well known evolution equations. Examples of second order are for instance provided by Burgers equation, Murray equation and Svinolupov-Sokolov equations. Higher order examples are provided by Kuramoto-Sivashinsky equation (see also [17]), Sawada-Kotera equation and Kaup-Kupershmidt equation (see also [29]) as well as by hierarchies of evolution equations written in conservation law form like the following ones:

(i) Burgers hierarchy

$$z_t = D_x \left[D_x (a_n) + \frac{z}{2} a_n \right], \qquad n \in \mathbb{N},$$

where $a_1 = z$ and $a_{n+1} = D_x (a_n) + \frac{z}{2} a_n$;

(ii) mKdV hierarchy

$$z_t = D_x \left[D_x \left(\frac{D_x (a_n)}{z} \right) + z a_n \right], \qquad n \in \mathbb{N},$$

where $a_1 = \frac{z^2}{2}$ and $\frac{D_x(a_{n+1})}{z} = D_x (z a_n) + D_x^2 \left[\frac{D_x(a_n)}{z} \right]$;

(iii) KdV hierarchy

$$z_t = D_x \left(\frac{a_{n+1}}{4} \right), \qquad n \in \mathbb{N},$$

where $a_1 = 4z$, $D_x \left(\frac{a_{n+1}}{4} \right) = D_x^3 \left(\frac{a_n}{4} \right) + z D_x (a_n) + \frac{z_1 a_n}{2}$.

Theorem 3.1.2 proves that, whenever the associated 1-forms ω_i and the solutions z are defined on a strip of the form (3.1.5) or (3.1.7), finite-order local isometric immersions for the described family of pseudospherical surfaces exist in all such cases.

Chapter 4

Nontrivial 1-parameter families of ZCRs obtained via symmetry actions

In this chapter we consider the problem of constructing nontrivial 1-parameter families of ZCRs for PS equations. This problem is of special interest for the application of the theory of ZCRs, for instance in the calculation of exact solutions and infinite hierarchies of conservation laws, and has been solved in the more general case of $\mathfrak{g}$-valued ZCRs, with $\mathfrak{g}$ a Lie sub-algebra of $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C})$, by using the theory of classical symmetries of differential equations and the cohomology defined by the horizontal gauge differential of a given ZCR. In particular we provide an infinitesimal criterion which permits to identify all infinitesimal classical symmetries of an equation $\mathcal{E}$ whose flow A_λ could be used to embed a given ZCR α of $\mathcal{E}$ into a nontrivial 1-parameter family α_λ of zero-curvature representations of $\mathcal{E}$. The results reported here have been recently published in the paper [15].

The chapter is organized as follows. In Section 4.1 we discuss the application of symmetries of an equation $\mathcal{E}$ in the construction of a 1-parameter family of ZCRs of $\mathcal{E}$. In Section 4.2 we prove the main theorem which allows one to identify infinitesimal gauge-like symmetries as well as non gauge-like ones, for a given ZCR. In view of this theorem, only infinitesimal symmetries which are non gauge-like, for a ZCR α, may be used to construct a nontrivial 1-parameter family α_λ. Then we illustrate the results of this chapter by means of some examples in Section 4.3.

4.1 Action of continuous symmetries on ZCRs

In this section we will show how the flows of infinitesimal classical symmetries of a differential equation $\mathcal{E}$ could be used to embed a given $\mathfrak{g}$-valued ZCR α of $\mathcal{E}$ into a 1-parameter family α_λ of ZCRs.

Since the flow of a classical infinitesimal symmetry of an equation $\mathcal{E} \subset J^k(\pi)$ is in particular a 1-parameter family of finite symmetries of $\mathcal{C}^k(\pi)$, it will be useful to recall that finite symmetries of $\mathcal{C}^k(\pi)$ can always be obtained by prolonging either a (local) diffeomorphism on $J^0(\pi)$ or a (local) diffeomorphism on $J^1(\pi)$. Indeed, in view of Bäcklund theorem [45, 63], finite symmetries of $\mathcal{C}^k(\pi)$ are of two distinct types: when $m > 1$ classical finite symmetries are prolongations of (local) diffeomorphisms on $J^0(\pi)$ (also called *point transformations*); on the contrary when $m = 1$ there are classical finite symmetries which are not prolongations of point transformations, but are prolongations of (local) diffeomorphisms on $J^1(\pi)$ (also called *contact transformations*). In practice, a contact or point transformation can be prolonged to a finite symmetry of $\mathcal{C}^k(\pi)$ on $J^k(\pi)$ as follows.

For ease of notation, denoting by $\mathbf{z}^{(h)}$ the totality of coordinate z^j_σ, with $j \in \{1, ..., m\}$ and $0 \leq |\sigma| \leq h$, the prolongations

$$\bar{x}_i = \xi_i(\mathbf{x}, \mathbf{z}^{(1)}), \qquad \bar{z}^j_\rho = \psi^j_\rho(\mathbf{x}, \mathbf{z}^{(k)}), \qquad 0 \leq |\rho| \leq k$$

to $J^k(\pi)$ of a contact transformation on $J^1(\pi)$

$$\bar{x}_i = \xi_i(\mathbf{x}, \mathbf{z}^{(1)}), \qquad \bar{z}^j_\sigma = \psi^j_\sigma(\mathbf{x}, \mathbf{z}^{(1)}), \qquad 0 \leq |\sigma| \leq 1$$

can be computed by using for any fixed σ and j the following recurrence formula

$$\left\| \bar{z}^j_{\sigma+1_i} \right\| = \Delta^{-1} \left\| D_i^{(k+1)} \psi^j_\sigma(\mathbf{x}, \mathbf{z}^{(k)}) \right\|,$$

where the $D_i^{(k+1)}$ denote the $(k+1)$-th order truncated total derivative operators

$$D_i^{(k+1)} := \partial_{x^i} + \sum_{|\sigma| \leq k} z^j_{\sigma+1_i} \partial_{z^j_\sigma},$$

and Δ is the nonsingular matrix

$$\Delta := \left\| \begin{array}{ccc} D_1^{(1)}(\xi_1) & ... & D_1^{(1)}(\xi_n) \\ \vdots & & \vdots \\ D_n^{(1)}(\xi_1) & ... & D_n^{(1)}(\xi_n) \end{array} \right\|. \tag{4.1.1}$$

The same formula could be used to prolong a point transformation $\{\bar{x}_i = \xi_i(\mathbf{x}, \mathbf{z}),\ \bar{z}^j = \psi^j(\mathbf{x}, \mathbf{z})\}$ on $J^0(\pi)$ to a (local) finite symmetry of $\mathcal{C}^k(\pi)$ on $J^k(\pi)$.

Now, since the infinite prolongation of a finite classical symmetry is a finite symmetry of $\mathcal{C}(\pi)$, by considering $\mathfrak{g}$-valued forms, with $\mathfrak{g}$ a sub-algebra of $\mathfrak{gl}(n, \mathbb{R})$, one has the following

Lemma 4.1.1. *Let F be the infinite prolongation of a point or contact transformation. For any pair (a,b) of natural numbers, the following diagram commutes:*

$$\begin{array}{ccc}
\mathfrak{g}\otimes\Lambda^{(a+1,b)}(\pi) & \overset{\pi^{(a+1,b)}\circ F^{*}}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a+1,b)}(\pi)\\
d_H\uparrow & \circlearrowleft & \uparrow d_H\\
\mathfrak{g}\otimes\Lambda^{(a,b)}(\pi) & \underset{\pi^{(a,b)}\circ F^{*}}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a,b)}(\pi).
\end{array}$$

In particular, if $\bar{F}$ is the restriction to $\mathcal{E}^{(\infty)}$ of the infinite prolongation F of a point or contact transformation which maps a formally integrable equation $\mathcal{E}\subset J^k(\pi)$ to a formally integrable equation $\mathcal{Y}\subset J^k(\pi)$, then for any pair (a,b) of natural numbers the following diagram commutes:

$$\begin{array}{ccc}
\mathfrak{g}\otimes\Lambda^{(a+1,b)}(\mathcal{Y}) & \overset{\bar{\pi}_{\mathcal{E}}^{(a+1,b)}\circ \bar{F}^{*}}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a+1,b)}(\mathcal{E})\\
\bar{d}_{\mathcal{Y}}\uparrow & \circlearrowleft & \uparrow \bar{d}_{\mathcal{E}}\\
\mathfrak{g}\otimes\Lambda^{(a,b)}(\mathcal{Y}) & \underset{\bar{\pi}_{\mathcal{E}}^{(a,b)}\circ \bar{F}^{*}}{\longrightarrow} & \mathfrak{g}\otimes\Lambda^{(a,b)}(\mathcal{E}).
\end{array}$$

Proof. We only give a proof of the commutativity of the first diagram, since the commutativity of the second diagram is obtained by restricting on $\mathcal{E}^{(\infty)}$ and $\mathcal{Y}^{(\infty)}$.

