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APPLICATION OF BOOLEAN PRE-ALGEBRAS TO THE FOUNDATIONS OF COMPUTER SCIENCE

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Monografia apresentada ao curso de Ciência da Computação, Departamento de Ciência da Computação, Instituto de Matemática, Universidade Federal da Bahia, como requisito parcial para obtenção do grau de Bacharel em Ciência da Computação.

Orientador: Steffen Lewitzka

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Resumo

Aumentar a expressividade de um sistema lógico é um objetivo de muitos campos na Ciência da Computação como o de Sistemas Formais, Construção de Conhecimento, Linguística, Lógica Universal e Teoria dos modelos. O aumento dessa expressividade pode ser obtido a partir do uso da Lógica não-Fregeana, uma lógica não-clássica. Nela, fórmulas com mesmo valor de verdade podem ter diferentes significados ou denotações (também chamadas situações). O seu uso corresponde à quebra do chamado Axioma de Frege, presente por exemplo, na lógica proposicional clássica. Pela não utilização do Axioma de Frege decorre o nome lógica não-Fregeana. Recentemente foi mostrado que há uma equivalência entre pré-algebras Booleanas e modelos na lógica não-Fregeana. Esse fato interligou áreas da lógica que já utilizavam essas pré-algebras como modelos. Nesta monografia é feita uma investigação da equivelência especificada e são expostas aplicações dessa semântica em áreas como: Lógica Modal, Teoria da Verdade, Lógica com Quantificadores e Lógica Epistêmica.

Palavras-chaves: pré-algebras Booleanas. lógica não-clássica. lógica não-Fregeana. semântica algébrica.

Abstract

Increasing the expressiveness of a logical system is a goal of many fields in Computer Science such as Formal Systems, Knowledge construction, Linguistics, Universal Logic and Model Theory. The increasing of this expressiveness can be reached by the use of non-Fregean Logic, a non classical logic. In non-Fregean Logic, formulas with the same truth value can have different denotations or meanings (also called situations). This concept breaks the Frege Axiom, reason for the name non-Fregean Logic. Recently, it was shown that there is an equivalence between Boolean pre-algebras and non-Fregean logic models. This fact linked fields which were already using Boolean pre-algebras to represent their semantic models. In this thesis, an investigation on this equivalence is done and applications are exposed in the fields of Modal Logic, Truth Theory, Logic with Quantifiers and Epistemic Logic.

Key-words: Boolean pre-algebras. non-classical logic. non-Fregean Logic. algebraic semantic.

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List of abbreviations and acronyms

AAL **A**bstract **A**lgebraic **L**ogic
SCI **S**etential **C**alculus with **I**dentity
TFA **T**ruth **F**uncitonal **A**xioms
IDA **I**dentity **A**xioms
PI **P**ropositional **I**dentity
SE **S**trict **E**quivalency
NFL **N**on-**F**regean **L**ogics

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1 Introduction

1.1 Motivation

Increasing the expressiveness of a logical system is a goal of many fields in Computer Science such as Formal Systems, Knowledge construction, Linguistics, Universal Logic and Model Theory. Regarding Linguistics, for example, the desire for an expressiveness close to the Natural Language motivated the non-Fregean Logic work.

Consider the sentences:
φ := “Salvador is the capital of Bahia”.
ψ := “The age of majority in Brazil is 18”.

Even though both formulas are true, they have very different meanings (denotations). It is the reason a person cannot say one of the sentences meaning the other. The problem is that inside the Classical Propositional Logic we cannot differentiate these two formulas. Actually, we cannot differentiate any formula from another one with the same truth value. Semantically, the Classical Propositional Logic has only two propositions: true and false (or 1 and 0) and valuation will map each true formula to the true proposition as well as each false formula to the false proposition.

In non-Fregean Logic, we can express more denotations. Formulas with the same truth value can have different denotations (also called situations). In this logic, true and false will not represent single propositions, but sets TRUE and FALSE, usually containing many propositions. The main feature introduced to this logic is an identity connective $\equiv$, such that $\psi_1 \equiv \psi_2$ are said to have the same denotation. Introducing the $\equiv$ operator, the informal idea discussed can be formalized saying: $(\psi_1 \equiv \psi_2) \rightarrow (\psi_1 \leftrightarrow \psi_2)$ is a theorem, but $(\psi_1 \leftrightarrow \psi_2) \rightarrow (\psi_1 \equiv \psi_2)$ is not valid. The last one is the called Frege Axiom.

1.1.1 More about intensions

Consider now a less distinct example:
φ := "In 2010, Salvador had a population of about 2.6 million.”
ψ := "In 2010, the capital of Bahia had a population of about 2.6 million.”

Even though these two formulas are equivalent for almost everyone in Bahia (who knows that Salvador is the capital of Bahia). For a tourist that comes to the city, they will not be. One can think in two different situations where one would be more appropriate than the other.

Finally, consider an even less distinct example :

φ := "I'm going with you and her"
ψ := "I'm going with her and you"

Maybe for different situations one of them could be preferable. For example, one could state the first person you say is more important than the second one.

1.2 Computer Science applications

1.2.1 Central applications

The construction of semantically closed languages (LEWITZKA, 2012e), the relationship between Godel's Incompleteness Theorem and the treatment of antinomies (such as Liar paradox) and many-valued logics (LEWITZKA, 2012d) are useful applications of non-Fregean Logic and the use of Boolean pre-algebras as a semantic model. On Linguistics, the formalization of a logical system which could express the Natural Language motivated the study of Non-Fregean logics (it can be found in (FOX; LAPPIN, 2001)). On the Theory of Computation, the isomorphism between in Intuitionistic Propositional Logic and Simply-typed Lambda (BRUCE; COSMO; LONGO, 1992) is one application.

1.2.2 Other applications

Besides the abstract algebraic interest on the Boolean pre-algebras in the field of Mathematical Logic, other Computer Science related fields are Artificial Intelligence, Multi-agent Systems, Distributed Systems and Security.

According to (DAVIS; MORGENSTERN, 1983), some applications of Epistemic Logic in Artificial Intelligence are: Planning, Motivation Analysis, Active Perception, Speech Acts, Speech Acts, Intelligent CAI, Non-monotonic logic, Design of Intelligent Systems. Furthermore, on Distributed Systems: a distributed system state can be characterized regarding epistemic logic too. On Security, epistemic logic is related to model access control issues on multi-agent systems. For example, we must guarantee an agent is not able to find out password information unless he owns the key. Finally, Epistemic logic can be defined in terms of Non-Fregean Logic, such as in (LEWITZKA, 2011).

1.3 Study proposal and scope

On this project, we perform a deeper analysis of algebraic semantics (called denotational semantics). Specifically, we will focus on Boolean pre-algebras, showing computer science applications and exploiting its relationship and equivalences with other logic semantics.

Specifically, on Chapter 2 we will introduce the Setential Calculus with Identity (SCI), the most basic non-Fregean Logic. Introducing its axiomatization, soundness and completeness proofs.

On Chapter 3, we will focus on the different ways to design a Logic Semantic. Specifying our central semantic, the algebraic. Then, we will detail our most important concept, the Boolean pre-algebras.

Finally, in Chapter 4, 5 and 6 are the applications of these Boolean pre-algebras. Specifically on Modal Logic, Truth theory, Epistemic Logic and Logic with quantifiers.

1.4 General and Specific Objectives

1.4.1 General

- Show the concept of Boolean pre-algebras as a logic's semantic representation.
- Show the high expressiveness of Boolean pre-algebra through the generalization power of a pre-order.

1.4.2 Specific

- Detail the equivalence between Boolean pre-algebra and other semantics such as Hyperintensional and SCI-models.
- Detail its use on different types of Logic such as Modal, Epistemic and Semantically closed Logics.
- Expose in more detail, the proofs for some results presented on (SUSZKO; BLOOM, 1972) and (LEWITZKA, 2014). Specifically, for the former: the SCI Completeness. For the second, the equivalence between SCI-models and Boolean pre-algebras and equivalence between alternative axiomatization utilized and the original one from the former.

2 Mathematical Logic Background: Non-Fregean Logic

2.1 Sentential Calculus with Identity

First, we will introduce the simplest Non-Fregean Logic, the Sentential Calculus with Identity (SCI). It was introduced in (SUSZKO; BLOOM, 1972). It does not have, for example, quantifiers, more complex concepts like that will be introduced in the next chapters. So, the idea behind the SCI is quite simple. The SCI is essentially a Classical Propositional Logic with an identity ($\equiv$) symbol. $\varphi \equiv \psi$ means that φ and ψ have the same denotation (also called by Suszko and others "situation"). It is a generalization from the classical propositional logic, where there are only two propositions: true and false (or 0,1). The semantic is given by a valuation that maps a formula to a proposition (we can have many true propositions and many false propositions). A model is given by a structured set of propositions and a valuation.

2.1.1 Language syntax

Definition 1. Let F_m be the set of formulas, it is the smallest set such as
(i) $V \cup \{\top, \bot\} \subseteq F_m$, where V is the set of propositional variables
(ii) If $\varphi, \psi \in F_m$, so $\neg\varphi, (\varphi \to \psi), (\varphi \equiv \psi), (\varphi \wedge \psi), (\varphi \vee \psi)$, $(\varphi \leftrightarrow \psi) \in F_m$. Being the connective basis $\{\neg, \to, \equiv\}$: and the following abbreviations: $\top := x_0 \to x_0$; $\bot := \neg\top$; $\varphi \wedge \psi := \neg(\varphi \to \neg\psi)$; $\varphi \vee \psi := \neg\varphi \to \psi$; $\varphi \leftrightarrow \psi := (\psi \to \psi) \wedge (\psi \to \varphi)$
(iii) The precedence of operators follows the order: $\neg, \equiv, \vee$ and $\wedge, \to, \leftrightarrow$, where $\to$ is right-associated while the others are left-associated, for $\leftrightarrow$ it can be any of them.

2.1.2 Deductive system

As pointed out from ($LEWITZKA$, 2012c), Frege is responsible for the formalization of the Propositional Logic as a formal calculus. Then, Bertrand Russell studied the classical logic as a deductive system. In the 20th century, Hilbert established the called Hilbert Calculus, where a deductive system is defined by many axioms and a few inference rules. For the propositional logic, there was only one rule (*Modus Ponens*) and four axioms (it can be reduced by three, using Lukasiewicz axiomatization). For the SCI, new axioms regarding the identity operator will be introduced as well as their semantic counterparts.

2.1.2.1 TFA: truth-functional axioms

They are the same axioms of the classical propositional logic:
A1. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
A2. $\varphi \to (\psi \to \varphi)$
A3. $\neg\varphi \to (\varphi \to \psi)$
A4. $(\varphi \to \psi) \to (\neg\varphi \to \psi) \to \psi$

2.1.2.2 IDA: identity axioms

(ID1) $\varphi \equiv \varphi$
(ID2) $\varphi \equiv \psi \to \neg\varphi \equiv \neg\psi$
(ID3) $\varphi_1 \equiv \psi_1 \to (\varphi_2 \equiv \psi_2 \to (\varphi_1 \to \varphi_2) \equiv (\psi_1 \to \psi_2))$
(ID4) $\varphi_1 \equiv \psi_1 \to (\varphi_2 \equiv \psi_2 \to (\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2))$

Inference Rule: Modus Ponens.

A derivation of a formula is defined in the same way as in classic propositional logic. **Definition 2.** A derivation of a formula φ (conclusion) from a set $\Phi \subseteq F_m$ (premises) is a series of formulas $\varphi_1 ... \varphi_n = \varphi$ such that for each formulas $i = 1...n$, one of the following is true:

I. φ_i is an axiom
II. $\varphi_i \in \Phi$ (premise)
III. φ_i is a result of Modus Ponens applied to derived formulas through

$$\frac{\varphi_j \qquad \varphi_j \to \varphi_i}{\varphi_i}$$

, where $j, k < i$

2.2 Semantic

A proposition domain can be defined as $\mathfrak{M} = (M, f_\perp, f_\top, f_\neg, f_\to, f_\equiv, \Gamma)$, where M is the universe of propositions and $f_\perp, f_\top, f_\neg, f_\to, f_\equiv$, operations with arity: $0, 0, 1, 2, 2$ respectively.

Definition 3. A valuation $\Gamma : C \mapsto M$ is a function which satisfies: $\Gamma(c) = f_c; \Gamma(\perp) = f_\perp; \Gamma(\top) = f_\top$; where f_c is the proposition referent to the constant c.

Definition 4. An assignment is a function $\gamma : V \mapsto M$ such that for all $\varphi, \psi \in V$, holds: $\gamma(x) = f_x$, being f_x a proposition on M correspondent to the variable x; $f_\neg(\gamma(\varphi))$; $\gamma(\varphi \to \psi) = f_\to(\gamma(\varphi), \gamma(\psi))$; $\gamma(\varphi \equiv \psi) = f_\equiv(\gamma(\varphi), \gamma(\psi))$

Definition 5. A valuation $\gamma : Fm(C)' \mapsto M$, such that:
$\gamma(\top) = \Gamma(\top) = f_\top$; $\gamma(\bot) = \Gamma(\bot) = f_\bot$; $\gamma(c) = \Gamma(c)$ and for the variables, it follows the previous definition.

A proposition is a denotation $\gamma(\varphi) \in \mathfrak{M}$ of a formula φ under valuation γ.

Definition 6. Let TRUE and FALSE sets such that $M = TRUE \cup FALSE$ and $TRUE \cap FALSE = \varnothing$. Then, $\mathfrak{M} = (M, f_\bot, f_\top, f_\neg, f_\rightarrow, f_\equiv, \Gamma)$ is a SCI-model if it satisfies:
(i): $f_\top \in TRUE, f_\bot \in FALSE$
(ii): $f_\neg(a) \in TRUE \iff a \in FALSE$
(iii): $f_\rightarrow(a, b) \in TRUE \iff a \in FALSE$ or $b \in TRUE$
(iv): $f_\equiv(a, b) \in TRUE \iff a = b$

2.2.1 Extensional and Intensional models

On Non-Fregean Logic, a proposition, also called an extension of a formula is the equivalence class $\overline{\varphi} = \{\varphi \equiv \psi\}$. So $\overline{\varphi}$ is the proposition correspondent to φ. Thus, the main concern is to define what we mean by formulas with propositional identity. It will change for the SCI-models we are considering.