Since $d=d_H+d_V$ and F^* commute, for any $\alpha\in\mathfrak{g}\otimes\Lambda^{(a,b)}(\pi)$ one gets that $F^*(d_H\alpha)+F^*(d_V\alpha)=d_H\left(F^*(\alpha)\right)+d_V\left(F^*(\alpha)\right)$. On the other hand, since F is a symmetry of $\mathcal{C}(\pi)$, it is not difficult to see that, for any $\rho\in\Lambda^{(p,q)}$, all terms in the decomposition of $F^*(\rho)$ on $\bigoplus_{r+s=p+q}\Lambda^{(r,s)}(\pi)$ have at least vertical degrees q. Hence $\pi^{(a+1,b)}\left(F^*(d_V\alpha)\right)=\pi^{(a+1,b)}\left(d_V\left(F^*(\alpha)\right)\right)=0$, and one has that $\pi^{(a+1,b)}\left(F^*(d_H\alpha)\right)=\pi^{(a+1,b)}\left(d_H\left(F^*(\alpha)\right)\right)$. But, again in view of the fact that F is a symmetry of $\mathcal{C}(\pi)$, one has $\pi^{(a+1,b)}\left(d_H\left(F^*(\alpha)\right)\right)=d_H\left(\pi^{(a,b)}\left(F^*(\alpha)\right)\right)$ then $\pi^{(a+1,b)}\left(F^*(d_H\alpha)\right)=d_H\left(\pi^{(a,b)}\left(F^*(\alpha)\right)\right)$. □

An analogous result holds for forms on $J^\infty(\pi)$ and $\mathcal{E}^{(\infty)}$.

We will adopt the following

Definition 4.1.2. Let F be the infinite prolongation of a point or contact transformation. By $F^{\#}$ we denote the map

$$F^{\#}=\pi^{(a,b)}\circ F^*:\ \mathfrak{g}\otimes\Lambda^{(a,b)}(\pi)\to\mathfrak{g}\otimes\Lambda^{(a,b)}(\pi).$$

Analogously, if $\bar{F}$ is the restriction to $\mathcal{E}^{(\infty)}$ of the infinite prolongation F of a point or contact transformation which maps a formally integrable equation $\mathcal{E}\subset J^k(\pi)$ to a formally integrable equation $\mathcal{Y} \subset J^k(\pi)$, by $\bar{F}^{\#}$ we will denote the map

$$\bar{F}^{\#} = \bar{\pi}_{\mathcal{E}}^{(a,b)} \circ \bar{F}^* : \ \mathfrak{g} \otimes \Lambda^{(a,b)}(\mathcal{Y}) \to \mathfrak{g} \otimes \Lambda^{(a,b)}(\mathcal{E}).$$

Notice that, if F is projectable (i.e., $F^*(C^\infty(M)) \subset C^\infty(M)$), then $F^{\#} = F^*$ and $\bar{F}^{\#} = \bar{F}^*$.

Now we can prove the following

Proposition 4.1.3. *If F is the infinite prolongation of a point or contact transformation, which maps a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ to a formally integrable equation $\mathcal{Y} \subset J^k(\pi)$, then*

$$\bar{F}^{\#} : \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{Y}) \to \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$$

maps any ZCR β of $\mathcal{Y}$ to a ZCR $\alpha = \bar{F}^{\#}(\beta)$ of $\mathcal{E}$.

Proof. It is not difficult to show that, in view of the non degeneracy of (4.1.1), α is a non-vanishing $\mathfrak{g}$-valued horizontal form on $\mathcal{E}^{(\infty)}$. Hence, one has to prove that $\bar{d}_{H,\mathcal{E}}\alpha - \frac{1}{2}[\alpha, \alpha] = 0$. To this end, it suffices to observe that in view of Lemma 4.1.1

$$\bar{F}^{\#}\left(\bar{d}_{H,\mathcal{Y}}\beta - \tfrac{1}{2}[\beta, \beta]\right) = \bar{d}_{H,\mathcal{E}}\left(\bar{F}^{\#}(\beta)\right) - \tfrac{1}{2}\left(\left[\bar{F}^{\#}(\beta), \bar{F}^{\#}(\beta)\right]\right) = \bar{d}_{H,\mathcal{E}}\alpha - \tfrac{1}{2}[\alpha, \alpha].$$

Hence the claim follows by the fact that $\bar{d}_{H,\mathcal{Y}}\beta - \frac{1}{2}[\beta, \beta] = 0$. □

Corollary 4.1.4. *If $\bar{F}$ is the restriction to $\mathcal{E}^{(\infty)}$ of a classical symmetry of a formally integrable equation $\mathcal{E}$, then $\bar{F}^{\sharp}$ maps any ZCR α of $\mathcal{E}$ to a ZCR $\bar{F}^{\#}(\alpha)$. In particular, if A_λ is the flow of a restricted classical generalized symmetry of $\mathcal{E}$, then $\alpha_\lambda := A_\lambda^{\#}(\alpha)$ is a 1-parameter family of ZCRs of $\mathcal{E}$.*

We close this section with the following examples illustrating the results of Proposition 4.1.3 and Corollary 4.1.4.

Example 4.1.5. The sine-Gordon equation

$$\mathcal{E} := \{z_{1,t} = sin\,(z)\}\,, \tag{4.1.2}$$

defines a submanifold of $J^2(\pi)$, with $\pi : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$, $(x, t, z) \mapsto (x, t)$, and admits the following $\mathfrak{sl}(2, \mathbb{R})$-valued ZCR

$$\alpha := \begin{pmatrix} 1 & -\frac{z_1}{2} \\ \frac{z_1}{2} & -1 \end{pmatrix} dx + \frac{1}{4}\begin{pmatrix} cos\,(z) & sin\,(z) \\ sin\,(z) & -cos\,(z) \end{pmatrix} dt. \tag{4.1.3}$$

The algebra of classical symmetries of $\mathcal{E}$ is generated by the prolongations of vector fields

$$Y_1 = \partial_x, \qquad Y_2 = \partial_t, \qquad Y_3 = x\partial_x - t\partial_t.$$

Symmetries Y_1 and Y_2 describe the obvious invariance of (4.1.2) under translations $x \mapsto x + c_1$ and $t \mapsto t + c_2$, $c_1, c_2 \in \mathbb{R}$. Hence their prolongations leave invariant the ZCR α and cannot be used to construct a 1-parameter family of $\mathfrak{sl}(2,\mathbb{R})$-valued ZCRs of (4.1.2). The same is not true for Y_3, and according to Corollary 4.1.4 one could use the flow A_λ of the restriction to $\mathcal{E}^{(\infty)}$ of $Y_3^{(\infty)}$ to generate a 1-parameter family of $\mathfrak{sl}(2,\mathbb{R})$-valued ZCRs of (4.1.2). Indeed, since α only involves first-order jet-coordinates and A_λ induces the following first-order transformation

$$t \mapsto e^{-\lambda}t, \quad x \mapsto e^{\lambda}x, \quad z \mapsto z, \quad z_1 \mapsto e^{-\lambda}z_1, \quad z_t \mapsto e^{\lambda}z_t,$$

one readily gets that

$$\alpha_\lambda = A_\lambda^{\#}(\alpha) = \begin{pmatrix} e^{\lambda} & -\frac{z_1}{2} \\ \frac{z_1}{2} & -e^{\lambda} \end{pmatrix} dx + \frac{1}{4e^{\lambda}} \begin{pmatrix} cos\,(z) & sin\,(z) \\ sin\,(z) & -cos\,(z) \end{pmatrix} dt,$$

which is the well known 1-parameter family of ZCRs for the sine-Gordon equation [56]. Using Theorem 1.6.3, one could check that λ is not removable, and hence that α_λ is a nontrivial 1-parameter family of $\mathfrak{sl}(2,\mathbb{R})$-valued ZCRs. Using the Theorem 4.2.8 of next section one could predict the non-removability of λ by the fact that the prolongation of Y_3 is non gauge-like for α.