On the one side, on **extensional models**, proposition identity is defined by formulas having the same truth value. So all formulas with the same truth-value denote the same proposition. It gives us a two-element Boolean Algebra with the elements: True and False. On the other side, an intension of a formula is defined by its syntax representation. We call an **intensional model** if extension and intension of a sentence we can be put in a 1-1 relationship. The main concern of Non-Fregean Logic is that we can easily design a model which different intensions are no longer indiscernible semantically. As an example, the following formulas do not denote the same proposition on intentional models: $(\varphi \vee \neg\varphi) \not\equiv \top$, $\varphi \not\equiv \neg\neg\varphi$, $(\neg\varphi \vee \psi) \not\equiv (\varphi \rightarrow \psi)$, $\neg(\varphi \vee \psi) \not\equiv (\neg\varphi \wedge \neg\psi)$, $\neg(\varphi \wedge \psi) \not\equiv (\neg\varphi \vee \neg\psi)$.

Summarily, SCI-models can be one of two types: Intensional or Extensional.
On extensional models, $(\mathfrak{M}, \gamma) \models \varphi \equiv \psi \iff \varphi \leftrightarrow \psi$
On intensional models, $(\mathfrak{M}, \gamma) \models \varphi \equiv \psi \iff \varphi = \psi$.

2.2.2 A note about SCI-models notation

The original notation for SCI-models, when introduced by Suszko (SUSZKO; BLOOM, 1972), were represented as: (1) an SCI-matrix $M = < \mathfrak{U}, B >$ such that $\mathfrak{U} =< A, \dot\neg, \dot\rightarrow, \dot\equiv >$ is the SCI-algebra with operations $\{\dot\neg, \dot\rightarrow, \dot\equiv\}$ and B $\subseteq A$ is a filter. (2) B is a prime filter (proper, closed and admissible according to his definitions).

In this senior thesis, we denote an SCI-model in an equivalent way, found in (LEWITZKA, 2013; LEWITZKA, 2014). An SCI-model is the tuple:
$\mathfrak{M} = (M, f_\bot, f_\top, f_\neg, f_\wedge, f_\vee, f_\rightarrow, f_\equiv, TRUE)$, being $TRUE \subseteq M$ an ultrafilter. Using our

notation $\mathfrak{M}$ represents M, TRUE represents B and $(M, f_{\neg}, f_{\rightarrow}, f_{\equiv})$ is the SCI-algebra, with M representing A and operations $\{f_{\neg}, f_{\rightarrow}, f_{\equiv}\}$ representing $\{\dot{\neg}, \dot{\rightarrow}, \dot{\equiv}\}$ respectively. Furthermore, the operations $f_{\bot}$, $f_{\top}$, $f_{\wedge}$ and $f_{\vee}$ are defined in terms of $\{f_{\neg}, f_{\top}\}$. The ultra-filter TRUE corresponds to the normal prime filter B used in (*SUSZKO*; *BLOOM*, 1972)

2.2.3 SCI Strong Completeness according to Suszko Notation

Strong completeness is divided into two steps: Soundness and Completeness. In order to show the Soundness, we must do the following: (I) prove all Truth Functional Axioms are correct, (II) prove all Identity Axioms are correct and (III) Prove Modus Ponens is correct.

For this section, we will use (SUSZKO; BLOOM, 1972) notation. So, we will use the framework proof for a formula θ: $\theta \in Sat_h(\Im, B)$, for every valuation h of $\Im$ $\iff$ $\theta \in TR(\Im, B)$, for every filter B in $\Im$ $\iff$ θ is valid in $\Im$, for every SCI-algebra $\Im$ $\iff$ θ is valid.

Furthermore, we must define the valuation function h behavior over the operators when the matrix $M =< \Im, B >$ is a model.

$h_{\neg} : h(\neg\varphi) \in B \iff h(\varphi) \notin B$ $h(\neg\varphi) \in B \iff h(\varphi) \notin B$
$h_{\rightarrow} : h(\varphi \rightarrow \psi) \in B \iff h(\varphi) \notin B$ or $h(\psi) \in B$.
$h_{\equiv} : h(\varphi \equiv \psi) \iff h(\varphi) = h(\psi)$

A proposition is the denotation $h(\varphi) \in M$ of a formula φ in a model $\mathfrak{M}$ under an assignment h.

2.2.3.1 Soundness ($\Phi \vdash \varphi \Rightarrow \Phi \Vdash \varphi$)

(I) TFA's are correct.
Here, the “truth-set"of an SCI-model is defined in terms of prime filters, as in (SUSZKO; BLOOM, 1972).

(A1) $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$.

Let h on $\Im$, where M = $< \Im, B >$ with B a normal prime filter, what makes M a model. Suppose $h(\varphi) \notin B$ (I). Then, $h(\varphi \rightarrow (\psi \rightarrow \chi)) \in B$ (II). Then we have that $h(\varphi \rightarrow \psi) \in B$ and $(\varphi \rightarrow \chi) \in B$. So, $h((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \in B$ (III). By (I), (III) and (h property 1.6 in Suszko1972-STEIIT) when M is a model, $h(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)) \in B$.

Suppose $h(\varphi) \in B$ (I). Now, $h(\psi) \in B$ or $h(\psi) \notin B$ (because B is prime). Therefore, let us analyze this valuation h on ψ.Suppose $h(\psi) \notin B$(II). Then $h(\psi \rightarrow \chi) \in B(III)$, by (II) and (II) $h(\varphi \to (\psi \to \chi))B(IV)$. Then, by (I) and (II) $h(\varphi \to \psi) \notin B(V)$, then $h((\varphi \to \psi) \to \varphi \to \chi) \in B$(VI). By VI and V, $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.

Suppose $h(\psi) \in B$. So, $\varphi \to \psi \in B$. Now, evaluating χ. Suppose $h(\chi) \in B$, so $h(\psi \to \chi) \in B$ and $h(\varphi \to \chi) \in B$. Then, $h((\varphi \to \psi) \to (\varphi \to \chi)) \in B$ as well as $(\varphi \to (\psi \to \chi)) \in B$. Therefore $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$. Finally, suppose $h(\chi) \notin B$, so $h(\psi \to \varphi) \notin B \Rightarrow h(\varphi \to (\psi \to \varphi)) \notin B$. By 1.6 h property on M model for implication connective $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.

(A2) $\varphi \to (\psi \to \varphi)$

Considering any SCI-matrix $M =< \Im, B >$, where $\Im$ is any SCI-algebra and B any filter (proper, closed and admissible set) and h a valuation of $\Im$. Assuming B prime and normal, making M a model ((SUSZKO; BLOOM, 1972), see 1.6).

Let h a valuation such that $h(\varphi) \in B$. It implies that $h(\psi \to \varphi) \in B$. We have $h(\varphi) \in B$ and $h(\psi \to \varphi) \in B$, so $h(\varphi \to \psi \to \varphi) \in B$. Suppose an h s.t. $h(\varphi) \notin B$. It implies that $h(\varphi \to \theta) \notin B$, for any θ. In special, for $\theta = (\psi \to \varphi)$ we have $h(\varphi \to (\psi \to \varphi)) \in B$.

(A3) $\neg\varphi \to (\varphi \to \psi)$

Let h a valuation on $\Im$. Suppose h satisfies ψ in $M =< \Im, B >$. So, h satisfies $\varphi \to \psi$. By 1.6 (SUSZKO; BLOOM, 1972), h satisfies $\omega \to (\varphi \to \psi)$, for any ω (in special, for $\omega = \neg\psi$ we have $\neg\varphi \to (\varphi \to \psi) \in B$).

Suppose a valuation h s.t. $h(\psi) \notin B$. How M is a model, $h(\neg\psi) \in B$. Then, as B is prime: $h(\varphi) \in B$ or $h(\varphi) \notin B$. Suppose $h(\varphi) \in B$. How $h(\varphi) \in B$ and $h(\varphi)\neg \in B$, $h(\varphi \to \psi) \in B$. Then, how $h(\neg\psi) \in B$, $h(\neg\psi \to (\varphi \to \psi)) \in B$. On the other side, suppose $h(\varphi) \notin B$, as $h(\psi) \notin B$, by 1.6 $h(\varphi)$ or $h(\psi) \in B$, $h(\varphi \to \psi) \in B$. By 1.6 again, $\omega \to (\varphi \to \psi)$, for any ω (in special for $\omega = \neg\varphi$ we achieve what we wanted).

(A4) $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$

Suppose $h(\varphi) \in B$. So, $h(\neg\varphi) \notin B$. So, $(\neg\varphi \to \psi) \in B$. Then, suppose $h(\varphi) \in B$, we have $(\varphi \to \psi)$. How $h(\neg\varphi \to \psi) \in B$, $h(\neg\varphi \to \psi) \to \psi) \in B$ and $(\varphi \to \psi) \in B$. Therefore $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)$. On the other hand, suppose $h(\varphi) \notin B$. By h property 1.6 when M is a model, $h(\varphi \to \psi) \notin B$. By 1.6 again $h((\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)) \in B$.

Suppose $h(\varphi) \notin B$. Then, $h(\varphi \to \psi) \in B$. Now in order to have $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi) \in B$, we must have $h((\neg\varphi \to \psi) \to \psi)$.Suppose $h(\psi) \in B$. As $h(\varphi) \notin B$ and

M is a model, $h(\neg\varphi) \in B$. So, $h(\neg\varphi \to \psi) \in B$. We have also that $h(\varphi) \in B$. Therefore, $h((\neg\varphi \to \psi) \to \psi) \in B$.

(II) IDA's are correct.
Consider an SCI-matrix M $=< \Im, B >$ with B a normal prime filter, what makes M a model.

(ID1) $\varphi \equiv \varphi$

We always have $h(\varphi) = h(\varphi)$. Then, by $h_{\equiv}$ $h(\varphi) = h(\varphi) \Rightarrow \varphi \equiv \varphi$.

(ID2) $\varphi \equiv \psi \to \neg\varphi \equiv \neg\psi$.

We will use here the explicitly the fact of the valuation is a homomorphism h from Language into the SCI-algebra $\Im$, therefore $h(\neg\varphi) = \neg h(\varphi)$ (I).
By the implication $h_{\to}$, it is sufficient show that if $\varphi \equiv \psi$ is valid, then it is valid $\neg\varphi \equiv \neg\psi$. Suppose $h(\varphi \equiv \psi) \in B$, so, $h(\varphi) = h(\psi)$(II). Then, by I and II: $h(\neg\varphi) = \neg h(\varphi) = \neg h(\psi) = h(\neg\psi)$. As $h(\neg\varphi) = h(\neg\psi)$, by $h_{\equiv}$, $\neg\varphi \equiv \neg\psi$.

(ID3) $\varphi_1 \equiv \psi_1 \to (\varphi_2 \equiv \psi_2 \to (\varphi_1 \to \varphi_2) \equiv (\psi_1 \to \psi_2))$

By the implication rule $h_{\to}$, it is sufficient show: given $\varphi_1 \equiv \psi_1(I)$ (for the first implication) and $\varphi_2 \equiv \psi_2(II)$ (for the second implication), show $(\varphi_1 \to \varphi_2) \equiv (\psi_1 \to \psi_2)$. Then, by (I), $h(\varphi) = h(\psi)$ and by (II) $h(\varphi) = h(\psi)$.

So, $h(\varphi_1 \to \varphi_2) =$
$h(\varphi_1) \to h(\varphi_2) =$, by the fact h is an homomorphism
$h(\psi_1) \to h(\varphi_2) =$, $h_{\equiv}$ applied to φ_1
$h(\psi_1) \to h(\psi_2) =$, $h_{\equiv}$ applied to φ_2
$h(\psi_1 \to \psi_2)$, by the fact h is an homomorphism
So, $h(\varphi_1 \to \varphi_2) = h(\psi_1 \to \psi_2)$, then considering $h \equiv$ we have: $(\varphi_1 \to \varphi_2) \equiv \psi_1 \to \psi_2$.

(ID4) $(\varphi_1 \equiv \psi_1) \to (\varphi_2 \equiv \psi_2) \to ((\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2))$

As done in ID3, by (I), $h(\varphi) = h(\psi)$ and by (II) $h(\varphi) = h(\psi)$ and it is sufficient to show that $(\varphi_1 \equiv \varphi_2) \equiv (\psi_1 \equiv \psi_2)$.

So, $h(\varphi_1 \equiv \varphi_2) =$
$h(\varphi_1) \equiv h(\varphi_2) =$, by the fact h is an homomorphism

$h(h(\varphi_1)) = h(h(\varphi_2)) =, h_{\equiv}$, by $h_{\equiv}$
$h(h(\psi_1)) = h(h(\varphi_2)) =, h_{\equiv}$, by $h_{\equiv}$, applied to φ_1
$h(h(\psi_1)) = h(h(\psi_2)) =, h_{\equiv}$, by $h_{\equiv}$, applied to φ_2
$h(\psi_1) \equiv h(\varphi_2) =, h_{\equiv}$ by $h_{\equiv}$
$h(\psi_1 \equiv \psi_2)$, by the fact h is an homomorphism
So, $h(\varphi_1 \equiv \varphi_2) = h(\psi_1 \equiv \psi_2)$, then considering $h \equiv$ we have: $(\varphi_1 \equiv \varphi_2) \equiv \psi_1 \equiv \psi_2$.

Remark: $(\varphi \equiv \psi) \to (\varphi \to \psi)$

Suppose $h(\varphi) \in B$. Then $\varphi \equiv \psi \Rightarrow$ (by $h_{\approx}$) $h(\varphi) = h(\varphi) \Rightarrow h(\psi) \in B$. So, $h(\varphi \to \psi) \in B$. By $(h_{\to})$ $h(\varphi \equiv \psi \to (\varphi \to \psi)) \in B$. Suppose $h(\varphi) \notin B$, by 1.6 also, $h(\varphi) = h(\psi)$. So $h(\varphi \to \psi) \in B$, (by $h_{\to}$) $\varphi \equiv \psi \to (\varphi \to \psi)$.

(III) Modus Ponens is correct.