Remark 4.1.6. We notice that, in the current literature, classical symmetries of nonlinear differential equations admitting ZCRs are usually projectable. However, as shown by the following example, non-projectable symmetries also may occur and hence exploited in the embedding of a given nonparametric ZCR α into a 1-parameter family α_λ of ZCRs.

Example 4.1.7. In the previous example, A_λ is projectable and hence $A_\lambda^{\#}(\alpha) = A_\lambda^{*}(\alpha)$. However, if one uses the non-projectable transformation F defined by the prolongation of the point transformation

$$\tau = \ t - z, \qquad \xi = x, \qquad v = z \ , \tag{4.1.4}$$

equation (4.1.2) and ZCR (4.1.3) transform to

$$\mathcal{Y} = \left\{ v_{\xi\tau} = \frac{1}{v_\tau - 1}\left(v_\xi v_{\tau\tau} + v_\tau^3 sin\,(v) - 3v_\tau^2 sin\,(v) + 3v_\tau sin\,(v) - sin\,(v)\right) \right\} \tag{4.1.5}$$

and

$$\beta = \left(\bar{F}^{-1}\right)^{\#}(\alpha) = \begin{pmatrix} 1 - \frac{v_\xi cos(v)}{4} & \frac{v_\xi}{2(v_\tau - 1)} - \frac{v_\xi sin(v)}{4} \\ \frac{v_\xi}{2(1-v_\tau)} - \frac{v_\xi sin(v)}{4} & \frac{v_\xi cos(v)}{4} - 1 \end{pmatrix} d\xi$$
$$+ \frac{1-v_\tau}{4} \begin{pmatrix} cos\,(v) & sin\,(v) \\ sin\,(v) & -cos\,(v) \end{pmatrix} d\tau,$$

respectively, where $\bar{F}$ is the restriction of F to $\mathcal{E}^{(\infty)}$. Consequently, Y_3 transforms to the non-projectable field $X_3 := F_*(Y_3) = \xi\partial_\xi + (v-\tau)\,\partial_\tau$ which generates a non-projectable classical symmetry of $\mathcal{Y}$. Hence the flow B_λ of the restriction to $\mathcal{Y}^{(\infty)}$ of $X_3^{(\infty)}$ is not projectable and $B_\lambda^{\#}(\beta)$ does not coincide with $B_\lambda^{*}(\beta)$.

Since β only involves first-order jet-coordinates and B_λ induces the following first-order transformation

$$\xi \mapsto e^\lambda \xi, \quad \tau \mapsto v + (\tau - v)\,e^{-\lambda}, \quad v \mapsto v, \quad v_\xi \mapsto \frac{e^{-2\lambda} v_\xi}{v_\tau + e^{-\lambda} - e^{-\lambda} v_\tau}, \quad v_\tau \mapsto \frac{v_\tau}{v_\tau + e^{-\lambda} - e^{-\lambda} v_\tau},$$

one gets that

$$\beta_\lambda = B_\lambda^{\#}(\beta) = \begin{pmatrix} e^\lambda - \frac{v_\xi cos(v)}{4e^\lambda} & \frac{v_\xi}{2(v_\tau - 1)} - \frac{v_\xi sin(v)}{4e^\lambda} \\ \frac{v_\xi}{2(1-v_\tau)} - \frac{v_\xi sin(v)}{4e^\lambda} & \frac{v_\xi cos(v)}{4e^\lambda} - e^\lambda \end{pmatrix} d\xi$$
$$+ \frac{1-v_\tau}{4e^\lambda} \begin{pmatrix} cos\,(v) & sin\,(v) \\ sin\,(v) & -cos\,(v) \end{pmatrix} d\tau.$$

Of course, since F transforms the flow of Y_3 to the flow of X_3, one has that $\bar{F}^{\#}(\beta_\lambda) = \alpha_\lambda$.

4.2 Infinitesimal criterion for gauge-like symmetries and nontrivial 1-parameter families of ZCRs

In this section we will prove an infinitesimal version of Theorem 1.6.3 (see Theorem 4.2.8 below), which will give a characterization of classical symmetries whose flows acts like gauge transformations for a ZCR α of $\mathcal{E}$. We will call these symmetries gauge-like and prove that they form a sub-algebra of the Lie algebra of symmetries of $\mathcal{E}$. Hence, $\alpha_\lambda := A_\lambda^{\#}(\alpha)$ is nontrivial if and only if A_λ is the flow of a restricted classical non gauge-like symmetry.

We begin by introducing the following

Definition 4.2.1. Let Z be a vector field on $J^\infty(\pi)$ and $\omega \in \mathfrak{g} \otimes \Lambda^{(p,q)}(\pi)$. By $Z(\omega)$ we denote the $\pi^{(p,q)}$-projected Lie derivative

$$Z(\omega) := \pi^{(p,q)}\left(L_Z(\omega)\right).$$

In particular, if Z is a generalized symmetry of $\mathcal{E}$ and $\bar{Z}$ its restriction to $\mathcal{E}^{(\infty)}$, for any $\omega \in \mathfrak{g} \otimes \Lambda^{(p,q)}(\mathcal{E})$ we denote by $\bar{Z}(\omega)$ the $\bar{\pi}^{(p,q)}$-projected Lie derivative

$$\bar{Z}(\omega) := \bar{\pi}^{(p,q)}\left(L_{\bar{Z}}(\omega)\right).$$

The following lemma gives an analogy of the standard commutation property between the Lie derivative and the exterior differential.

Lemma 4.2.2. *If Z is a generalized symmetry of $\mathcal{E}$ and $\bar{Z}$ its restriction to $\mathcal{E}^{(\infty)}$, then $\bar{Z}(\bar{d}_H(\omega)) = \bar{d}_H(\bar{Z}(\omega))$ for any $\omega \in \mathfrak{g} \otimes \Lambda^{(a,b)}(\mathcal{E})$.*

Proof. Since L_Z and d commute on $\mathfrak{g} \otimes \Lambda^*(\mathcal{E})$ and Z is tangent to $\mathcal{E}^{(\infty)}$, one gets that $\left(L_{\bar{Z}} \circ d|_{\mathcal{E}^{(\infty)}} - d|_{\mathcal{E}^{(\infty)}} \circ L_{\bar{Z}}\right)(\omega) = 0$, for any $\omega \in \mathfrak{g} \otimes \Lambda^{(a,b)}(\mathcal{E})$. On the other hand, since $d|_{\mathcal{E}^{(\infty)}} = \bar{d}_H + \bar{d}_V$, Lemma 4.1.1 allows one to rewrite

$$\bar{\pi}^{(a+1,b)} \circ \left(L_{\bar{Z}} \circ d|_{\mathcal{E}^{(\infty)}} - d|_{\mathcal{E}^{(\infty)}} \circ L_{\bar{Z}}\right)(\omega) = 0$$

as

$$\left(\left(\bar{\pi}^{(a+1,b)} \circ L_{\bar{Z}}\right) \circ \bar{d}_H - \bar{d}_H \circ \left(\bar{\pi}^{(a,b)} \circ L_{\bar{Z}}\right)\right)(\omega) = \left(\bar{\pi}^{(a+1,b)} \circ \left[\bar{d}_V, L_{\bar{Z}}\right]\right)(\omega).$$

Then, since $\bar{Z}$ is a symmetry of $\mathcal{C}(\mathcal{E})$, $\left[\bar{d}_V, L_{\bar{Z}}\right](\omega)$ cannot have horizontal degree greater than a and $\left(\bar{\pi}^{(a+1,b)} \circ \left[\bar{d}_V, L_{\bar{Z}}\right]\right)(\omega) = 0$.