Using the definition of Derivation, it follows from an induction in the length of the derivation φ from Φ.

2.2.3.2 Completeness ($\Phi \Vdash \varphi \Rightarrow \Phi \vdash \varphi$)

As in the classical SC, we will prove it by contraposition, where $\Phi \nvdash \varphi \Rightarrow \Phi \nVdash \varphi$. Suppose $\Phi \nvdash \varphi$, then by the deduction theorem $\Phi \cup \{\neg\varphi\}$ is consistent (step 1). Then, by the Lindenbaum's theorem, every consistent set admit an extension to a maximally consistent set (step 2). After that, a maximally consistent set admits a model (step 3). Considering the SCI definition of completeness, it must hold for all models and valuations h. By its contrapositive form, it is sufficient to find a specific valuation h on a specific Model M such that h(superset) $\in$ B. Then, every subset of a satisfiable set is satisfiable (step 4). It means there is a valuation h of $M_{\approx_B}$ such that h(φ) $\in B$.

Using the semantic from ($SUSZKO; BLOOM$, 1972). Let $M =< \Im, B >$ any SCI-matrix. Let's define $\approx_B$ on $\Im$ a relation such as: $a \approx_B b \iff a \equiv b \in B$, for all a,b $\in \Im$.

Proposition 1. This a relation of equivalence: Reflexivity: $a \approx_B a$, i.e. $a \equiv a \in B$, by I1, $a \equiv a \in B$ is an axiom, specifically it holds in any filter (admissible, closed, proper set) of any SCI-algebra $\Im$.

Symmetry: Suppose $a \approx_B b$. By def. $a \equiv b$. So, by 1.3 in $\Im$, I can substitute a by b. Substituting 'a' for 'b', we have $a \equiv a \in B$. Now, we can replace all occurrences of a by b, in specific, we can only substitute the first one, $b \equiv a$.

Transitivity: Suppose $a \approx_B b$ and $b \approx_B c$. By 1.3 ((SUSZKO; BLOOM, 1972)) G(a) $\equiv$ G(b) and I1, I can substitute a for b in the second expression, i.e. $\chi[b := a] \equiv c$, where $\chi = a\chi$ (a constant). So, $a \equiv c$.

We will consider a quotient structure $M/_{\approx_B} =< \Im/_{\approx_B}, B/_{\approx_B} >$ an SCI-matrix and an homomorphism h from the SCI-algebra $\Im$ to the SCI-algebra $\Im/_{\approx_B}$ ($h : \Im \rightarrow \Im/_{\approx_B}$).

$B/_{\approx_B}$ is a filter: it is closed, proper and admissible.

$\Im/_{\approx_B}$ is an SCI-algebra

A natural map of an element to its class of equivalence $h_{\approx B} \rightarrow C_M = C_{M/\approx}$. Natural map makes the completeness.

By defining this binary congruence relation that is an equivalence relation, SCI completeness follows the called Lindenbaum-Tarski method. It constructs a quotient matrix obtained by factoring the algebra by its congruence relation $Q_{/\approx} =< M_{/\approx}, B_{/\approx} >$, where $M_{/\approx}$ is the quotient algebra of M modulo and $B_{/\approx}$ is filter on $M_{/\approx}$ that represents the set of equivalence classes of B. Then, we have a matrix homomorphism of M onto $M_{/\approx}$ which maps each element to its equivalence class, as found in (SUSZKO; BLOOM, 1972): $\varphi \in B \iff B \vdash \varphi \iff [\varphi] \in B_{/\approx}$. So, for an element B $\nvdash \varphi$, there is a model s.t. for an algebra $M/\approx$ and a valuation v, $< M, v >$. Then, the completeness proof is based on the fact of:

(i) This Lindenbaum Algebra is a maximally consistent set. It is clearly a Boolean algebra that is a maximally consistent set.
Let a Lindenbaum Algebra S over a logic L; a set Φ is an L-maximal consistent set of formulas if and only if $B_{/\approx} = \{|\varphi_L| : |\varphi \in \Phi\}$ is an ultrafilter of S.

(ii) Every maximal consistent set has a model.
It is supported by the fact that this Ultra-filter in fact always exists. It is called Ultrafilter's Theorem, a model based proof can be found in (BLACKBURN; BENTHEM; WOLTER, 2006): Any proper filter of a Boolean algebra B can be extended to an ultrafilter of B. More useful information on Ultrafilters can be found on (GALVIN, 2009),komjath2008ultrafilters and (ODIFREDDI, 2000).

(iii) A model of the maximal consistent set corresponds to a model to the initial set B.
A subset of a boolean pre-algebra is an ultrafilter if and only its characteristic function is a Boolean homomorphism to the two-element Boolean pre-algebra (POLLARD, 2008), chapter. 20. Furthermore, given the homomorphism to the boolean algebra on the $B_{/\approx}$ that corresponds to true,false, the inverse image of true is the ultrafilter B (BLACKBURN; BENTHEM; WOLTER, 2006).

2.2.3.3 Notes on Factorization and Quotient structures

Let us consider a very basic example to grasp the concept of factorization of an algebra and its quotient structure. We will construct a two-element Boolean Algebra.

Consider the arithmetic operation $\equiv (mod\, 2)$ congruence. Let $(Z_{10}, +, 0)$ an algebra (in this case a group), where $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then, we define $a \equiv b \ (mod\ 2)$ as the usual modulo operation for this algebra. Then, the factorization of the algebra Z_{10} by the congruence relation ($a \approx b \iff a \equiv b \ (mod\ 2)$), generates the quotient structure $Z_{10/\approx} = \{\overline{0}, \overline{1}\}$, with equivalence classes $\overline{0} = \{0, 2, 4, 6, 8\}$ and $\overline{1} = \{1, 3, 5, 7, 9\}$.

We can see then each equivalence class as an element w.r.t the quotient structure or a set w.r.t the original structure. For example, $\overline{0}$ represents an element w.r.t. $Z_{10}/\approx$ and $\overline{0}$ represents a set of element w.r.t. Z_{10}, where $\overline{0} = \{0, 2, 4, 6, 8\}$.

Now, let us change back to our context. In the Completeness Theorem, we define a congruence relation, $\varphi \approx \psi \iff \Phi \vdash \varphi \equiv \psi$, that means in semantic counterpart $f_{\equiv}(\varphi, \psi) \in \Phi$. Then, we specify the TRUE set as the elements of the quotient algebra that is in the original set Φ, where Φ is a maximal theory. It follows $\varphi \in \Phi \iff \overline{\varphi} \in TRUE \iff \alpha(\varphi) \in TRUE \iff (\mathfrak{M}, \varphi) \models \varphi$. Then, considering the quotient structure (algebra factorized by the congruence created), the new algebra will be a Boolean Algebra, a maximally consistent set. It will ensure the completeness of our Logic, and then we can work extending it. It is, we create an algebra such that $\{\overline{0}, \overline{1}\}$, where $\overline{0}$ represents the elements which are not in the TRUE set, and $\overline{1}$ elements that are in the TRUE set.

2.3 Equivalence proof of another possible axiomatization for IDA

Another IDA axiomatization is the following:
(ALT1) $\varphi \equiv \varphi$,
(ALT2) $(\varphi \equiv \psi) \to (\varphi \to \psi)$ and
(ALT3) $(\psi \equiv \psi') \to (\varphi[x := \psi] \equiv \varphi[x := \psi'])$

Given (ALT1) and (ALT2) are axioms on (ID) also. So, we just have to show the equivalence of the above formula (ALT3) with (ID2) (ID3) and (ID4).

(ID) ⇒ (ALT): structural induction proof in the construction of formulas.

For the base case, $\varphi := x$. So, we have $\psi \equiv \psi' \to \psi \equiv \psi'$, that is trivially valid i.e. $h(\psi \equiv \psi' \to \psi \equiv \psi') \in B$. Then, we must define constructively, the other possible cases:
$\varphi := \neg x$, valid by (ID2)
$\varphi := x \to \theta$, valid by (ID3)
$\varphi := \theta \to x$, valid by (ID3)
$\varphi := x \equiv \theta$, valid by (ID4)

Note: $\theta \equiv x$, is trivally valid also, since $x \equiv \theta$ is valid. Therefore, we proved for the possible φ substitutions.

(ALT) ⇒ (ID): On (ALT3), φ can be any consistent formula and x a free variable on φ (important we we will handle quantifiers in the logic). So, we can map values of φ from (ALT3) to map to the sentences (ID2), (ID3) and (ID4) through $\{\neg, \to, \equiv\}$.

(I) From (ALT3) to ID2: pick $\varphi := \neg x$.
(II) From (ALT3) to ID3: pick $\varphi := x \to \theta$
(III) From (ALT3) to ID4: pick $\varphi := x \equiv \theta$

Formulated sentences:
(result I) $\psi \equiv \psi' \to \neg\varphi \equiv \neg\psi'$
(result II) $\psi \equiv \psi' \to (\psi \to \theta) \equiv (\psi' \to \theta)$
(result III) $\psi \equiv \psi' \to (\psi \equiv \theta) \equiv (\psi' \equiv \theta)$

For (II) and (III), we should notice that the main change in the formula from (ID) formulas to (a) constructed formulas is $\varphi_2 \equiv \psi_2$, is now mapped for the same variable θ. It is possible, once $\varphi_2 \equiv \psi_2$ implies $h(\varphi_2) = h(\psi_2)$, we just mapped it to the same variable θ, s.t. $h(\theta) = h(\varphi) = h(\psi)$.

3 Boolean pre-algebras and a close view on Semantic

3.1 Algebraic Semantic

3.1.1 Abstract Algebraic Logic (AAL)

The AAL is the general theory of the algebraization of logical systems. One of the precursors was George Boole, who names the most important algebraic structure of this senior thesis. According to (FONT; JANSANA; PIGOZZI, 2003), Boole, De Morgan and others were the precursors to the development, on 19th century, of logical systems focused in the logical equivalence rather than the truth as the primitive for a logical predicate. The algebraic view of the logic started to be explored. His work, specifically, resulted in the theory of Boolean Algebras, despite De Morgan, Schroder and others work end up on the Relation Algebras. Then, the first connection between the metalogical properties of a logic and the algebraic properties of its algebraic counterpart (a process known as a bridge theorem) was done by Alfred Tarski. Specifically, he developed the connection between a Boolean algebra and the Classical Propositional Calculus, by the Lindenbaum-Tarski method. Tarski worked on the Lindenbaum idea of "viewing a set of formulas as an algebra with operations induced by the logical connectives. The logical equivalence is a congruence relation on the algebra of formulas. Then, the quotient algebra turns out to be a free Boolean Algebra, as pointed out on (FONT; JANSANA; PIGOZZI, 2003).

Finally, this senior project meets the current interest of AAL in the investigation of the relationship between meta-logical and algebraic properties of logic and its algebra semantic counterpart. This and others current AAL interests were born after the development of such connection by Tarski. According to (FONT; JANSANA; PIGOZZI, 2003), the traditional algebraic investigation changed from the systematic study of a broad class of logics to a process which associates a class of algebras with an arbitrary logic.

3.2 Boolean pre-algebras

According to (LEWITZKA, 2013), Boolean algebras are usually defined as a non-empty set, three operations: meet, join, complement, and two elements: supremum (greatest) and infimum (smallest) such that certain equations are satisfied. We will define Boolean pre-algebra as an algebra formed by a pre-order such that its quotient structure forms a Boolean algebra. In the following subsections, we will clarify these concepts. A remark here is we can introduce the algebra operators also in terms of ($\rightarrow$) and ($\neg$). In the following definitions, we use a convenient way to define the necessary background to define a Boolean pre-algebra. Then, we present some results found in (LEWITZKA, 2012b).

Definition 7. A pre-order $\leq$ is a binary relation defined over a set which for any a,b,c elements, it is valid:
$a \leq a$ (Reflexivity)
If $a \leq b$ and $b \leq c$, then $a \leq c$ (Transitivity)

Definition 8. A pre-ordered set is the pair $(S, \leq)$, where S is a set and $\leq$ a preorder over it.

Definition 9. A partially ordered set is the pair $(S, \leq)$, where S is a set and $\leq$ a partial order (antisymmetric pre-order). An antisymmetric relation on a set can be defined as for all a,b elements: $a \leq b$ and $b \leq a \Rightarrow a = b$.

Definition 10. A lattice is a partially-ordered set which each two elements have a unique supremum $(\top)$ and a unique infimum $(\bot)$.

Definition 11. Let the structure $\Psi = (M, f_\top, f_\bot, f_\neg, f_\vee, f_\wedge, f_\top, \leq_\Psi)$ which
1. M is the universe
2. $f_\top, f_\bot, f_\neg, f_\vee, f_\wedge, f_\rightarrow$ are functions on $M \rightarrow M$ (operations on M) with arity 0,0,1,2,2,2 respectively.
3. $\leq_\Psi onM$ is a pre-order.
It is a Boolean prealgebra if (according to (LEWITZKA, 2014)):
Defining the **the congruence relation** $\approx_\Psi$ as $a \approx_\Psi b :\iff a \leq_\Psi b$ and $b \leq_\Psi a$. It **is an equivalence relation on M** and the **quotient algebra**:
$\Psi_{/\approx_\Psi} = (M_{/\approx_\Psi}, \overline{f}_\top, \overline{f}_\bot, \overline{f}_\neg, \overline{f}_\vee, \overline{f}_\wedge, \overline{f}_\top, \leq_{M_{/\approx_\Psi}})$ **is a Boolean algebra** with lattice order $\leq' a \leq_{\Psi_{/\approx_\Psi}} b \iff a \leq_\Psi b$.
4. $\overline{f}_\top, \overline{f}_\bot, \overline{f}_\neg, \overline{f}_\vee, \overline{f}_\wedge, \overline{f}_\top$ are induced operations for supremum, infimum elements, complement and implication respectively on the set of congruence classes $\in \Psi_{/_M}$.

Remark: the word algebra in the context above is referring to the mathematical structure lattice.