It is not difficult to prove also the following two results □

Lemma 4.2.3. *If A, B are infinite prolongations of point or contact transformations, then $B^{\#} \circ A^* = B^{\#} \circ A^{\#}$. In particular, if A and B are symmetries of $\mathcal{E}$, then their restrictions $\bar{A}, \bar{B}$ to $\mathcal{E}^{(\infty)}$ are such that $\bar{B}^{\#} \circ \bar{A}^* = \bar{B}^{\#} \circ \bar{A}^{\#}$.*

Lemma 4.2.4. *If Z_1, Z_2 are generalized symmetries of $\mathcal{E}$, then for any $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ their restrictions $\bar{Z}_1$ and $\bar{Z}_2$ to $\mathcal{E}^{(\infty)}$ are such that $\bar{Z}_1\left(L_{\bar{Z}_2}\alpha\right) = \bar{Z}_1\left(\bar{Z}_2(\alpha)\right)$.*

Lemma 4.2.4 will be used in the proof of Proposition 4.2.10, whereas Lemma 4.2.3 is needed in the proof of the following

Proposition 4.2.5. *Let $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ be a ZCR of $\mathcal{E}$ and Z be a classical symmetry of $\mathcal{E}$. If A_λ is the flow of the restriction $\bar{Z}$ of Z to $\mathcal{E}^{(\infty)}$, i.e., $\bar{Z} = \frac{d}{d\lambda}\big|_{\lambda=0} A_\lambda^*$, then the 1-parameter family of ZCRs $\alpha_\lambda := A_\lambda^{\#}(\alpha)$ is such that:*

(i) $\bar{Z}(\alpha) = \frac{d}{d\lambda}\big|_{\lambda=0} A_\lambda^\#(\alpha)$;

(ii) $\frac{d\alpha_\lambda}{d\lambda} = A_\lambda^\#\left(\bar{Z}(\alpha)\right)$.

Proof. (i) By definition of Lie derivative $L_{\bar{Z}}(\alpha) = \frac{d}{d\lambda}\big|_{\lambda=0} A_\lambda^*(\alpha)$, hence the claim follows by observing that $\bar{Z}(\alpha) = \bar{\pi}^{(1,0)}\left(\frac{d}{d\lambda}\big|_{\lambda=0} A_\lambda^*(\alpha)\right) = \left(\frac{d}{d\lambda}\big|_{\lambda=0} \bar{\pi}^{(1,0)}\left(A_\lambda^*(\alpha)\right)\right) = \frac{d}{d\lambda}\big|_{\lambda=0} A_\lambda^\#(\alpha)$.

(ii) In view of Lemma 4.2.3, $A_{\lambda+\delta}^\# = \bar{\pi}^{(1,0)} \circ A_\lambda^* \circ A_\delta^* = A_\lambda^\# \circ A_\delta^* = A_\lambda^\# \circ A_\delta^\#$ hence

$$\frac{d\alpha_\lambda}{d\lambda} = \lim_{\delta\to 0} \frac{A_\lambda^\#\left(A_\delta^\#(\alpha) - \alpha\right)}{\delta} = A_\lambda^\#\left(\lim_{\delta\to 0}\frac{A_\delta^\#(\alpha) - \alpha}{\delta}\right) = A_\lambda^\#\left(\bar{Z}(\alpha)\right),$$

in view of (i). □

Then one can also prove the following

Proposition 4.2.6. *Let $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ be a ZCR of $\mathcal{E}$. If $\bar{Z}$ is a restricted generalized symmetry of $\mathcal{E}$, then $\bar{Z}(\alpha)$ is a 1-cocycle with respect to $\bar{\partial}_\alpha$, i.e.,*

$$\bar{\partial}_\alpha \bar{Z}(\alpha) = 0.$$

Proof. Since $\bar{d}_H\alpha - \frac{1}{2}[\alpha,\alpha] = 0$ and $\bar{Z}$ is a vector field on $\mathcal{E}^{(\infty)}$, one still has

$$L_{\bar{Z}}\left(\bar{d}_H\alpha - \frac{1}{2}[\alpha,\alpha]\right) = 0$$

identically on $\mathcal{E}^{(\infty)}$. Hence, by using Lemma 4.2.2 and formula (1.5.1), the derivative of $\bar{Z}\left(\bar{d}_H\alpha - \frac{1}{2}[\alpha,\alpha]\right)$ returns

$$0 = \bar{d}_H\left(\bar{Z}(\alpha)\right) - \frac{1}{2}[\bar{Z}(\alpha),\alpha] - \frac{1}{2}[\alpha,\bar{Z}(\alpha)] = \bar{d}_H\left(\bar{Z}(\alpha)\right) - [\alpha,\bar{Z}(\alpha)] = \bar{\partial}_\alpha\bar{Z}(\alpha).$$

□

Remark 4.2.7. If $\bar{Z}$ is the restriction to $\mathcal{E}^{(\infty)}$ of a classical symmetry Z of $\mathcal{E}$ with flow A_λ, one can prove Proposition 4.2.6 by using Proposition 4.2.5. Indeed, by considering $\alpha_\lambda = \bar{A}_\lambda^\#(\alpha)$ and differentiating the identity $\bar{d}\alpha_\lambda - \alpha_\lambda \wedge \alpha_\lambda = 0$ at $\lambda = 0$, one gets

$$\begin{aligned} 0 &= \bar{d}_H\left(\bar{Z}(\alpha)\right) - \bar{Z}(\alpha)\wedge\alpha - \alpha\wedge\bar{Z}(\alpha) = \bar{d}_H\left(\bar{Z}(\alpha)\right) - 2\alpha\wedge\bar{Z}(\alpha) \\ &= \bar{d}_H\left(\bar{Z}(\alpha)\right) - [\alpha,\bar{Z}(\alpha)] = \bar{\partial}_\alpha\left(\bar{Z}(\alpha)\right). \end{aligned}$$

The following result, together with Proposition 4.2.6, provides a cohomological obstruction to the removability of λ from the 1-parameter family of ZCRs obtained by using the flow of a classical symmetry.

Theorem 4.2.8. *Let $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ be a ZCR of $\mathcal{E}$, Z a classical symmetry of $\mathcal{E}$ and A_λ the flow of its restriction $\bar{Z}$ to $\mathcal{E}^{(\infty)}$. Then the parameter λ in $\alpha_\lambda = A_\lambda^{\#}(\alpha)$ is removable if, and only if, $\bar{Z}(\alpha)$ is a coboundary with respect to $\bar{\partial}_\alpha$, i.e.,*

$$\bar{Z}(\alpha) = \bar{\partial}_\alpha K, \tag{4.2.1}$$

for some $\mathfrak{g}$-valued smooth function K on $\mathcal{E}^{(\infty)}$.

Proof. If the parameter λ is removable, then for $\lambda_0 = 0$ there exists some G-valued function S_λ such that $S_0 = \mathbb{I}$ (identity) and

$$\alpha = \alpha_\lambda^{S_\lambda^{-1}} = \left(A_\lambda^{\#}\alpha\right)^{S_\lambda^{-1}} = \bar{d}_H\left(S_\lambda^{-1}\right)S_\lambda + S_\lambda^{-1}\left(A_\lambda^{\#}\alpha\right)S_\lambda.$$

Hence, by differentiating with respect to λ

$$\begin{aligned}0 = {}& \bar{d}_H\left(\tfrac{d}{d\lambda}\left(S_\lambda^{-1}\right)\right)S_\lambda + \bar{d}_H\left(S_\lambda^{-1}\right)\tfrac{d}{d\lambda}S_\lambda + \left(\tfrac{d}{d\lambda}S_\lambda^{-1}\right)\left(A_\lambda^{\#}\alpha\right)S_\lambda + S_\lambda^{-1}\left(\tfrac{d}{d\lambda}A_\lambda^{\#}\alpha\right)S_\lambda \\ &+ S_\lambda^{-1}\left(A_\lambda^{\#}\alpha\right)\tfrac{d}{d\lambda}S_\lambda,\end{aligned}$$

and further evaluating at $\lambda_0 = 0$, by Proposition 4.2.5 one gets

$$0 = \bar{d}_H\left(\left.\frac{d}{d\lambda}\right|_{\lambda=0}S_\lambda^{-1}\right) + \left(\left.\frac{d}{d\lambda}\right|_{\lambda=0}S_\lambda^{-1}\right)\alpha + \bar{Z}(\alpha) + \alpha\left.\frac{d}{d\lambda}\right|_{\lambda=0}S_\lambda. \tag{4.2.2}$$

On the other hand $\frac{d}{d\lambda}\left(S_\lambda^{-1}S_\lambda\right) = 0$ entails that $\left(\frac{d}{d\lambda}S_\lambda^{-1}\right)S_\lambda + S_\lambda^{-1}\left(\frac{d}{d\lambda}S_\lambda\right) = 0$ and hence $\left.\frac{d}{d\lambda}\right|_{\lambda=0}S_\lambda^{-1} = -\left.\frac{d}{d\lambda}\right|_{\lambda=0}S_\lambda$. Therefore, by choosing $K = \left.\frac{d}{d\lambda}\right|_{\lambda=0}S_\lambda$, (4.2.2) can be rewritten as $\bar{Z}(\alpha) = \left(\bar{d}_H - [\alpha, .]\right)(K) = \bar{\partial}_\alpha(K)$.

Conversely, assume that $\bar{Z}(\alpha) = \bar{\partial}_\alpha(K)$ and consider a solution S_λ of

$$\begin{cases} \dot{S}_\lambda = A_\lambda^{*}(K)\,S_\lambda, \\ S_0 = \mathbb{I}, \end{cases} \tag{4.2.3}$$

where $\dot{S}_\lambda = \frac{d}{d\lambda}S_\lambda$. In a neighborhood I of $\lambda_0 = 0$, S_λ defines the gauge transformation $\alpha^{S_\lambda} = \bar{d}_H S_\lambda S_\lambda^{-1} + S_\lambda \alpha S_\lambda^{-1}$ which can be rewritten as $\bar{d}_H S_\lambda = \alpha^{S_\lambda}S_\lambda - S_\lambda\alpha$. Then, by defining

$$z_\lambda := \bar{d}_H S_\lambda + S_\lambda\alpha - \alpha_\lambda S_\lambda, \tag{4.2.4}$$

one readily gets

$$z_\lambda = \left(\alpha^{S_\lambda} - \alpha_\lambda\right)S_\lambda. \tag{4.2.5}$$

Of course $z_0 = 0$, and it can be proved that $z_\lambda = 0$, for any $\lambda \in I$. To this end, one may first consider the derivative of (4.2.4) with respect to λ

$$\dot{z}_\lambda = \bar{d}_H \dot{S}_\lambda + \dot{S}_\lambda \alpha - \dot{\alpha}_\lambda S_\lambda - \alpha \dot{S}_\lambda,$$

and, by using equation (4.2.3), rewrite it as

$$\dot{z}_\lambda = \bar{d}_H \left(A^*_\lambda (K) S_\lambda\right) + \left(A^*_\lambda (K) S_\lambda\right) \alpha - \dot{\alpha}_\lambda S_\lambda - \alpha_\lambda A^*_\lambda (K) S_\lambda.$$

On the other hand, in view of Proposition 4.2.5, one has

$$\dot{\alpha}_\lambda = \frac{d}{d\lambda} A^{\#}_\lambda (\alpha) = A^{\#}_\lambda Z(\alpha) = A^{\#}_\lambda \left(\bar{\partial}_\alpha (K)\right).$$

Hence

$$\begin{aligned}
\dot{z}_\lambda &= \bar{d}_H \left(A^*_\lambda (K)\right) S_\lambda + A^*_\lambda (K) \bar{d}_H S_\lambda + A^*_\lambda (K) S_\lambda \alpha - A^{\#}_\lambda \left(\bar{\partial}_\alpha (K)\right) S_\lambda \\
&\quad - A^{\#}_\lambda (\alpha) A^*_\lambda (K) S_\lambda \\
&= \bar{d}_H \left(A^*_\lambda (K)\right) S_\lambda + A^*_\lambda (K) \bar{d}_H S_\lambda + A^*_\lambda (K) S_\lambda \alpha - A^{\#}_\lambda \left(\bar{d}_H K - [\alpha, K]\right) S_\lambda \\
&\quad - A^{\#}_\lambda (\alpha) A^*_\lambda (K) S_\lambda \\
&= \bar{d}_H \left(A^*_\lambda (K)\right) S_\lambda + A^*_\lambda (K) \left(\bar{d}_H S_\lambda + S_\lambda \alpha\right) - \bar{d}_H \left(A^*_\lambda (K)\right) S_\lambda \\
&\quad + A^{\#}_\lambda (\alpha) A^*_\lambda (K) S_\lambda - A^*_\lambda (K) A^{\#}_\lambda (\alpha) S_\lambda - A^{\#}_\lambda (\alpha) A^*_\lambda (K) S_\lambda \\
&= A^*_\lambda (K) \left(\bar{d}_H S_\lambda + S_\lambda \alpha - A^{\#}_\lambda (\alpha) S_\lambda\right) = A^*_\lambda (K) z_\lambda
\end{aligned}$$

and z_λ must be the solution of the Cauchy problem

$$\begin{cases} \dot{z}_\lambda = A^*_\lambda (K) z_\lambda \\ z_0 = 0. \end{cases}$$

It follows, by the existence and uniqueness of solutions to such a Cauchy problem, that z_λ must be identically zero. Then, since S_λ is invertible, by (4.2.5) one gets $\alpha = \alpha_\lambda^{S_\lambda^{-1}}$ and hence that λ is removable . □

This theorem justifies the following

Definition 4.2.9. A generalized symmetry Z of $\mathcal{E}$ is called **gauge-like** for the ZCR $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ if its restriction $\bar{Z}$ to $\mathcal{E}^{(\infty)}$ satisfies the condition $\bar{Z}(\alpha) = \bar{\partial}_\alpha K$ for some $\mathfrak{g}$-valued smooth function K on $\mathcal{E}^{(\infty)}$. If in addition Z is a classical symmetry, then Z will be called a **classical gauge-like** symmetry for α.

We have the following

Proposition 4.2.10. *Let Z_1 and Z_2 be two gauge-like symmetries for the same ZCR $\alpha \in \mathfrak{g} \otimes \bar{\Lambda}^1(\mathcal{E})$ of $\mathcal{E}$. Then also $[Z_1, Z_2]$ is gauge-like for α. In particular, if*

$$\bar{Z}_1(\alpha) = \bar{\partial}_\alpha K_1, \quad \bar{Z}_2(\alpha) = \bar{\partial}_\alpha K_2, \tag{4.2.6}$$

then

$$\overline{[Z_1, Z_2]}(\alpha) = \bar{\partial}_\alpha \left(K_{12}\right),$$

with

$$K_{12} = \bar{Z}_1(K_2) - \bar{Z}_2(K_1) - [K_1, K_2] .$$

Proof. First observe that $\overline{[Z_1, Z_2]} = \left[\bar{Z}_1, \bar{Z}_2\right]$ and $L_{[\bar{Z}_1, \bar{Z}_2]} = [L_{\bar{Z}_1}, L_{\bar{Z}_2}]$. Hence, in view of Lemma 4.2.4, one gets

$$\left[\bar{Z}_1, \bar{Z}_2\right](\alpha) = \bar{Z}_1(\bar{Z}_2(\alpha)) - \bar{Z}_2(\bar{Z}_1(\alpha)),$$

and a direct computation gives

$$\begin{aligned}\bar{Z}_1(\bar{Z}_2(\alpha)) - \bar{Z}_2(\bar{Z}_1(\alpha)) = \bar{d}_H \left(\bar{Z}_1(K_2) - \bar{Z}_2(K_1)\right) - \left[\alpha, \bar{Z}_1(K_2) - \bar{Z}_2(K_1)\right] \\ - \left\{ \left[\bar{Z}_1(\alpha), K_2\right] - \left[\bar{Z}_2(\alpha), K_1\right] \right\} .\end{aligned}$$

Then using again (4.2.6), and formulas (1.5.1,1.5.2,1.5.4), one readily gets that

$$\bar{Z}_1(\bar{Z}_2(\alpha)) - \bar{Z}_2(\bar{Z}_1(\alpha)) = \bar{\partial}_\alpha \left(\bar{Z}_1(K_2) - \bar{Z}_2(K_1) - [K_1, K_2]\right) .$$

□

Hence one gets the following

Corollary 4.2.11. *Gauge-like symmetries, for the same ZCR α of $\mathcal{E}$, form a Lie subalgebra of the Lie algebra of generalized symmetries of $\mathcal{E}$. In particular, classical gauge-like symmetries form a Lie sub algebra of the Lie algebra of classical symmetries.*

Remark 4.2.12. It is worth to remark here that, in general, two non gauge-like symmetries Z_1, Z_2 lead to two nontrivial 1-parameter families of ZCRs α^1_λ and α^2_η. However, one should consider α^1_λ and α^2_η as being two distinct 1-parameter families only if they are not equivalent, according to Definition 1.6.2.