Definition 12. A filter F with respect to $\leq_\Psi$, according to (LEWITZKA, 2014), in a Boolean pre-algebra Ψ is a non-empty subset F $\subseteq M$, s.t. for all a,b $\in M$ the filter axioms holds:

if $a \in F$ and $a \leq_\Psi b$, then $b \in F$
if a,b $\in F$, then $f_\wedge(a, b) \in F$
$f\bot \notin F$

On boolean algebras, the concept of prime filters and maximal filters coincide. Also the concept of ultrafilters and prime filters. So, an **ultrafilter** with regard to $\leq_\Psi$ is a maximal filter with regard to $\leq_\Psi$.

3.2.1 Equivalence of Boolean pre-algebras and SCI-models

Definition 13. Let $\Psi = (M, \leq_\Psi, f_\perp, f_\top, f_\wedge, f_\vee, f_\rightarrow, f_\rightarrow)$ a Boolean pre-lattice with a pre-ordered set ($\leq_\Psi$, M) and induced operations infimum ($f_\perp$), supremum ($f_\top$), complement ($f_\neg$), join ($f_\vee$), meet ($f_\wedge$) and implication ($f_\rightarrow$).

Definition 14. $\mathfrak{M} = (M, TRUE, f_\perp, f_\top, f_\wedge, f_\vee, f_\rightarrow, f_\equiv)$ a SCI-model.

We want to show that Ψ and $\mathfrak{M}$ are equivalent. It is done in two parts. First, proving a supporting lemma, then proving the desired statement. This result is found in the literature on (LEWITZKA, 2014) and it is re-explained here.

Part I - Supporting Lemma: Let Ψ a Boolean pre-lattice with preorder (so it will be valid for partial and total orders) $\leq_\Psi$ and F a filter w.r.t. $\leq_\Psi$. The following conditions are equivalents:

(i) $a \leq_\Psi b \iff f_\rightarrow(a, b) \in F$, for all a, b $\in M$

(ii) $F = \{a \in M |\ a \approx_\Psi f_\top\}$

(iii) F is the smallest filter. It is the intersection of all (ultra)filters with regard to $\leq_\Psi$.

Proof: (i) $\Rightarrow$ (ii)
We want to show $a \approx_\Psi b \iff$ (def. Boolean pre-algebra equivalence relation) $a \leq_\Psi b$ and $a \leq_\Psi b \iff$ (by 3) $f_\rightarrow(a, f_\top) \in F$ and $f_\rightarrow(f_\top, a) \in F$. So we have to show $f_\rightarrow(a, f_\top) \in F$ (I) and $f_\rightarrow(f_\top, a) \in F$ (II).
(I): Let "a" $\in F$. We have a tautology $f_\rightarrow(a, f_\rightarrow(f_\top, a))$. As F is a filter and "a"$\in F$, by the first filter property ($a \leq b$ and a $\in F$, then b $\in F$) and $a \leq b \iff f_\rightarrow(a, b) \in F$, then $f_\rightarrow(f_\top, a) \in F$.
(II): We have $\Lambda = f_\rightarrow(a, f_\rightarrow(f_\top, a))$ such that $f_\rightarrow(a, \Lambda) \in F$, because Λ is a tautology. Then, we have $f_\top \equiv \Lambda$. Therefore, $f_\top(a, \top) \in F$.

Proof: (ii) $\Rightarrow$ (iii)
Let G be any filter with regard to $\leq_\Psi$.
For every a $\in F$. by (ii), $a \approx_\Psi f_\top$, so $f_\rightarrow(a, f_\top) \in F$ and $f_\rightarrow(f_\top, a) \in F$ (I). Then, $f_\top$ is in any filter, so $f_\top \in G$ (II). Now, by (I), (II) and the fact G is a filter, $a \in G$. So, for every $a \in F$, $a \in G$, for any filter G. Therefore $F \subseteq G$. Thus, F is the smallest filter.

Proof: (iii) $\Rightarrow$ (i)
We have the property that $a \leq_\Psi b \iff \overline{a} \leq' \overline{b}$ Let the quotient algebra Ψ/F, where F is a filter and the quotient algebra has lattice order $\leq'$. Let us define $\overline{a} \leq' \overline{b} \iff \overline{f}_\rightarrow(a, \overline{b}) \in \Psi/\approx_\Psi$. It is Boolean algebra, so it is valid $f_\rightarrow(\overline{a}, \overline{b}) = \overline{f}_\top$.
Then, we define a homomorphism (canonical isomorphism) mapping $a \mapsto \overline{a}$, where $\overline{a}$

means a's equivalent class, where $a \leq b \iff \bar{a} \leq' \bar{b}$. Then, by (iii), F is the smallest filter of the Boolean pre-algebra. This map, maps F to the smallest filter of the quotient algebra that is $\overline{f}_{\top}$, i.e. $f_{\rightarrow}(a,b) \in F$.

Part II - Theorem: Ψ and $\mathfrak{M}$ are equivalent

(a) Representing a Boolean pre-algebra from a SCI-model:
Let $\mathfrak{M} = (M, TRUE, f_{\perp}, f_{\top}, f_{\wedge}, f_{\vee}, f_{\rightarrow}, f_{\equiv})$ a SCI-model.

1. Consider $F \subseteq M$, such that $F = \cap\{T \subseteq M | (M, T, f_{\perp}, f_{\top}, f_{\wedge}, f_{\vee}, f_{\rightarrow}, f_{\equiv})$ is also a SCI-model$\}$. It means, F is the intersection of all T's on M that forms also a SCI-model.

2. Define a pre-order $\leq_{\Psi}$ s.t. $a \leq_{\Psi} b :\iff f_{\rightarrow}(a,b) \in F$, defining a pre-ordered set $(M, \leq_{\Psi})$.

3. $\Psi = (M, f_{\perp}, f_{\top}, f_{\neg}, f_{\vee}, f_{\wedge,\vee}, \leq_{\Psi})$ is a Boolean pre-algebra.

Proposition 2. The sets T are ultra-filters and F is the smallest filter with regard to $\leq_{\Psi}$. Proved on the Supporting Lemma.

Proposition 3. Ψ is a pre-algebra.

Proposition 4. Given, Ψ a boolean (pre)algera. Let TRUE' an ultra-filter w.r.t. $\leq_M$ and $f_{\equiv}$ s.t. $f_{\equiv}(a,b) \in TRUE \iff a = b$, for all a,b $\in M$.
Let $\mathfrak{M} = (M, TRUE, f_{\perp}, f_{\top}, f_{\vee}, f_{\wedge}, f_{\rightarrow}, f_{\equiv})$, $\mathfrak{M}$ is a SCI-model.

(b) Representing a Boolean pre-algebra from an SCI-model

Let $\Psi = (M, f_{\perp}, f_{\top}, f_{\neg}, f_{\vee}, f_{\wedge,\vee}, \leq_{\Psi})$ a Boolean pre-algebra. Take a filter TRUE on pre-ordered set $(M, \leq_{\Psi})$ and define a binary operRation $f_{\equiv}$ on M, such that $f_{\equiv}(a,b) \in TRUE \iff a = b$, for all a,b $\in M$.

Proposition 5. $\leq'_{\mathfrak{M}}$ is a pre-order.

Reflexive: $f_{\rightarrow}(a,a)$ represents a tautology. A tautology is part of any filter, so $f_{\rightarrow}(a,a) \in F$. Transitive: Let $f_{\rightarrow}(a,b) \in F$ and $f_{\rightarrow}(b,c) \in F$, we must prove: $f_{\rightarrow}(a,c) \in F$. Suppose $a \in F$. Then, $f_{\rightarrow}(a,b) \in F$. Let $a \in F$, $a \leq b$ and F is a filter, so $b \in F$. Now, we do the same for b: $b \in F$ and $f_{\rightarrow}(b,c) \in F$, therefore $c \in F$. Finally, $a \in F$ and $c \in F$, $f_{\rightarrow}(a,c) \in F$. Suppose now $a \notin F$, then it follows directly that $f_{\rightarrow}(a,c) \in F$.

Proposition 6. If the following conditions hold, F is a filter:
$a \in F$ and $f_{\rightarrow}(a,b) \in F$, then $a \in F$
If a,b $\in F$, then $f_{\wedge}(a,b) \in F$
$f_{\perp} \notin F$

Proposition 7. All sets are ultrafilters with regard to $\leq_{\mathfrak{M}}$.
It follows from the definition of SCI-models.

Now, $F\ \{a \in M | a \approx_\Psi f_\top$ and $\approx$ is the congruence relation.

(c) The SCI-model generated on (a) corresponds to the stated on (b). The Boolean pre-algebra stated on (a) corresponds to the one generated on (a).

Let Ψ a Boolean pre-algebra. Then obtain an SCI-model $\mathfrak{M}$ following (a). From $\mathfrak{M}$, use (b) to obtain a Boolean pre-algebra Ψ'. By the Supporting Lemma, the pre-order $\leq_\Psi$ is the same as Ψ', also $\Psi = \Psi'$ and the same for the operations. Then, $\mathfrak{M} = (M, TRUE, f_\perp, f_\top, f_\wedge, f_\vee, f_\rightarrow, f_\equiv)$ is a SCI-model. A remark here is that on some logics (e.g. modal logics), we will add NEC in the representation of our SCI-model. NEC will correspond to the F filter formed.

3.2.2 Heyting Algebra

The explicit use of Heyting Algebras is considered in the algebraic semantic of the Intuitionist Logic. The Heyting Algebra is the algebraic semantic representation to the Intuitionistic Logic, providing soundness and completeness. While in the classical logic $\neg\neg\varphi \equiv \varphi$ is valid, it is not valid in the Intuitionistic Logic. Furthermore, the Heyting Algebra used as semantic for Intuitionistic Logic is a Boolean pre-algebra in the sense that the quotient algebra of the Heying algebra is a Boolean algebra. Another interesting point of Heyting Algebras is that a Boolean algebra is also a Heyting algebra, but it is a very specific case. More information can be found in (LEWITZKA, 2015).

4 Application: Modal Logic

According to (GOLDBLATT, 2003), “modality is any word or phrase that can be applied to a given statement S to create a new statement that makes an assertion about the mode of the truth of S: about when, where or how S is true, or about the circumstances under which S may be true.” A pair of modalities usually characterizes a Modal Logic System, which one modality usually means a universal property and the other an existential property. The most common modalities pairs are: “necessarily"and “possibly". Other interesting pairs are: “after the program finishes"and “the program enables”, seen on (GOLDBLATT, 2003). For the universal modality usually, the symbol used is $\Box$ and for the dual existential $\Diamond$. It is a non-classical logic in terms that it extends the classical logics by the modality operators. We can express any of those operators as the other, plus some operators of Classical Propositional Logic. We will introduce in our Logic the necessary operator $\Box$. According to (GARSON, 2016), "The applications of modal logic to mathematics and computer science have become increasingly important.” Some modalities application are on Artificial Intelligence, Distributed Systems and Theory of Computation. They are usually related to Epistemic Logic, which is a subfield of modal logic.

The traditional link between propositional identity ($\varphi \equiv \psi$) and modal logic’s necessity ($\Box\varphi$) is defining one in terms of the other. However, these two concepts can also be defined independently, and this was the main result of the NFL and Boolean pre-algebras approach. This new denotational semantic will refine Modal logics expressiveness using the Propositional Identity ($\equiv$).

4.1 History on Strict Equivalence and S1-S4 Lewis Systems

According to (HUGHES; CRESSWELL, 1996), even though what we call nowadays Modal Logic were discussed by ancient authors such as Aristotle and medieval logicians, the first steps toward modern modal logic and its concept of strict equivalence seems to have been taken in the 19th century by the Hugh Mac Coll. However, a formal axiomatization of a system with a strict implication seems only to be constructed by C.I. Lewis after the Principia Mathematica (1910) which was used as a base axiomatization.

The undesired result on Propositional Calculus is due the theorem $(p \to q) \vee (q \to p)$ (3) obtained from the two theorems $p \to (q \to p)$ (1), $\neg p \to (p \to q)$ (2). These two theorems can be found on any axiomatization of Propositional Calculus. The theorem (3) is sometimes called the paradox of implication and wanted to be avoided by logicians. It is not actually a paradox, but it is a not desired result for a generalized logic. So, in order to avoid this, instead of rejecting the axioms (1) and (2), Lewis axiomatized a new class of systems considering a new implication, called strict implication (also called necessary implication). It created a strong sense of implying. There are some propositions which neither of them implies the other. Lewis used to use the symbol p - q, representing: "It is impossible that p should be true without q's being true too": p -3 q. It was then reformulated equivalently: "Is necessary that if p is true, so is q": $\Box(p \to q).So$p -3 q := $\Box(p \to q)$. As the $\{\to, \neg, \Box$ are connective basis for modal logic and strict implication can be defined as $\Box(\to \phi)$, the study of a strict implication is a way to see the whole modal logic.

4.2 Linking Propositional Identity (PI) and Strict Equivalence (SE)

On the one hand, the Propositional Identity ($\varphi \equiv \psi$) is the main concept for the development of the NFL. In the other hand, the necessity operator ($\Box\varphi$) is the main concept on Modal Logics. Historically, the two main ways to relate these two logic notions were defining: the necessity (unary) operator in terms of PI ($\Box\varphi := \varphi \equiv \top$, where $\top$ is any tautology) or defining the identity (binary) operator in terms of the called SE ($\varphi \equiv \psi := \Box((\varphi \to \psi) \wedge (\psi \to \varphi))$). Examples of these uses can be seen in (LEWITZKA, 2013), (LEWITZKA, 2015).

4.2.1 Example: Modal System S3 (Lewis Axiomatization)

We will first consider the Lewis system $S3$ which is, as proved on (LEWITZKA, 2013), the smallest Lewis System which PI can be expressed in terms of SE.

In the first moment, we will define PI in terms of SE, i.e. $\varphi \equiv \psi : \Box(\varphi \leftrightarrow \psi)$ based on (LEWITZKA, 2013) and see that the semantic of this logic is done using Boolean algebras. In the second moment, we will consider PI as part of the language, following (SUSZKO; BLOOM, 1972) approach, and see that actually PI refines SE, i.e. $\varphi \equiv \psi \to \Box(\varphi \leftrightarrow \psi)$, as seen on (LEWITZKA, 2014). At this logic, the semantic will be represented using Boolean pre-algebra, a semantic generalization of the past example.