We conclude this section by observing that the Lie algebra of gauge-like symmetries for a ZCR α of $\mathcal{E}$ is invariantly associated to any equation equivalent to $\mathcal{E}$, modulo some contact transformation. Indeed one has the following

Proposition 4.2.13. *If F is the infinite prolongation of a point or contact transformation which maps a formally integrable equation $\mathcal{E} \subset J^k(\pi)$ to a formally integrable equation $\mathcal{Y} \subset J^k(\pi)$, then the push-forward F_* transforms the Lie algebra of gauge-like symmetries for a ZCR α of $\mathcal{E}$ to the Lie algebra of gauge-like symmetries for the ZCR $\beta = (F^{-1})^{\#}(\alpha)$ of $\mathcal{Y}$.*

Proof. Consequence of the formula $(F^{-1})^*(L_{\bar{Z}}\alpha) = L_{F_*(\bar{Z})}(F^{-1})^*(\alpha)$, which holds for any restricted generalized symmetry $\bar{Z}$ of $\mathcal{E}$, and of formulas $\bar{\pi}_{\mathcal{Y}}^{(a,b)} \circ (F^{-1})^* = (F^{-1})^* \circ \bar{\pi}_{\mathcal{E}}^{(a,b)}$ and $\bar{d}_{H,\mathcal{Y}} \circ (F^{-1})^{\#} = (F^{-1})^{\#} \circ \bar{d}_{H,\mathcal{E}}$. □

4.3 Examples

Here we illustrate some examples of how, starting from a given ZCR α of $\mathcal{E}$, one may use the flow of an infinitesimal classical symmetry which is non gauge-like for α to construct a nontrivial 1-parameter family α_λ of ZCRs of $\mathcal{E}$.

Example 4.3.1. Burgers equation

$$z_t = z_2 + z z_1, \tag{4.3.1}$$

is one of the better-known nonlinear differential equations. In the paper [20] it has been observed that (4.3.1) can be embedded into a huge class of pseudospherical equations. In particular, the $\mathfrak{sl}(2,\mathbb{R})$-valued ZCR of (4.3.1) found in that paper is

$$\beta_\eta := \begin{pmatrix} \frac{\eta}{2} & \frac{z}{4} + \frac{\eta}{2} \\ \frac{z}{4} - \frac{\eta}{2} & -\frac{\eta}{2} \end{pmatrix} dx + \begin{pmatrix} \frac{\eta z}{4} & \frac{z_1}{4} + \frac{z^2}{8} + \frac{\eta z}{4} \\ \frac{z_1}{4} + \frac{z^2}{8} - \frac{\eta z}{4} & -\frac{\eta z}{4} \end{pmatrix} dt,$$

where η is a nonzero parameter. However, by using Theorem 1.6.3 one can see that η is removable through the gauge transformation defined by

$$S = \begin{pmatrix} \frac{1}{2\sqrt{\eta}} & -\sqrt{\eta} \\ \frac{1}{2\sqrt{\eta}} & \sqrt{\eta} \end{pmatrix}.$$

Indeed, one has that

$$\alpha = S^{-1}\beta_\eta S - S^{-1}\bar{d}_H S = \begin{pmatrix} \frac{z}{4} & 0 \\ -\frac{1}{2} & -\frac{z}{4} \end{pmatrix} dx + \begin{pmatrix} \frac{z_1}{4} + \frac{z^2}{8} & 0 \\ -\frac{z}{4} & -\frac{z_1}{4} - \frac{z^2}{8} \end{pmatrix} dt.$$

Here, by applying the results of Sections 4.1 and 4.2, we will show how use α to construct a nontrivial 1-parameter family of ZCRs of (4.3.1).

To this end, we first observe that the algebra of classical symmetries of (4.3.1) is 5-dimensional and generated by the prolongations of vector fields

$$\begin{aligned}
&Y_1 = \partial_x, \quad Y_2 = \partial_t, \quad Y_3 = x\partial_x + 2t\partial_t - z\partial_z, \quad Y_4 = t\partial_x - \partial_z,\\
&Y_5 = -xt\partial_x - t^2\partial_t + (x + tz)\,\partial_z.
\end{aligned}$$

In particular, the structure of the algebra of classical symmetries is

$$\begin{aligned}
&[Y_1, Y_2] = [Y_1, Y_4] = [Y_4, Y_5] = 0, \;\; [Y_1, Y_3] = [Y_2, Y_4] = Y_1,\\
&[Y_2, Y_3] = 2Y_2, \;\; [Y_5, Y_2] = Y_3, \;\; [Y_3, Y_5] = 2Y_5, \;\; [Y_5, Y_1] = [Y_3, Y_4] = Y_4.
\end{aligned}$$

Then, using Theorem 4.2.8, one can check that Y_1, Y_2 and Y_3 generate the algebra of gauge-like symmetries for α. On the contrary Y_4 and Y_5 are non gauge-like for α. Notice that the sub-algebra of gauge symmetries, with respect to α is not an ideal. For instance this is evident from the commutator $[Y_3, Y_5] = 2Y_5$.

By way of illustration, we explicitly prove these properties for Y_3 and Y_5.

For instance, denoting by $\bar{Z}$ the restriction of $Y_3^{(\infty)}$ to the infinite prolongation $\mathcal{E}^{(\infty)}$ of the Burgers equation. Equation (4.2.1) is equivalent to

$$\begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} dx + \begin{pmatrix} 0 & 0 \\ -\frac{z}{4} & 0 \end{pmatrix} dt = \bar{d}_H K - [\alpha, K]\,, \tag{4.3.2}$$

where $\bar{d}_H$ is the horizontal differential on $\mathcal{E}^{(\infty)}$ and $K = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ an $\mathfrak{sl}(2,\mathbb{R})$-valued function on $\mathcal{E}^{(\infty)}$. Then, it is not difficult to check that (4.3.2) is satisfied by $K = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, and hence that Y_3 is gauge-like for α.

On the other hand, denoting by $\bar{Z}$ the restriction of $Y_5^{(\infty)}$ to $\mathcal{E}^{(\infty)}$, one can readily check that the resulting equation (4.2.1) does not admit any solution K. Indeed, assuming that $K = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is an $\mathfrak{sl}(2,\mathbb{R})$-valued function on $\mathcal{E}^{(\infty)}$, then the coefficient of dx in the 1-form $\bar{Z}(\alpha) - \bar{d}_H K + [\alpha, K]$ is

$$i_{\partial_x}\left(\bar{Z}(\alpha) - \bar{d}_H K + [\alpha, K]\right) = \begin{pmatrix} \frac{x+2b}{4} - \bar{D}_x a & \frac{zb}{2} + \bar{D}_x b \\ \frac{t-zc}{2} - a - \bar{D}_x c & -\frac{x+2b}{4} + \bar{D}_x a \end{pmatrix}. \tag{4.3.3}$$

Now, it is straightforward to check that for (4.3.3) being identically zero it is necessary that the functions a, b, c depend only on (x,t). But, even in such a case (4.3.3) would never vanish due to its dependence on z. Hence Y_5 is non gauge-like for α.

Hence, for instance, one may use the flow A_λ of the restricted symmetry $\bar{Y}_5^{(\infty)}$, to construct a nontrivial 1-parameter family of ZCRs of (4.3.1).

Indeed, since α only involves first-order jet-coordinates and A_λ induces the following first-order transformation

$$x \mapsto \frac{x}{1+\lambda t}, \quad t \mapsto \frac{t}{1+\lambda t}, \quad z \mapsto (1+\lambda t)\, z + \lambda x, \quad z_1 \mapsto (1+\lambda t)^2 z_1 + \lambda (1+\lambda t)\,,$$

one gets the following nontrivial 1-parameter family of ZCRs of (4.3.1):

$$\alpha_\lambda := A_\lambda^{\#}(\alpha) = \begin{pmatrix} \frac{z}{4} + \frac{\lambda x}{4(1+\lambda t)} & 0 \\ -\frac{1}{2(1+\lambda t)} & -\frac{z}{4} - \frac{\lambda x}{4(1+\lambda t)} \end{pmatrix} dx$$
$$+ \begin{pmatrix} \frac{z_1}{4} + \frac{z^2}{8} + \frac{\lambda}{4(1+\lambda t)} - \frac{\lambda^2 x^2}{8(1+\lambda t)^2} & 0 \\ -\frac{z}{4(1+\lambda t)} + \frac{\lambda x}{4(1+\lambda t)^2} & -\frac{z_1}{4} - \frac{z^2}{8} - \frac{\lambda}{4(1+\lambda t)} + \frac{\lambda^2 x^2}{8(1+\lambda t)^2} \end{pmatrix} dt.$$

Another nontrivial 1-parameter family of ZCRs of (4.3.1) is

$$\beta_\eta := \begin{pmatrix} \frac{z}{4} - \frac{\eta}{4} & 0 \\ -\frac{1}{2} & \frac{\eta}{4} - \frac{z}{4} \end{pmatrix} dx + \begin{pmatrix} \frac{z_1}{4} + \frac{(z-\eta)^2}{8} + \frac{\eta(z-\eta)}{4} & 0 \\ -\frac{\eta}{4} - \frac{z}{4} & -\frac{z_1}{4} - \frac{(z-\eta)^2}{8} - \frac{\eta(z-\eta)}{4} \end{pmatrix} dt,$$

and arises from the non gauge-like symmetry generated by Y_4. One can see that α_λ and β_η are not equivalent, in the sense of Definition 1.6.2.