4.2.1.1 PI in terms of SE: Deductive system

Axioms:
(i) formulas which have the form of a propositional tautology
(ii) $\Box\varphi \to \varphi)$
(iii) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
(iv) $\Box(\varphi \to \psi) \to \Box(\Box\varphi \to \Box\psi)$

Definition 15. $\varphi \equiv \psi := \Box((\varphi \to \psi) \wedge (\psi \to \varphi))$

Inference rules.
Modus Ponens (MP): $\varphi, \varphi \to \psi$, then $\vdash \psi$
Axiom of Necessitation (AN): Let φ an axiom, then $\vdash \Box\varphi$.

Two important results to this system are:
Lemma 1. Deduction Theorem: If $\Phi \cup \{\varphi\} \vdash \psi$, then $\Phi \vdash \varphi \to \psi$
Lemma 2. $\Box\varphi \leftrightarrow (\varphi \equiv \top)$ is a theorem, for any φ

An important note on Lemma 2. is that it says the proposition φ is necessary if and only if φ and $\top$ denotes the same proposition. So, for each φ necessary, it denotes the same proposition as $\top$. In other words, as pointed out by (LEWITZKA, 2013), it means "there is exactly one necessary proposition", the proposition denoted by $\top$ and all the others necessary propositions. It was called “principle N"by (LEWITZKA, 2013).

4.2.1.2 PI in terms of SE: Semantic

A S3 model can be defined as the Boolean algebra:
$\Psi = (M, f_\perp, f_\top, f_\neg, f_\wedge, f_\vee)$ a Boolean algebra with a pre-ordered set (M,$\leq$), where M is a non-empty set of propositions and $\leq$ is the lattice order, operations complement ($f_\neg$), meet ($f_\wedge$), join ($f_\vee$), bound elements infimum ($f_\top$) and supremum ($f_\perp$) and with an ultrafilter TRUE $\subseteq M$ with regard to $\leq$ (the set of true propositions) and induced unary operation $f_\Box$, satisfying (i)-(vi) for all $a, b, c \in M$.

(i) $f_\top \in$ TRUE, $f_\perp \notin$ TRUE
(ii)$f_\neg(a) \in TRUE \iff a \notin TRUE$
(iii)$f_\to(a,b) \in TRUE \iff a \notin TRUE$ or $b \in TRUE$. It also satisfy conditions for the other operators
(iv) $f_\Box(a) \in TRUE \iff a = f_\top$
(v) $f_\Box(a) \leq a$
(iv) $f_\Box(f_\to(a,b)) \leq f_\to(f_\Box(a), f_\Box(b))$
(vi) $f_\Box(f_\to(a,b)) \leq f_\Box(f_\to(f_\Box(a), f_\Box(b)))$
It also satisfy Boolean algebra properties such as $f_\to(a,b) = f_\top$.

4.2.2 PI in terms of SE: representing through SCI-model

The Boolean Algebra Ψ defined can be represented as a SCI-based S3-model, doing for all $a, b \in M$, $a \leq b \iff f_\to(a,b) \in TRUE$, where $TRUE = f_\top$. Then, we define the SCI-model $\mathfrak{M} = (M, TRUE, f_\perp, f_\top, f_\neg, f_\wedge, f_\vee, f_\Box)$ that is the S3-model.

4.3 Logic $S3_{\equiv}$: PI and SE independently, deductive system

A set of formulas $Fm(C)$ defined over:
V, set of variables
C, set of constants $\top, \bot \in C$
Connectives $\{\bot, \top, \neg, \rightarrow, \equiv, \Box\}$, with logical connectives $\neg, \rightarrow, \bot, \top$, identity connective $\equiv$ and modal connective $\Box$ for necessity.

Axioms:
(i) propositional tautologies
Modal Lewis System S3 axioms:
(ii) $\Box\varphi \rightarrow \varphi)$
(iii) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(iv) $\Box(\varphi \rightarrow \psi) \rightarrow \Box(\Box\varphi \rightarrow \Box\psi)$
IDA axioms (in the a axiomatization):
(v) $\varphi \equiv \varphi$
(vi) $(\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \psi)$
(vii) $(\varphi \equiv \psi') \rightarrow (\varphi[x := \psi] \equiv \varphi[x := \psi'])$, if x $\in V$, also caleld (SP).

Inference rules: also (MP) Modus Ponens and (AN) Axiom Necessitation. AN application is limited to (i)-(iv).

Derivation is defined in the same way as in classic propositional logic.

4.3.1 PI and SE indepedently, denotational semantics

As an example, we will define the denotational semantics for S4, being the one not direct defined yet in the current literature. We will follow the same process as done in (SUSZKO; BLOOM, 1972).

Definition 16. Propositional domain

Let us represent the universe of the propositions
$\Psi = (M, TRUE, NEC, f_{\bot}, f_{\top}, f_{\Box}, f_{\neg}, f_{\vee}, f_{\wedge}, f_{\rightarrow}, f_{\equiv}, \Gamma)$, where
1. M is a non-empty set of propositions. 2. TRUE $\subseteq$ M is a set of true propositions 3. NEC $\subseteq$ set of necessary propositions 4. $f_{\bot}, f_{\top}, f_{\Box}, f_{\neg}, f_{\vee}, f_{\wedge}, f_{\rightarrow}, f_{\equiv}$, functions with arity (operations of type) 0,0,1,1,2,2,2,2 respectively. 5. $\Gamma : C \rightarrow M$ is a function Gamma, satisfying $\Gamma(\top) = f_{\top}$, $\Gamma(\bot) = f_{\bot}$.

Definition 17. An assignment is a function $\gamma : V \rightarrow M$, where $\gamma \in M^V$.
An assignment ${\gamma^a}_x$ is the assignment which maps x to a and maps variables $y \neq x$ to $\gamma(y)$ (substituition).

We extends this assignment with the homonym function $\gamma : Fm(C) \to M$, where:
$\gamma(c) = \Gamma(c)$, for $c \in C$
$\gamma(\Box(\varphi)) = f_{\Box}(\gamma(\varphi))$
$\gamma(\neg(\varphi)) = f_{\neg}(\gamma(\varphi))$
$\gamma(\varphi @ \psi) = f_{@}(\gamma(\varphi), \gamma(\psi))$, where $@\{\wedge, \vee, \to, \equiv\}$

Definition 18. A proposition domain is a $S3_{\equiv}$-model if the following conditions hold:

if $NEC \neq \varnothing$, then the relation on M defined by

$a \leq_{\Psi} b :\iff f_{\to}(a,b) \in NEC$ is a preorder and $(M, f_{\bot}, f_{\top}, f_{\neg}, f_{\vee}, f_{\wedge}, f_{\to}, \leq_{\Psi})$ is a Boolean pre-lattice.

(ii) The following truth conditions hold for all $a, b, \in M$:
(a) $f_{\top} \in TRUE$ and $f_{\bot} \in M$
$TRUE$
(b) $f_{\neg}(a) \in TRUE \iff a \notin TRUE$
(c) $f_{\to}(a,b) \in TRUE \iff a \notin TRUE$ or $b \in TRUE$
(d) $f_{\wedge}(a,b) \in TRUE \iff a \in TRUE$ and $b \in TRUE$
(e) $f_{\vee}(a,b) \in TRUE \iff a \in TRUE$ or $b \in TRUE$
(f) $f_{\Box}(a) \in TRUE \iff a \in NEC$
(g) $f_{\equiv}(a,b) \in TRUE \iff a = b$

(iii) If $NEC \notin \varnothing$, then $NEC \subseteq TRUE$ is a filter on M, i.e. for all a,b $\in M$:

(a) if $a \in NEC$ and $a \leq_{\Psi} b$, then $b \in NEC$

(b) if a,b $\in NEC$, then $f_{\wedge}(a,b) \in NEC$

(iv) If NEC , then the following hold for all a, b $\in M$:
(a) $f_{\top} \leq_{\Psi} f_{\equiv}(a,a)$
(b) $f_{\equiv}(a,b) \leq_{\Psi} f_{\to}(a,b)$
(c) $f_{\Box}(a) \leq_{\Psi} a$
(d) $f_{\Box}(f_{\to}(a,b)) \leq_{\Psi} f_{\to}(f_{\Box}(a), f_{\Box}(b))$, distributivity over implication
(e) $f_{\Box}(f_{\to}(a,b)) \leq_{\Psi} f_{\Box}(f_{\to}(f_{\Box}(a), f_{\Box}(b)))$

Proposition 8. If NEC, then TRUE is an ultrafilter.

Part I: TRUE is a filter
1.If a $\in TRUE$ and $a \leq_{\Psi} b$, then $b \in TRUE$
Let $a \leq_{\Psi} b$, by (i) $f_{\to}(a,b) \in NEC \subseteq TRUE$. Let also $a \in TRUE$, by (ii.c), $b \in TRUE$.
2.If a,b $\in TRUE$, then $f_{\wedge}(a,b) \in TRUE$
It holds according to (ii.d)
3.$f_{\bot} \notin TRUE$
By (ii.a) $f_{\bot} \notin TRUE$.

Part II. TRUE is maximal

According to ii.b, we have the prime property which corresponds to the maximal property.

Definition 19. Satisfaction: $(\Pi, \gamma) \vDash \varphi :\Longleftrightarrow \gamma(\varphi) \in TRUE$

Definition 20. A logic consequence is defined as: $\Phi \vdash \psi :\Longleftrightarrow Mod(\Phi) \subseteq Mod(\{\varphi\})$, where $Mod(\Phi) = \{(\Psi, \gamma) | \Psi anormalS3_{\equiv} - model, \gamma \in M^V and (\Psi, \gamma) \vDash \Phi\}$

4.3.2 Refining Modal Logic expressiveness with the Propositional Identity

The Modal Logic is more expressive than the Classical Logic. Considering our objectives of representing different intensions, the development of the Modal Logic was an evolution. We could express, for example, a statement that was “necessary"or “possible"other than only “true"or “false". However, formulas such as $\neg\neg\varphi$ and φ still denote the same propositions. As we have seen through this work, we do not want that and due to this fact we will refine the Modal logics based on Strict Equivalence (SE) through the Propositional Identity (PI).

Lemma 3. **PI refines SE:** $(\varphi \equiv \psi) \rightarrow (\Box(\varphi \rightarrow \psi) \wedge \Box(\psi \rightarrow \varphi))$

Suppose $(\Psi, \gamma) \vDash \Box(\varphi \equiv \psi)$(I).It implies $\gamma(\varphi) = \gamma(\psi)$(II) , it means they denote the same preposition. On the other hand, $\Box(\varphi \rightarrow \varphi)$ is valid, so $f_{\Box}(f_{\rightarrow}(\gamma(\varphi), \gamma(\varphi)))$ is valid. Now, by (I) and definition of satisfaction, $f_{\Box}(f_{\rightarrow}(\gamma(\varphi), \gamma(\varphi))) \in TRUE$, by (II), $f_{\Box}(f_{\rightarrow}(\gamma(\varphi), \gamma(\psi))) \in TRUE$. By the satisfaction definition, $\Box(\varphi \rightarrow \psi)$. Same for $\Box(\psi \rightarrow \varphi)$.

4.3.3 Collapse Axiom and Collapse Theorem

These two ways to define PI and SE are closely related by was called by (LEWITZKA, 2014), Collapse Axiom (CA): $(\varphi \equiv \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$. It can be interpreted semantically as "There is only one necessary proposition", or algebraically "In every normal model, the smallest filter is $\{f_{\top}\}$. A theorem that adapts the semantic Boolean pre-algebra on NFL to express the modalities introduced by the NEC prepositional subset is related with the Collapse Axiom and stated bellow:

Collapse Theorem:

(i) M is a boolean algebra and satisfies the Collapse Axiom

(ii) M is a boolean algebra with NEC = $\{f_{\top}\}$.

(iii) $\leq_M$ is a partial order.

(iv) Strict equivalence coincides with propositiional identity , i.e. $M \models (x \equiv y) \leftrightarrow \Box((x \rightarrow y) \wedge (y \rightarrow x))$

4.4 Relational (also known as Kripke's semantics)

4.4.1 Hyperintesions

A related study is the Hyperintensions (POLLARD, 2008), where the propose of a boolean pre-algebra and ultrafilters are used for the Linguistic proposal of exploit the semantic of the Natural Language. It was created based on a semantic introduced by Kripke in 1972, called "Soft Actualism". An interesting result shown on (LEWITZKA, 2014) is that the concept of pre-algebra and ultrafilters, as used, coincides with the SCI-model definition. Furthermore, SCI-model, Denotational semantic are equivalents to model the Hyperintensions.

In (Pollard, 2008) it is argued that the Standard Possible World Semantic is not semantically sufficient for Natural Language expressiveness. More specifically, it suffers from the granularity problem, i.e. not semantically sufficient to represent distinct meaning to make consistent predictions with robust intensions. Two main problems divide the granularity: (1) anti-symmetry of entailment and (2) handle the existence of non-principal ultrafilters given the natural language expressions. The former, makes the Frege axiom to be true and, thus, reduce the value of a denotation of a formula to its truth-value. The last one makes sentences that follow from each other, to have the same meaning. For the problem (1), the solution was consider a pre-algebra instead of the usual Boolean algebra (created by the binary order of the inclusion of a subset of). For (2), a change of the semantic, where primitives worlds and constructed propositions to primitives propositions and constructed worlds (ultrafilters) over a pre-algebra solve it. This approach following called a revised Soft Actualism, following (Kripke, 1959), instead of (Kripke, 1963). Compounding the use of the pre-algebra with the Soft Actualism (1) and (2) it is constructed a high-order logic with subtypes that solve the granularity problem.

5 Application: Truth theory

5.1 Motivation

In a highly expressive language, as the Natural Language, we have the power to:

1. make references about the truth of sentences, like:
Foo's sentence is not true
That sentence we have heard in the bar is false [1]

2. make references about the own sentence, like:
This sentence means the same as that Foo's sentence.[2]
That Foo's sentence means exactly the opposite that Bar's sentence [3]

3. and consequently make statements as a composition of them, like:
I'm saying the truth right now.