Example 4.3.2. The well-known 1-parameter family of ZCRs [56] of KdV equation

$$z_t = z_3 + 6 z z_1, \tag{4.3.4}$$

can be obtained by the following nonparametric $\mathfrak{sl}(2,\mathbb{R})$-valued ZCR

$$\alpha := \begin{pmatrix} 0 & z-1 \\ -1 & 0 \end{pmatrix} dx + \begin{pmatrix} z_1 & -4+2z+z_2+2z^2 \\ -4-2z & -z_1 \end{pmatrix} dt,$$

with the use of a symmetry which is non gauge-like for α.

Indeed the algebra of classical symmetries of (4.3.4) is 4-dimensional and generated by the prolongations of the vector fields

$$Y_1 = \partial_x, \quad Y_2 = -t\partial_x + \frac{1}{6}\partial_z, \quad Y_3 = \partial_t, \quad Y_4 = x\partial_x + 3t\partial_t - 2z\partial_z.$$

In particular, the structure of the algebra of classical symmetries is

$$[Y_1,Y_2]=[Y_1,Y_3]=0,\;\;[Y_1,Y_4]=[Y_2,Y_3]=Y_1,\;\;[Y_2,Y_4]=-2Y_2,\;\;[Y_3,Y_4]=3Y_3.$$

Now, in view of Theorem 4.2.8, one can check that Y_1 and Y_3 generate the sub-algebra of gauge-like symmetries for α. On the contrary Y_2 and Y_4 are non gauge-like for α.

By way of illustration, here we will explicitly prove that Y_4 is non gauge-like.

For instance, denoting by $\bar{Z}$ the restriction of $Y_4^{(\infty)}$ to the infinite prolongation $\mathcal{E}^{(\infty)}$ of the KdV equation, one can readily check that the resulting equation (4.2.1) does not admit any solution K. Indeed, assuming that $K=\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is an $\mathfrak{sl}(2,\mathbb{R})$-valued function on $\mathcal{E}^{(\infty)}$, then the coefficient of dx in the 1-form $\bar{Z}(\alpha)-\bar{d}_H K+[\alpha,K]$ is

$$\begin{aligned} & i_{\partial_x}\left(\bar{Z}(\alpha)-\bar{d}_H K+[\alpha,K]\right) \\ &= \begin{pmatrix} b+(z-1)c-\bar{D}_x a & -z-1-2(z-1)a-\bar{D}_x b \\ -1-2a-\bar{D}_x c & -b-(z-1)c+\bar{D}_x a \end{pmatrix}. \end{aligned} \tag{4.3.5}$$

Now, it is straightforward to check that for (4.3.5) being identically zero it is necessary that the functions a, b, c depend only on (x,t). But, even in such a case (4.3.5) would never vanish due to its dependence on z. Hence Y_4 is non gauge-like for α.

Hence, for instance, one may use the flow A_λ of the restricted symmetry $\bar{Y}_4^{(\infty)}$ to construct a nontrivial 1-parameter family $\alpha_\lambda := A_\lambda^{\#}(\alpha)$ of ZCRs of (4.3.4). To this end, since α only involves second-order jet-coordinates and A_λ induces the following second-order transformation

$$x\mapsto e^{\lambda}x,\quad t\mapsto e^{3\lambda}t,\quad z\mapsto e^{-2\lambda}z,\quad z_1\mapsto e^{-3\lambda}z_1,\quad z_2\mapsto e^{-4\lambda}z_2,$$

one gets that

$$\alpha_\lambda=\begin{pmatrix} 0 & e^{-\lambda}z-e^{\lambda} \\ -e^{\lambda} & 0 \end{pmatrix}dx+\begin{pmatrix} z_1 & -4e^{3\lambda}+2e^{-\lambda}z^2+e^{-\lambda}z_2+2e^{\lambda}z \\ -4e^{3\lambda}-2e^{\lambda}z & -z_1 \end{pmatrix}dt.$$

Up to a gauge transformation α_λ is equivalent to the already known 1-parameter family of ZCRs [56]

$$\tilde{\alpha}_\eta:=\begin{pmatrix} \eta & z \\ -1 & -\eta \end{pmatrix}dx+\begin{pmatrix} 4\eta^3+2\eta z+z_1 & z_2+2\eta z_1+4\eta^2 z+2z^2 \\ -4\eta^2-2z & -4\eta^3-2\eta z-z_1 \end{pmatrix}dt,$$

where $\eta \neq 0$. Indeed, if one chooses $\eta = e^{\lambda}$ and $S = \begin{pmatrix} e^{\lambda} & -e^{\lambda} \\ 0 & 1 \end{pmatrix}$, then $\left(\tilde{\alpha}_{e^{\lambda}}\right)^S = A_{\lambda}^{\sharp}(\alpha)$.

On the contrary, by using the non gauge-like symmetry generated by Y_2, one would get another nontrivial 1-parameter family

$$\begin{aligned} \beta_{\eta} :=& \begin{pmatrix} 0 & z + \frac{\eta}{6} - 1 \\ -1 & 0 \end{pmatrix} dx \\ &+ \begin{pmatrix} z_1 & z_2 + (2-\eta)\left(z + \frac{\eta}{6} - 1\right) + 2\left(z + \frac{\eta}{6}\right)^2 - 2 \\ \frac{2\eta}{3} - 4 - 2z & -z_1 \end{pmatrix} dt. \end{aligned}$$

However α_{λ} and β_{η} are equivalent (according to Definition 1.6.2), since for $\eta = 6\left(1 - e^{\lambda}\right)$ and $S = \begin{pmatrix} e^{\lambda/2} & 0 \\ 0 & e^{-\lambda/2} \end{pmatrix}$ one has that $\left(\alpha_{\lambda}\right)^S = \beta_{\eta}$.

Example 4.3.3. The known 1-parameter family of ZCRs [19] of the Chen-Lee-Liu system

$$\begin{cases} z_t = z_2 + 2zvz_1 \\ v_t = -v_2 + 2zvv_1 \end{cases} \tag{4.3.6}$$

can be obtained by the following nonparametric $\mathfrak{sl}(2,\mathbb{R})$-valued ZCR

$$\alpha := \begin{pmatrix} \frac{zv-1}{2} & z \\ v & \frac{1-zv}{2} \end{pmatrix} dx + \begin{pmatrix} \frac{z_1 v - z v_1 + (zv-1)^2}{2} & z^2 v - z + z_1 \\ z v^2 - v - v_1 & \frac{z v_1 - z_1 v - (zv-1)^2}{2} \end{pmatrix} dt,$$

with the use of a symmetry which is non gauge-like for α. Indeed, the algebra of classical symmetries of (4.3.6) is 4-dimensional and generated by the prolongations of vector fields

$$Y_1 = \partial_x, \quad Y_2 = \partial_t, \quad Y_3 = -z\partial_z + v\partial_v, \quad Y_4 = x\partial_x + 2t\partial_t - v\partial_v.$$

In particular, the structure of the algebra of classical symmetries is

$$[Y_1, Y_2] = [Y_1, Y_3] = [Y_2, Y_3] = [Y_3, Y_4] = 0, \quad [Y_1, Y_4] = Y_1, \quad [Y_2, Y_4] = 2Y_2.$$