5.1.1 Paradoxes

A language strong enough to express the self-reference, a truth predicate (true) and use of negation, as we have seen above, can lead us to contradictions such as:
I'm lying right now
Foo: What Bar is going to say next is true
Bar: What Foo just said is false

One could argue why we would like to express, or we should worry about sentences as "I'm lying right now"(a version of the famous Liar sentence). Some authors consider these constructions important to the proper understanding of a language.

5.1.2 Avoiding paradoxes

The different kinds of contradictions, also called antinomies, break the semantic consistency of our language. Once we know for example a formula is true; inferences rules could be applied to produce other formulas. This process becomes weird, in classical logic, when propositions are true and false at the same time (para-consistent interpretation allowing it is just a one more way to handle the paradoxes).

Therefore, in order to solve the semantic consistency, we can have two actions related to the contradictions. First, avoid them at maximum, do not permitting self-

[1] Informally, "What Foo said is not true"and "That thing we have heard in the bar is false".
[2] Informally,"What Foo said is exactly the opposite of what Bar said".
[3] Informally,"What I'm saying is the same as Foo said".

reference and truth predicates operators on the language. So, any contradiction sentence could be possible to be expressed, but also none of the above valid ones. It will drastically reduce the expressiveness of our language. Another way is to handle them.

Logicians have been trying different ways to deal wit them and maintain the semantic consistency of the language. According to (HODGES, 2014), the five main ways to get out of antinomies, such as Liar Paradox are:
a. Russell Type Theory
b. Tarski's Hierarchy of Meta-languages
c. Kripke's Hierarchy of Interpretations
d. Barwise and Etchemendy method
e. Para-consistency.

On some primitive methods such as the Russel Type Theory, Tarski's Hierarchy of Metalanguages the expressiveness of the language is most of the times reduced. On (BARWISE; ETCHEMENDY, 1987), it is given an example where Tarski's object language/metalanguage and Russell ban of "vicious cycles" exclude from consideration a non-problematic cycling proposition.

5.1.3 Example: Tarski's T-Schema and his Hierarchy of Metalanguages

The Tarski T-schema was a truth schema used to provide a definition of truth from the universe of sentences while the Hierarchy of Languages was an alternative to solve Liar Paradox (as a type of semantic paradoxes) on its truth schema. The T-schema was built to provide an inductive construction of a semantic theory. It translates the syntactical structure to its semantical counterpart. It stated since complex syntax structures made of atomic sentences and operations that range over the universe of the sentences. Finally, a paradox would not be possible to satisfy the T-schema. A paradox would only a valid paradox on the T-schema if the meta-language is higher in the hierarchy than the object language. Then, with the Hierarchy of Languages, Tarski stated the separation of the object language (the language which is being talked about) and the metalanguage (the language which is using to talk) as a necessity to produce a linguistic theory without semantic paradoxes.

An example would be the Liar (BARWISE; ETCHEMENDY, 1987):
"The proposition that (this proposition is not true) is true if and only if (this proposition is not true)."It is not a contradiction in itself. To get the contradiction, we must state from an above instance: "The proposition that (this proposition is not true) is true if and only if the proposition that (this proposition is not true) is not true."

5.2 Chapter work

The main concern of this chapter is produce a logic with truth predicates and self-referential capacity. Then, we provide based on (LEWITZKA, 2012e), an elegant approach to avoid paradoxes. It was based on a previous work of (STRÄTER, 1992). A logic with these two properties was called, by Alfred Tarski, a semantically closed language, as pointed out on (LEWITZKA, 2012e).

5.2.1 Truth-predicates and self-reference with Non-Fregean Logic

Without getting into the formalization details yet, we introduce the solution for the Antinomies using Non-Fregean Logics. The truth predicate by the introduction of truth predicates $T\varphi$, for example. Then, by the association of truth predicates to semantic structures, in these case, propositions and not to the syntactic structures (in this case sentences). The semantic structure that carries the denotation of a formula was attached to operators mapping propositions to propositions, which operators are seen as bearing truth values. The self-reference is then achieved by the truth predicate associated with a $f_{\equiv}$.

A semantically closed intuitionistic abstract logics can be found in (LEWITZKA, 2012e). A many-valued approach can be found in (LEWITZKA, 2012d).

Examples of the Non-Fregean Logic approach, would be:

1. “This proposition is false."$\Rightarrow c \equiv Fc$ (using false self-referential). Paradox
2. “This proposition is not true."$\Rightarrow c \equiv \neg Tc$. Paradox. On the paper of ϵ_I, it couldn't be defined.
3. “This proposition is true."$\Rightarrow c \equiv Tc$ It is the called truth-teller.
4. “The next proposition is true". $\Rightarrow$ (1) $c \equiv Td$ (2) $d \equiv \varphi$

, To be or not to be a paradox depends of the next proposition. Then, it becomes circular dependency, a paradox. However, if it is clear its truth-value, then it is not a paradox.

5.“This previous proposition is not true."$\Rightarrow c \equiv \varphi \quad d \equiv Tc$, Again, be or not to be a paradox depend of the previous proposition.

6. “This sentence is false or φ is true."$\Rightarrow c \equiv (F_c \vee T\varphi)$. To be or not to be a paradox depends of the φ value

5.2.2 A close look at the Liar

Propositional paradoxes similar to the Liar Paradox can are vast in the literature. Some of them found in (BARWISE; ETCHEMENDY, 1987) are Liar Cycles, Contingent Liars, Contingent Liar Cycles, Lobs Paradox.

5.2.2.1 Liar Cycles

The next proposition is true. ($c_0 \equiv Tc_1$)
The next proposition is true. ($c_1 \equiv Tc_2$)
...
The first proposition is false. ($c_n \equiv Fc_0$)

5.2.2.2 Contingent Liar

Your name is Marcelo and this proposition is false. $c \equiv (\varphi \wedge Fc)$

The paradox depends if your name is Marcelo. If your name is not Marcelo (φ is false), the sentence is just false. If your name is Marcelo (φ is true), then it is a paradox.

5.2.2.3 Contingent Liar Cycles

As expected, it is the combination of the Contingent Liar and Liar Cycles.
Your name is Marcelo and this proposition is false.
The previous proposition is true.
At least one of the two previous propositions is false.

5.2.2.4 Lobs Paradox

The idea is the perception that you can prove using an inference rule (in the case, Modus Ponens) and conditional proof.
If (this proposition is true), then (your name is Marcelo). $c \equiv (Tc \rightarrow \varphi)$

By the conditional proof, it is sufficient to ensure the first and see that the second must be true also. So, assume $Tc =$ this proposition is true. Then, assuming alpha true, by the auto-reference "this proposition", we have $c \equiv (Tc \rightarrow \varphi)$ also true. Now, by modus ponens, we have (your name is Marcelo) is true.

5.2.2.5 Relationship between Liar and Theory of Computation results

A relationship between Godel Incompleteness and Liar Paradox is that in fact, they state similar assertions. While Liar Paradox says "This proposition is not true."A simple reformulation of the Godel Incompleteness says "G is not provable in the theory T where G is the Godel number of a false formula". Furthermore, both are also closely related with the Tarski Undefinability Theorem proved in 1936 that can be stated as the concept of truth in arithmetic cannot be definable using arithmetic expressiveness. In other words, the diagonal lemma holds on a language with negation and self-reference.

5.2.3 Example: Liar Paradox solution

Let $c \equiv Fc$ the Liar equation. Consider c a true proposition regarding a model. Then, by definition of the truth-operator F, $Fc \notin TRUE \Rightarrow \in FALSE$. So, $c \in TRUE$, $Fc \in FALSE$. However we have that $c \equiv Fc$ and $TRUE \cap FALSE = \varnothing$. It is a contradiction. Consider a valuation which maps c to the set of FALSE propositions. Then, by definition of the truth-operator F, $Fc \in TRUE$. Then, by the facts $FALSE \cap TRUE = \varnothing$ and $c \equiv Fc$, we have another contradiction.

5.2.4 Example: Arithmetic informal similarity

Consider the set of real numbers and his usual operations multiplication, sum, subtraction. It forms an algebra $\omega = (IR, f_*, 1, 0, f_+, f_-)$.

Now, consider we define the equation $i^2 = -1$, in the algebra notation $f_*(x,x) = f_-(0,1)$. Even though this equation can be defined, there is no “i"in the range of real values (i.e. $i \in IR$) that satisfies this equation (in arithmetic terms, only the imaginary number $i = \sqrt[2]{-1}$ would satisfy it.

This is the same basic idea of Non-Fregean Logic solution. It will be possible to express equations such as $c \equiv Fc$ (Liar). However, it will not exist any proposition in the universe of propositions that correspond to it, because there is no model that satisfy this equation.

5.3 A formal logic with truth predicates

This formalization follows (LEWITZKA, 2011) and (LEWITZKA, 2012e). Both of them have their bases on (STRÄTER, 1992).

5.3.1 Syntax

Let V a set of variables, C a set of constants containing the usual $\top$ and $\bot$. Then, let Fm(C)’ be the smallest set of formulas with V,C and closed on: if φ and ψ are expressions, then $(\mathfrak{T}\varphi)$, $(F\varphi)$, $(\varphi \rightarrow \psi)$,$(\neg\varphi)$, $(\varphi \equiv \psi)$. The usual abbreviations hold.

5.3.2 Deductive System

Axioms:
Propositional logic axioms:
(i): $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
(ii): $\varphi \rightarrow (\psi \rightarrow \varphi)$
(iii): $\neg\varphi \rightarrow (\varphi \rightarrow \psi)$

(iv): $(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi))$
SCI axioms;
(v): $\varphi \equiv \varphi$
(vi): $\varphi \equiv \psi \to (\varphi \to \psi)$
(vii) $\varphi \equiv \psi \to \chi[x := \varphi] \equiv \chi[x := \psi]$
Truth-predicate axioms:
(viii): $\varphi \to (\mathfrak{T}\varphi)$ and $(\mathfrak{T}\varphi) \to \varphi$
(ix): $\neg\varphi \to (F\varphi)$ and $(F\varphi) \to \varphi$

Inference Rule: Modus Ponens (MP).

5.3.3 Semantic: ϵ_T' algebra

Definition 21. A ϵ_T' algebra (propositional domain) $\mathfrak{M} = (M, f_\neg, f_\to, f_\equiv, f_\mathfrak{T}, f_F)$, where M is the universe of propositions and $f_\neg, f_\to, f_\equiv, f_\mathfrak{T}, f_F$, operations with arity: $1, 2, 2, 1, 1$ respectively.
$\Gamma : C \mapsto M$ is a function which satisfies: $\Gamma(c) = f_c; \Gamma(\bot) = \bot; \Gamma(\top) = \top$ where f_c is the proposition referent to the constant c. An assignment is a function $\gamma : V \mapsto M$ such that for all $\varphi, \psi \in V$, holds:
$\gamma(x) = f_x$, being f_x a proposition on M correspondent to the variable x.
$\gamma(\mathfrak{T}\varphi) = f_\mathfrak{T}(\gamma(\varphi))$; $\gamma(F\varphi) = f_F(\gamma(\varphi))$; $\gamma(\neg\varphi) = f_\neg(\gamma(\varphi))$
$\gamma(\varphi \to \psi) = f_\to(\gamma(\varphi), \gamma(\psi))$; $\gamma(\varphi \equiv \psi) = f_\equiv(\gamma(\varphi), \gamma(\psi))$

An homonym extension of the assignment is the valuation $\gamma : Fm(C)' \mapsto M$, such that:
$\gamma(\top) = \Gamma(\top) = f_\top$; $\gamma(\bot) = \Gamma(\bot) = f_\bot$; $\gamma(c) = \Gamma(c)$. For the variables, it maintains the same behaviour above.

Definition 22. Let TRUE and FALSE sets such that $M = TRUE \cup FALSE$. Then, $\mathfrak{M} = (M, TRUE, FALSE, \Gamma)$ is a model if it satisfies the following truth conditions:
(i): $f_\top \in TRUE, f_\bot \in FALSE$
(ii): $f_\mathfrak{T}(a) \in TRUE \iff a \in TRUE$
(iii): $f_F(a) \in TRUE \iff a \in FALSE$
(iv): $f_\neg(a) \in TRUE \iff a \in FALSE$
(v): $f_\to(a, b) \in TRUE \iff a \in FALSE$ or $b \in TRUE$
(vi): $f_\equiv(a, b) \in TRUE \iff a = b$

There is also a necessary structural condition that these models are transitives.

Furthermore, $\Gamma : C \mapsto M$ and the valuations $\gamma : Fm(C)' \mapsto M$ works as specified in the propositional domain.

5.3.4 A model construction satisfying the truth-teller

On this section, we will construct a model that has the self-referential proposition Truth-teller, as an example of the approach suggested on (STRÄTER, 1992) and further developed on (LEWITZKA, 2012e). Consider a formal system that has a self-referential proposition truth teller and then prove its completeness.

In more details, let Fm the classic language (the language of well-formed formulas). Extending it by a set C = {c} and the unary connectives T and F we generate $Fm(C)'$. Let $\Phi \subseteq Fm(C)'$, let us define the congruence relation $\approx_\Phi$ om $Fm(\{c\})'$, such that $\varphi \approx \psi \iff \Phi \vdash \varphi \equiv \psi$. We will use just $\approx$ for simplification. Furthermore, $\approx$ is the smallest congruence relation on $Fm(\{c\})$ that contains (c, Tc), i.e. $c \approx Tc$. Then, we will construct a model that has exactly one self-referential proposition, the: **Truth-Teller** $(c \equiv Tc)$.

5.3.5 Model construction

Definition 23. Let the set C = {c} and the unary connectives T and F on the language $Fm(\{c\})'$. Let us define a congruence relation $\varphi \approx \psi \iff \varphi \equiv \psi$, where $\approx$ is the smallest congruence relation on $Fm(\{c\})$ that contains (c, Tc), i.e. $c \approx Tc$. Let us construct a model that satisfies the self-referential proposition the **Truth-Teller** $(c \approx Tc)$.

Consider the algebra $\mathfrak{M} = (M, TRUE, FALSE, \Gamma)$ such that: $M = \{\overline{\varphi} | \varphi\}$.