Now, in view of Theorem 4.2.8, one can check that Y_1, Y_2 and Y_3 generate the sub-algebra of gauge-like symmetries for α. On the contrary Y_4 is non gauge-like for α. Hence, by using the flow A_{λ} of the restricted symmetry $\bar{Y}_4^{(\infty)}$, one can construct a nontrivial 1-parameter family $\alpha_{\lambda} := A_{\lambda}^{\#}(\alpha)$ of ZCRs of (4.3.6). To this end, since α only involves first-order jet-coordinates and A_λ induces the following transformation

$$x \mapsto e^{\lambda}x, \quad t \mapsto e^{2\lambda}t, \quad z \mapsto z, \quad v \mapsto e^{-\lambda}v, \quad z_1 \mapsto e^{-\lambda}z_1, \quad v_1 \mapsto e^{-2\lambda}v_1,$$

one gets that

$$\alpha_\lambda := \begin{pmatrix} \frac{zv-e^{\lambda}}{2} & e^{\lambda}z \\ v & \frac{e^{\lambda}-zv}{2} \end{pmatrix} dx + \begin{pmatrix} \frac{z_1v-zv_1+\left(zv-e^{\lambda}\right)^2}{2} & e^{\lambda}\left(z^2v-e^{\lambda}z+z_1\right) \\ zv^2-e^{\lambda}v-v_1 & \frac{zv_1-z_1v-\left(zv-e^{\lambda}\right)^2}{2} \end{pmatrix} dt.$$

The already known 1-parameter family $\tilde{\alpha}_\lambda$ of ZCRs of (4.3.6) can be obtained in a similar way by using the flow B_λ of the restricted symmetry $\bar{X}^{(\infty)} = 2\bar{Y}_4^{(\infty)} + \bar{Y}_3^{(\infty)}$, i.e., $\tilde{\alpha}_\lambda = B_\lambda^{\#}(\alpha)$. On the other hand, since $B_\lambda = C_\lambda \circ A_{2\lambda}$ with C_λ being the flow of the restricted symmetry $\frac{1}{2}\bar{Y}_3^{(\infty)}$, then $\tilde{\alpha}_\lambda = B_\lambda^{\sharp}(\alpha) = A_{2\lambda}^{\#}\left(C_\lambda^{\#}(\alpha)\right)$.

Example 4.3.4. The known 1-parameter family of ZCRs [24, 64] of the $DNSL^-$ Schrödinger system

$$\begin{cases} z_t = -v_2 + 6z^2z_1 + 2z_1v^2 + 4zvv_1 \\ v_t = z_2 + 4zz_1v + 2z^2v_1 + 6v^2v_1 \end{cases} \tag{4.3.7}$$

can be obtained by the following nonparametric $\mathfrak{sl}(2,\mathbb{C})$-valued ZCR

$$\alpha := \begin{pmatrix} -iv & z+\frac{i}{2} \\ \frac{i}{2}-z & iv \end{pmatrix} dx$$
$$+ \begin{pmatrix} i\left[\left(1-2z^2-2v^2\right)v-z_1\right] & (i+2z)\left(z^2+v^2\right)-v_1-z-\frac{i}{2} \\ (i-2z)\left(z^2+v^2\right)+v_1+z-\frac{i}{2} & i\left[z_1-\left(1-2z^2-2v^2\right)v\right] \end{pmatrix} dt,$$

with the use of a symmetry which is non gauge-like for α. Indeed, the algebra of classical symmetries of (4.3.7) is 4-dimensional and generated by the prolongations of vector fields

$$Y_1 = \partial_x, \quad Y_2 = \partial_t, \quad Y_3 = 2x\partial_x + 4t\partial_t - z\partial_z - v\partial_v, \quad Y_4 = -v\partial_z + z\partial_v.$$

In particular, the structure of the algebra of classical symmetries is

$$[Y_1, Y_2] = [Y_1, Y_4] = [Y_2, Y_4] = [Y_3, Y_4] = 0, \quad [Y_1, Y_3] = 2Y_1, \quad [Y_2, Y_3] = 4Y_2.$$

Now, in view of Theorem 4.2.8, one can check that Y_1, Y_2 and Y_4 generate the sub-algebra of gauge-like symmetries for α. On the contrary Y_3 is non gauge-like for α. Hence, by using the flow A_λ of the restricted symmetry $\bar{Y}_3^{(\infty)}$, one can construct a nontrivial 1-parameter family $\alpha_\eta := A_\lambda^{\#}(\alpha)$ of ZCRs of (4.3.7). To this end, since α only involves first-order jet-coordinates and A_λ induces the following transformation

$$x \mapsto e^{2\lambda}x,\ t \mapsto e^{4\lambda}t,\ z \mapsto e^{-\lambda}z,\ v \mapsto e^{-\lambda}v,\ z_1 \mapsto e^{-3\lambda}z_1,\ v_1 \mapsto e^{-3\lambda}v_1,$$

one gets

$$\alpha_\lambda := \begin{pmatrix} -i\eta v & z\eta + \frac{i\eta^2}{2} \\ \frac{i\eta^2}{2} - \eta z & i\eta v \end{pmatrix} dx +$$

$$\begin{pmatrix} i\eta\left[\left(\eta^2 - 2z^2 - 2v^2\right)v - z_1\right] & \eta\left[\left(i\eta + 2z\right)\left(z^2 + v^2\right) - v_1 - \eta^2 z - \frac{i\eta^3}{2}\right] \\ \eta\left[\left(i\eta - 2z\right)\left(z^2 + v^2\right) + v_1 + \eta^2 z - \frac{i\eta^3}{2}\right] & i\eta\left[z_1 - \left(\eta^2 - 2z^2 - 2v^2\right)v\right] \end{pmatrix} dt.$$

with $\eta = e^\lambda$, which is the already known 1-parameter family of ZCRs given in [24, 64].

Example 4.3.5. The Sawada-Kotera equation [57]

$$z_t = z_5 + 5z^2 z_1 + 5zz_3 + 5z_1 z_2, \tag{4.3.8}$$

admits the following nonparametric $\mathfrak{sl}(3,\mathbb{R})$-valued ZCR

$$\alpha := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{3} & -z & 0 \end{pmatrix} dx$$

$$+ \begin{pmatrix} -2z & z^2 - z_2 & 3z_1 - 3 \\ -1 - z_1 & z - zz_1 - z_3 & z^2 + 2z_2 \\ \frac{z^2}{3} - \frac{z_2}{3} & -z^3 - 1 - 3zz_2 - z_1^2 - z_4 & z_3 + z + zz_1 \end{pmatrix} dt.$$

Here, by applying the results of Sections 4.1 and 4.2, we will show how use α to construct a nontrivial 1-parameter family of ZCRs of (4.3.8).

To this end, we first observe that the algebra of classical symmetries of (4.3.8) is 3-dimensional and generated by the prolongations of vector fields

$$Y_1 = \partial_x, \quad Y_2 = \partial_t, \quad Y_3 = x\partial_x + 5t\partial_t - 2z\partial_z.$$

In particular, the structure of the algebra of classical symmetries is

$$[Y_1, Y_2] = 0, \quad [Y_1, Y_3] = Y_1, \quad [Y_2, Y_3] = 5Y_2.$$

Now, in view of Theorem 4.2.8, one can check that Y_1 and Y_2 generate the sub-algebra of infinitesimal symmetries which are gauge-like for α. On the contrary Y_3 is non gauge-like for α. Hence, by using the flow A_λ of the restricted symmetry $\bar{Y}_3^{(\infty)}$, one can construct a nontrivial 1-parameter family $\alpha_\lambda := A_\lambda^{\#}(\alpha)$ of ZCRs of (4.3.8). To this end, since α only involves fourth-order jet-coordinates and A_λ induces the following transformation

$$\begin{aligned} &t \mapsto e^{5\lambda} t, \quad x \mapsto e^{\lambda} x, \quad z \mapsto e^{-2\lambda}, \quad z_1 \mapsto e^{-3\lambda} z_1, \\ &z_2 \mapsto e^{-4\lambda} z_2, \quad z_3 \mapsto e^{-5\lambda} z_3, \quad z_4 \mapsto e^{-6\lambda} z_4, \end{aligned}$$

one gets that

$$\begin{aligned} \alpha_\lambda := &\begin{pmatrix} 0 & \mu & 0 \\ 0 & 0 & \mu \\ \frac{\mu}{3} & -\frac{z}{\mu} & 0 \end{pmatrix} dx \\ &+ \begin{pmatrix} -2\mu^3 z & \mu\left(z^2 - z_2\right) & 3\mu^2 z_1 - 3\mu^5 \\ -\mu^5 - \mu^2 z_1 & \mu^3 z - z z_1 - z_3 & \mu\left(z^2 + 2z_2\right) \\ \mu\left(\frac{z^2}{3} - \frac{z_2}{3}\right) & -\mu^5 - \frac{z^3 + 3zz_2 + z_1^2 + z_4}{\mu} & z_3 + \mu^3 z + z z_1 \end{pmatrix} dt, \end{aligned}$$

where $\mu = e^\lambda$.

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