TRUE and FALSE are defined inductively:
$\overline{c} \in TRUE$
$\overline{x} \in TRUE :\iff$ “x is true"
$\overline{x} \in FALSE :\iff$ “x is false"
$\overline{\varphi \wedge \psi} \in TRUE :\iff \overline{\varphi} \in TRUE$ and $\overline{\psi} \in TRUE$
$\overline{\varphi \wedge \psi} \in FALSE :\iff \overline{\varphi} \in FALSE$ and $\bar{\psi} \in FALSE$
$\overline{T\varphi} \in TRUE :\iff \overline{\varphi} \in TRUE$
$\overline{T\varphi} \in FALSE :\iff \overline{\varphi} \in FALSE$
$\overline{\neg\varphi} \in TRUE :\iff \overline{\varphi} \notin TRUE$
$\overline{\neg\varphi} \in FALSE :\iff \overline{\varphi} \notin FALSE$
$\overline{\varphi \to \psi} \in TRUE :\iff$ If $\overline{\varphi} \notin TRUE$, then $\psi \in TRUE$
$\overline{\varphi \to \psi} \in TRUE :\iff \overline{\varphi} \notin TRUE$ or $\psi \in TRUE$ (TODO)

Operations: $f_\neg, f_\to, f_\wedge, f_\vee, f_T, f_F$ defined as: $f_\neg(\neg\overline{\varphi}) = \overline{\neg\varphi}$; $f_\to(\overline{\varphi}, \overline{\psi}) = \overline{\varphi \to \psi}$; $f_\wedge(\overline{\varphi}, \overline{\psi}) = \overline{\varphi \wedge \psi}$; $f_\vee(\overline{\varphi}, \overline{\psi}) = \overline{\varphi \vee \psi}$; $f_T(\overline{T}\varphi) = \overline{T\varphi}$; $f_F(\overline{F}\varphi) = \overline{F\varphi}$

A Gamma function $\Gamma : C \mapsto M$ that satisfies: $\Gamma(c) = \overline{c}$;

Now, consider the assignment $\alpha : V \mapsto M$, such that $\alpha(x) = \overline{x}$. We extend it to $\alpha : Fm(\{c\})' \mapsto M$, consider the homonym assignment $\alpha : Fm(\{c\})' \mapsto M$, such that:

$\alpha(\top) = \Gamma(f_\top)$; $\alpha(\bot) = \Gamma(f_\bot)$; $\alpha(c) = \Gamma(c)$; $\alpha(x) = \overline{x}$; $\alpha(\neg\varphi) = f_\neg(\overline{\varphi})$; $\alpha(\varphi \to \psi) = f_\to(\overline{\varphi}, \overline{\psi})$; $\alpha(\varphi \wedge \psi) = f_\wedge(\overline{\varphi}, \overline{\psi})$; $\alpha(\varphi \vee \psi) = f_\vee(\overline{\varphi}, \overline{\psi})$

Lemma. The congruence relation of satisfaction do not depend of the class instance: $\overline{\varphi} \in TRUE$ and $\varphi \approx \psi$, then $\overline{\varphi} \in TRUE$.

Proposition 9. $\varphi := c$
(a): $c \in \overline{c}$. By congruence, $c \approx Tc$, so $Tc \in \overline{c}$.By transitivity, $c \approx Tc$, substituting c by its equivalence Tc, $TcTTc$. Applying transitivity: $c \approx TTc$.Thus, we can see $\overline{c} = \{c, Tc, TTc, TTTc, ...\}$.
(b): $c \in TRUE$ by definition of our true truth-teller. Then, $c \in TRUE \implies Tc \in TRUE$ $Tc \implies TTc \in TRUE$ and so on.

Proposition 10. $\varphi :=$ "x is false"
(a): $\overline{x} = \{x\}$
(b): By the definition of implication. Suppose "x is false", we must ensure that $x \in TRUE.But"xisfalse" \Rightarrow x \in TRUE, bydefinition.$

Proposition 11. $\varphi :=$ "x is true"
(a): $\overline{x} = \{x\}$
(b): By the definition of implication. Suppose "x is true", we must ensure that $x \in TRUE$. But "x is true"$\Rightarrow x \in TRUE$, by definition.

Proposition 12. $\varphi := \neg\psi_1$
$\chi \approx \varphi$, i.e. $\chi \approx \neg\psi_1$. Analyzing χ: If $\neg\psi_1 \in TRUE$, $\chi \approx \neg\psi_1$. By the fact χ is the smallest congruence relationship, $\chi = \neg\chi_1$, where $\chi_1 \approx \psi_1$. Then, $\chi_1 \approx \psi_1 \Rightarrow \overline{\chi}_1 = \overline{\psi}_1$ (1). It means, χ must have a negation form that reflects the syntax form of φ. By the Induction Hypothesis: $\overline{\varphi} \in TRUE \iff \overline{\neg\psi_1} \in TRUE \iff \overline{\psi}_1 \notin TRUE$. By the equality of its congruence class (1), $\overline{\psi}_1 \notin TRUE \approx \overline{\chi}_1 \notin TRUE \iff \overline{\neg}\chi_1 \in TRUE \iff \overline{\chi_1 \vee \chi_2} \in TRUE$.

Proposition 13. $\varphi := \neg\psi_1$
$\chi \approx \varphi$, i.e. $\chi \approx \neg\psi_1$. Analyzing χ: If $\neg\psi_1 \in TRUE$, $\chi \approx \neg\psi_1$. By the fact χ is the smallest congruence relationship, $\chi = \neg\chi_1$, where $\chi_1 \approx \psi_1$. Thus, $\chi_1 \approx \psi_1 \Rightarrow \overline{\chi}_1 = \overline{\psi}_1$ (1). It means, χ must have a negation form that reflects the syntax form of φ. By the Induction Hypothesis: $\overline{\varphi} \in TRUE \iff \overline{\neg\psi_1} \in TRUE \iff \overline{\psi}_1 \notin TRUE$. By the equality of its congruence class (1), $\overline{\psi}_1 \notin TRUE \approx \overline{\chi}_1 \notin TRUE \iff \overline{\neg}\chi_1 \in TRUE \iff \overline{\chi_1 \vee \chi_2} \in TRUE$.

Proposition 14. $\varphi := Tc$
As we have seen, $\overline{Tc} = \overline{c} = \{c, Tc, TTc, ...\}$. By Hypothesis of Induction $Tc \in TRUE \iff c \in TRUE$. By $\approx$ transitivity, $TcTTc$, $TTc \in TRUE \iff Tc \in TRUE$, but we saw $Tc \in TRUE$, so $TTc \in TRUE$, and so on. So, $\overline{c} \in TRUE$.

Proposition 15. $\varphi := \psi_1 \vee \psi_2$

$\chi \approx \varphi$, i.e. $\chi \approx \psi_1 \vee \psi_2$. Analyzing χ: If $\psi_1 \in TRUE$, if $\chi \approx \psi_1$, $\chi \approx \psi_2$. By the fact χ is the smallest congruence relationship, $\chi = \chi_1 \vee \chi_2$, where $\chi_1 \approx \psi_1$ and $\chi_2 \approx \psi_2$. Thus, $\chi_1 \approx \psi_1 \Rightarrow \overline{\chi}_1 = \overline{\psi}_1$ (1) and $\chi_2 \approx \psi_2 \Rightarrow \overline{\chi}_2 \approx \overline{\chi}_2$ (2). It means, χ must have a disjunctive form that reflects the syntax form of φ. By the Induction Hypothesis: $\overline{\varphi} \in TRUE \iff \overline{(\psi_1 \psi_2)} \in TRUE \iff \overline{\psi}_1 \in TRUE$ or $\overline{\psi}_2 \in TRUE$. By congruence relations results (1) and (2), $\overline{\psi}_1 \in TRUE$ or $\overline{\psi}_2 \in TRUE \approx \overline{\chi}_1 \in TRUE$ or $\overline{\chi}_2 \in TRUE \iff \overline{\chi_1 \vee \chi_2} \in TRUE$.

Proposition 16. $\varphi := \psi_1 \wedge \psi_2$
Similar proof as above.

Proposition 17. $\varphi := \psi_1 \rightarrow \psi_2$
$\chi \approx \varphi$, i.e. $\chi \approx \psi_1 \rightarrow \psi_2$. Analyzing χ: If $\psi_1 \in TRUE$, if $\chi \approx \psi_1$, $\chi \approx \psi_2$. By the fact χ is the smallest congruence relationship, $\chi = \chi_1 \rightarrow \chi_2$, where $\chi_1 \approx \psi_1$ and $\chi_2 \approx \psi_2$. Thus, $\chi_1 \approx \psi_1 \Rightarrow \overline{\chi}_1 = \overline{\psi}_1$ (1) and $\chi_2 \approx \psi_2 \Rightarrow \overline{\chi}_2 \approx \overline{\chi}_2$ (2). It means, χ must have a implication form that reflects the syntax form of φ. By the Induction Hypothesis: $\overline{\varphi} \in TRUE \iff \overline{(\psi_1 \rightarrow \psi_2)} \in TRUE \iff$ If $\overline{\psi}_1 \in TRUE$, then $\overline{\psi}_2 \in TRUE$. So, suppose $\overline{\psi}_1 \in TRUE$, as we had $\overline{(\psi_1 \rightarrow \psi_2)}$, then $\overline{\psi}_2 \in TRUE$. By congruence (1), $\overline{\chi}_1 \in TRUE$. Then, by congruence (2): $\overline{\chi}_2 \in TRUE$. So, finally we have: If $\overline{\chi_1} \in TRUE$, then $\overline{\chi_2} \in TRUE \iff \overline{\chi_1 \rightarrow \chi_2} \in TRUE$.

6 Application: Logic with Quantifiers and Epistemic Logic

6.1 Logic with Quantifiers

6.1.1 Motivation

Building a logic with quantifiers ranging over a model-theoretic universe of propositions is a useful application of Boolean pre-algebras semantic representation. Besides the natural interest on First-order and High-order logics, it is also important for other related computer science applications such as epistemic logic and truth-theory (LEWITZKA, 2012a). On Epistemic Logic, knowledge axioms for agents K_i can be summarily expressed by a single sentence such as $\forall x.(K_i x \rightarrow x)$. On the truth-theoretic field, a Tarski's T-scheme can also be summarily expressed through by a single axiom $\forall x(Tx \leftrightarrow x)$, where T is a truth operator meaning "x is true".

6.1.2 Important concepts

6.1.2.1 Quantifier operators

We can introduce quantifier operators on an NFL, for example, $\forall$. Then, a quantifier such that $\forall$ is mapped semantically to a high order functions $f_\forall : M^M \mapsto M$. Then, we introduce new axioms regarding the new operator as well as their semantic counterpart. An application of $\forall$ can be seen on ($LEWITZKA$, 2014). An application of $\forall$ and $\exists$ can be seen on ($LEWITZKA$, $2012a$).

6.1.2.2 Alpha-congruence

Two formulas are said to be alpha-congruent if they differ at most on their bound variables. Two equations between alpha-congruent formulas are expected to be valid. Finally, the existence of models such that formulas that are not alpha-congruent denote different propositions are necessary for the different logics we have been representing. For example, for a first-order non-Fregean logic with self-referential and total truth-predicate it was proved by (STRÄTER, 1992) and then by (LEWITZKA, 2012a). On lambda calculus, two alpha-congruent formulas are seen when we see two expressions become the same after applying a beta reduction in one of them.

6.1.3 Propositional reference and canonical models

On (STRÄTER, 1992; LEWITZKA, 2012a), it was introduced a logic connective $\varphi <^M \psi$, where M is the propositional universe, in the semantics which means "φ says something about ψ". It is a counterpart to a syntactical reference $a \prec b$, meaning sentence "a speaks about b"sentence. We have seen a similar kind of reference, the self-referential propositions given by truth-predicates and the operator $\equiv$ on Truth-theory. In the first-order logic with referential propositions and truth-predicates (LEWITZKA, 2012a), this is achieved by the concept of a canonical model. A canonical model is a partially ordered structure of objects which have a truth value (either true or false). So, informally, given the ordered set (Syntactic universe, $\prec$) and the ordered set (Semantic universe, $<^M$), it was constructed an order isomorphism between them. The goal of this order isomorphism is that it guarantees that the reference, made syntactically will well behave semantically. In other words, in canonical models, it is valid $a < b \iff \alpha(a) \prec \alpha(b)$, where α is an assignment function, which maps sentences to its denotations.

6.2 Epistemic Logic

6.2.1 Motivation

On (LEWITZKA, 2011), propositions are interpreted as "facts" that have a truth value and represent a semantic content. The semantic content is determined by (Φ, α), where Φ is the proposition domain and α a function (called valuation) which maps formulas to propositions.

We follow ϵ_K defined on (LEWITZKA, 2011) that is an extension of the ϵ_E defined on (STRÄTER, 1992) and simplified and further developed on (ZEITZ, 2000).

- Propositions are now objects of knowledge, if $K_i\varphi$ is true, than φ denotes a true proposition and the agent i knows the proposition denoted by φ.
- The TRUE set, is the set of facts
- Let $KNOWN = \cup TRUE_i$, for all i agents. It is the set of all the facts known. It is obviously and probably not all the facts existent, i.e. $KNOWN \subseteq TRUE$. We usually expect $KNOWN \subset TRUE$.
- It is introduced the concept of a group of agents.
- There are facts known by a group of agents. Every agent in the group knows the group facts. Furthermore, every agent knows that the others agents know it, and so on.

- It is introduced the notion of Common Knowledge, a common knowledge among all the agents that satisfies some closure properties.

6.2.2 Syntax

Fm(C)' will be the smallest set of formulas that contains C and V and is closed under:
If φ and ϕ are expression, then $(\varphi \rightarrow \psi), (\neg\varphi), (\varphi \equiv \psi), (K_i\ \varphi), (C_G\varphi)$. Note: parenthesis precedence follows: $\neg \equiv, \rightarrow, K_i m C_G, \rightarrow$. Another important note is that $\wedge$ and $\vee$ are the usual abbreviations constructed with $\neg$ and $\rightarrow$.

6.2.3 Deductive System

Axioms:
Propositional logic axioms:
$\varphi \rightarrow (\psi \rightarrow \varphi)$
$(\varphi \rightarrow (\psi \rightarrow \omega)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \omega))$
$\neg\varphi \rightarrow (\varphi \rightarrow \psi)$
$(\varphi \rightarrow \psi) \rightarrow ((\neg\varphi \rightarrow \psi) \rightarrow \psi)$ SCI axioms:
$\varphi \equiv \varphi$
$\varphi \equiv \psi \rightarrow (\varphi \rightarrow \psi)$
$\varphi \equiv \psi \rightarrow \chi[x := \varphi] \equiv chi[x := \psi]$
Epistemic axioms:
$K_i c_i$, whenever the constant c_i is in the language.
$K_i\varphi \rightarrow \varphi$
$C_G\varphi \rightarrow C_G K_i\varphi$, whenever $i \in G$
$C_G\varphi \rightarrow C'_G\varphi$, whenever G' $\subseteq G$
We could add the truth-predicates in the language in order to avoid paradoxes on the epistemic level.

Inference rules: Only MP is considered.

6.2.4 Semantic: ϵ'_K algebra

Definition 24. A ϵ'_K algebra (propositional domain) is the tuple:
$\mathfrak{M} = (M, TRUE, (TRUE_i), FALSE, f_\neg, f_\rightarrow, f_\equiv, f_{K_i}, f_{C_G})$, where M is the universe of propositions and $f_\neg, f_\rightarrow, f_\equiv, f_\mathfrak{T}, f_F$, operations with arity: $1, 2, 2, 1, 1$ respectively.
$\Gamma : C \mapsto M$ is a function which satisfies: $\Gamma(c) = f_c; \Gamma(\bot) = \bot; \Gamma(\top) = \top$ where f_c is the proposition referent to the constant c. An assignment is a function $\gamma : V \mapsto M$ such that for all $\varphi, \psi \in V$, holds:
$\gamma(x) = f_x$, being f_x a proposition on M correspondent to the variable x.

$(\gamma(\varphi))$; $\gamma(F\varphi) = f_F(\gamma(\varphi))$; $\gamma(\neg\varphi) = f_\neg(\gamma(\varphi))$
$\gamma(\varphi \to \psi) = f_\to(\gamma(\varphi), \gamma(\psi))$; $\gamma(\varphi \equiv \psi) = f_\equiv(\gamma(\varphi), \gamma(\psi))$

An homonym extension of the assignment is the valuation $\gamma : Fm(C)' \mapsto M$, such that: $\gamma(\top) = \Gamma(\top) = f_\top$; $\gamma(\bot) = \Gamma(\bot) = f_\bot$; $\gamma(c) = \Gamma(c)$. For the variables, it maintains the same behaviour above.

Definition 25. Let TRUE, $TRUE_i(1 \leq i \leq n)$ and FALSE sets such that $M = TRUE \cup FALSE$, TRUE $\cap$ FALSE $= \varnothing$. Then, $\mathfrak{M} = (M, TRUE, (TRUE_i), FALSE, \Gamma)$ is a model if it satisfies the following truth conditions:
(i): $f_\top \in TRUE, f_\bot \in FALSE$
(ii): $f_\neg(a) \in TRUE \iff a \in FALSE$
(iii): $f_\to(a, b) \in TRUE \iff a \in FALSE$ or $b \in TRUE$
(iv): $f_\equiv(a, b) \in TRUE \iff a = b$
(v): $f_{K_i(a)} \in TRUE \iff a \in TRUE_i$
(vi): $f_{C_G(a)} \in TRUE \Rightarrow f_{Ki}(a) \in TRUE$ and $f_{C_G(f_{K_i(a)})} \in TRUE$, for all i $\in G$.
(vi): $f_{C_G(a)} \in TRUE \Rightarrow f_{G'_C(a)} \in TRUE$, whenever $G' \subseteq G$.
There is also a necessary structural condition that these models are transitives and a theorem, called Substitution Principle: $\vDash \varphi \equiv \psi \to \chi[x := \varphi] \equiv \chi[x := \psi]$.

7 Conclusion

In this senior thesis it was exposed the use of Boolean pre-algebras to the foundations of Computer Science. Specifically, we could see its relationship with Non-Fregean Logic and some applications as a semantic representation to many other logics derived from the Setential Calculus with Identity.

7.1 Main Contributions

- A more detailed SCI Strong Completeness Theorem proof, a result on (SUSZKO; BLOOM, 1972) and (LEWITZKA, 2014).

- Demonstration of the equivalence between original Suszko SCI axioms and an alternative SCI axiomatization used on (LEWITZKA, 2014),(LEWITZKA, 2013) and (LEWITZKA, 2012b).

- More detailed demonstration of equivalence between SCI-models and Boolean pre-algebras.

- A model construction which satisfies the Truth-Teller (true truth-teller).

7.2 Limitations

Some SCI-models could be shown as pre-Boolean algebras (as we saw they are equivalent). It would be probably more adequate for the work. Another limitation was not comment about the drawbacks of the situation semantics (also called denotational semantics). A reference to an argument against it, called Slingshot Argument, can be found in the section 8 on (WÓJCICKI, 1986).

7.3 Possible Future Work

- Approach other logical systems such as Intuitionistic Logic, which semantic is represented as Heyting Algebras.

- Construct some models for the Epistemic Logic ϵ'_K

7.4 Other Proposals

- Classify operators according to its contribution to the formal logic (granular or refiner).
- Make a semantic graphic set representation considering the intern order representation.
- Maybe handle different modalities as different subsets and handle them.

References

BARWISE, J.; ETCHEMENDY, J. *The liar: An essay on truth and circularity*. [S.l.]: Oxford University Press, 1987. Citado 2 vezes nas páginas 48 e 49.

BLACKBURN, P.; BENTHEM, J. F. A. K. v.; WOLTER, F. *Handbook of Modal Logic, Volume 3 (Studies in Logic and Practical Reasoning)*. New York, NY, USA: Elsevier Science Inc., 2006. ISBN 0444516905. Citado na página 30.

BRUCE, K. B.; COSMO, R. D.; LONGO, G. Provable isomorphisms of types. *Mathematical Structures in Computer Science*, Cambridge Univ Press, v. 2, n. 02, p. 231–247, 1992. Citado na página 20.

DAVIS, E.; MORGENSTERN, L. Epistemic logic and its applications: Tutorial notes. In: *International Joint Conferences on Artificial Intelligence*. [S.l.: s.n.], 1983. v. 93. Citado na página 20.

FONT, J. M.; JANSANA, R.; PIGOZZI, D. A survey of abstract algebraic logic. *Studia Logica*, Springer, v. 74, n. 1-2, p. 13–97, 2003. Citado na página 33.

FOX, C.; LAPPIN, S. A framework for the hyperintensional semantics of natural language with two implementations. In: *Logical aspects of computational linguistics*. [S.l.]: Springer, 2001. p. 175–192. Citado na página 20.

GALVIN, D. Ultrafilters, with applications to analysis, social choice and combinatorics. *unpublished notes*, 2009. Citado na página 30.

GARSON, J. Modal logic. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Spring 2016. [S.l.: s.n.], 2016. Citado na página 39.

GOLDBLATT, R. Mathematical modal logic: a view of its evolution. *Journal of Applied Logic*, Elsevier, v. 1, n. 5, p. 309–392, 2003. Citado na página 39.

HODGES, W. Tarski's truth definitions. In: ZALTA, E. N. (Ed.). *The Stanford Encyclopedia of Philosophy*. Fall 2014. [S.l.: s.n.], 2014. Citado na página 48.

HUGHES, G. E.; CRESSWELL, M. J. *A new introduction to modal logic*. [S.l.]: Psychology Press, 1996. Citado na página 39.

LEWITZKA, S. A non-fregean logic of explicit knowledge. *Studia Logica*, v. 97, n. 2, p. 233–264, 2011. ISSN 1572-8730. Disponível em: <http://dx.doi.org/10.1007/s11225-011-9304-8>. Citado 3 vezes nas páginas 20, 51 e 58.

LEWITZKA, S. Construction of a canonical model for a first-order non-fregean logic with a connective for reference and a total truth predicate. *Logic Journal of IGPL*, Oxford Univ Press, v. 20, n. 6, p. 1083–1109, 2012. Citado 2 vezes nas páginas 57 e 58.

LEWITZKA, S. Denotational semantics for normal modal logics with propositional quantifiers and identity. *CoRR*, abs/1212.6576, 2012. Disponível em: <http://arxiv.org-/abs/1212.6576>. Citado 2 vezes nas páginas 34 e 61.

LEWITZKA, S. *Lecture Notes: Logic for Computer Science*. Universidade Federal da Bahia, Brasil: [s.n.], 2012. Citado na página 23.

LEWITZKA, S. On some many-valued abstract logics and their epsilon-t-style extensions. *CoRR*, abs/1209.5364, 2012. Disponível em: <http://arxiv.org/abs/1209.5364>. Citado 2 vezes nas páginas 20 e 49.

LEWITZKA, S. Semantically closed intuitionistic abstract logics. *Journal of Logic and Computation*, v. 22, n. 3, p. 351–374, 2012. Disponível em: <http://logcom-.oxfordjournals.org/content/22/3/351.abstract>. Citado 4 vezes nas páginas 20, 49, 51 e 53.

LEWITZKA, S. A denotational semantics for a lewis-style modal system close to S1. *CoRR*, abs/1304.6983, 2013. Disponível em: <http://arxiv.org/abs/1304.6983>. Citado 5 vezes nas páginas 25, 33, 40, 41 e 61.

LEWITZKA, S. Denotational semantics for modal systems s3–s5 extended by axioms for propositional quantifiers and identity. *Studia Logica*, v. 103, n. 3, p. 507–544, 2014. ISSN 1572-8730. Disponível em: <http://dx.doi.org/10.1007/s11225-014-9577-9>. Citado 9 vezes nas páginas 21, 25, 34, 35, 40, 44, 45, 57 e 61.

LEWITZKA, S. A modal logic amalgam of classical and intuitionistic propositional logic. *Journal of Logic and Computation*, 2015. Disponível em: <http://logcom.oxfordjournals-.org/content/early/2015/07/20/logcom.exv048.abstract>. Citado 2 vezes nas páginas 37 e 40.

ODIFREDDI, P. *Ultrafilters, dictators, and Gods*. [S.l.]: Springer, 2000. Citado na página 30.

POLLARD, C. Hyperintensions. *J. Log. and Comput.*, Oxford University Press, Oxford, UK, v. 18, n. 2, p. 257–282, abr. 2008. ISSN 0955-792X. Disponível em: <http://dx.doi.org/10.1093/logcom/exm003>. Citado 2 vezes nas páginas 30 e 45.

STRÄTER, W. *ϵ_T: Eine Logik erster Stufe mit Selbstreferenz und totalem Wahrheitsprädikat*. [s.n.], 1992. (Technische Universität Berlin, Fachbereich Informatik). Disponível em: <https://books.google.com.br/books?id=8zDTjgEACAAJ>. Citado 5 vezes nas páginas 49, 51, 53, 57 e 58.

SUSZKO, R.; BLOOM, S. L. Investigations Into the Sentential Calculus with Identity. *Notre Dame Journal of Formal Logic*, Duke University Press, v. 13, n. 3, p. 289–308, 1972. Citado 10 vezes nas páginas 21, 23, 25, 26, 27, 29, 30, 40, 42 e 61.

WÓJCICKI, R. Situation semantics for non-fregean logic. *Journal of Non-Classical Logic*, v. 3, p. 33–67, 1986. Citado na página 61.

ZEITZ, P. *Parametrisierte Epsilon-T-Logik: eine Theorie der Erweiterung abstrakter Logiken um die Konzepte Wahrheit, Referenz und klassische Negation, Dissertation, Logos Verlag*. [S.l.: s.n.], 2000. (Berlin). Citado na página 58.

Appendix

.1 Some graphical representation

Let $F_m(C)$ the set of sentences (syntactical elements) and M is the universe of propositions (semantic elements). We have a valuation function that maps sentences to propositions, according to a certain behavior.

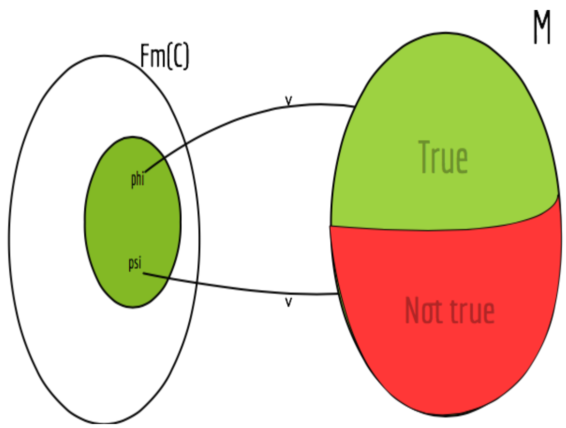

Figura 1 – Classical Logic

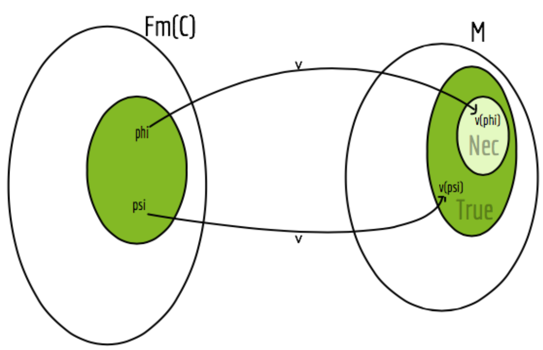

Figura 2 – Modal Logic

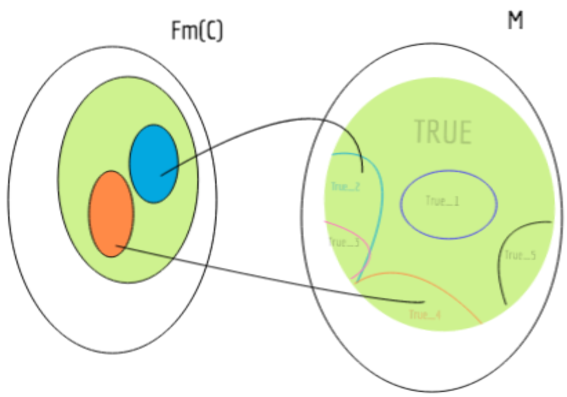

Figura 3 – Epistemic Logic