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Gaussian distributions, Jacobi group, and Siegel-Jacobi space

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Let $\mathcal{N}$ be the space of Gaussian distribution functions over $\mathbb{R}$, regarded as a 2-dimensional statistical manifold parameterized by the mean μ and the deviation σ. In this paper, we show that the tangent bundle of $\mathcal{N}$, endowed with its natural Kähler structure, is the Siegel-Jacobi space appearing in the context of Number Theory and Jacobi forms. Geometrical aspects of the Siegel-Jacobi space are discussed in detail (completeness, curvature, group of holomorphic isometries, space of Kähler functions, and relationship to the Jacobi group), and are related to the quantum formalism in its geometrical form, i.e., based on the Kähler structure of the complex projective space. This paper is a continuation of our previous work [M. Molitor, "Remarks on the statistical origin of the geometrical formulation of quantum mechanics," Int. J. Geom. Methods Mod. Phys. **9**(3), 1220001, 9 (2012); M. Molitor, "Information geometry and the hydrodynamical formulation of quantum mechanics," e-print arXiv (2012); M. Molitor, "Exponential families, Kähler geometry and quantum mechanics," J. Geom. Phys. **70**, 54–80 (2013)], where we studied the quantum formalism from a geometric and information-theoretical point of view. © *2014 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4903182]

I. MOTIVATION: THE QUANTUM FORMALISM

It was recently suggested that the quantum formalism might be "*grounded on the Kähler geometry which naturally emerges from statistics*."[3] What motivates this claim comes from the following facts (see also Refs. 1 and 2).

There exists a large class of statistical manifolds, called *exponential families* (see Definition 2.29 and 2.31), whose tangent bundles possess automatically a Kähler structure of information-theoretical origin (see Sec. II F). For example, the space $\mathcal{B}(n)$ of binomial distributions $p(k) = \binom{n}{k} q^k(1-q)^{n-k}$ defined over $\{0,\ldots,n\}$ forms a 1-dimensional exponential family parameterized by $q \in (0,1)$. Therefore, its tangent bundle is a Kähler manifold of real dimension 2, and one can show that it is locally isomorphic to the natural Kähler structure of the sphere S^2 multiplied by n. Another important example is the following. Take a finite set $\Omega := \{x_1,\ldots,x_n\}$ and consider the space $\mathcal{P}_n^\times$ of nowhere vanishing probabilities $p : \Omega \to \mathbb{R}, p > 0, \sum_{k=1}^n p(x_k) = 1$. This is a $(n-1)$-dimensional exponential family, and it can be shown (see Ref. 1) that $T\mathcal{P}_n^\times$ is locally isomorphic to the complex projective space $\mathbb{P}(\mathbb{C}^n)$ (see also Ref. 3 for a refinement of this statement using the concept of "Kählerification").

Many authors have stressed the importance of Kähler geometry in relation to the quantum formalism.[4–9] It is known that a quantum system, with Hilbert space $\mathbb{C}^n$, can be entirely described by means of the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$; this is the so-called *geometrical formulation of quantum mechanics*.[10] Therefore, by recovering the Kähler structure of $\mathbb{P}(\mathbb{C}^n)$ from a purely statistical object like $\mathcal{P}_n^\times$, one may legitimately suspect that the quantum formalism has an information-theoretical origin, at least for finite-dimensional Hilbert spaces.

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In Ref. 3, we pursued this line of thought and observed that, in finite dimension, all the ingredients of the geometrical formulation of quantum mechanics (quantum state space, observables, probabilistic interpretation, etc.) can be expressed in terms of the statistical structure of $\mathcal{P}_n^\times$ (+ completion arguments). This is a crucial observation, for it allows to somewhat enlarge the geometrical formulation of quantum mechanics and gives new geometrical insight. For example, we characterized the so-called *spin coherent states*[11,92–96] in terms of the *Veronese embedding* $S^2 \hookrightarrow \mathbb{P}(\mathbb{C}^{n+1})$, simply by studying the derivative of the canonical injection $\mathcal{B}(n) \hookrightarrow \mathcal{P}_{n+1}^\times$ (see Refs. 3 and 12).

It is important to note that the above "statistical-Kähler" geometry is not related to quantum mechanics in the same way as symplectic manifolds are related to quantum mechanics via a quantization scheme (e.g., geometric quantization[13,14]). In some sense, the above geometry is "quantum" right from the start due to its statistical origin. Let us illustrate this point by the following result (see Corollary 2.34). Let $\mathcal{E}$ be an exponential family (like $\mathcal{B}(n)$ or $\mathcal{P}_n^\times$) defined over a measure space (Ω, dx), with canonical projection $\pi : T\mathcal{E} \to \mathcal{E}$. Fix an arbitrary holomorphic isometry Φ of $T\mathcal{E}$. In this situation, it can be shown that there exists a vector space $\mathcal{A}_{\mathcal{E}}$ of random variables $X : \Omega \to \mathbb{R}$ such that: (1) $\dim(\mathcal{A}_{\mathcal{E}}) = \dim(\mathcal{E}) + 1$, and (2) functions of the form $T\mathcal{E} \to \mathbb{R},\ \ p \mapsto \int_\Omega X(x)[(\pi \circ \Phi)(p)](x)dx$ are automatically *Kähler functions*, that is, they preserve the Kähler structure of $T\mathcal{E}$ (see Definition 2.20). Kähler functions are important in relation to the geometrical formulation of quantum mechanics, for they play the role of observables.[10] The geometrical formalism of quantum mechanics analysed in Ref. 3 under the light of the above "Kähler decomposition" led naturally to the following definition: the spectrum of a Kähler function $f : T\mathcal{E} \to \mathbb{R}$ of the form $\int_\Omega X(x)[(\pi \circ \Phi)(p)](x)dx$ is $\mathrm{Spec}(f) := \mathrm{Im}(X)$, where $\mathrm{Im}(X)$ is the image of the random variable $X \in \mathcal{A}_{\mathcal{E}}$. Using this definition, we described the spin of a particle passing through two consecutive Stern-Gerlach devices, *without* using physicists' standard approach based on the unitary representations of $\mathfrak{su}(2)$.

It is on the basis of the above facts (together with others that are collected in Refs. 1–3) that we arrived at the conclusion that the quantum formalism might have an information-theoretical origin. Now there are two possibilities:

1. The quantum formalism has indeed an information-theoretical origin. In this case, the formalism should be rewritten and the role of the above statistical-Kähler geometry should be fully clarified. Recently, many authors have tried to derive (or "reconstruct") the quantum formalism from purely information-theoretical principles.[15–22] These attempts have their own merits and respective successes, but to our knowledge, no consensus has emerged yet.
2. Quantum mechanics cannot be derived from information-theoretical principles. In this case, one should still explain the relationship between the above definition of $\mathrm{Spec}(f)$, which is a priori independent of representation theory, and the definition of the spectrum of an operator. It may well be that there is some (obscure) geometrical content hidden behind the main results of functional analysis that goes beyond the well-known correspondence between the space of Kähler functions of the complex projective space and the space of Hermitian operators (as described for example in Ref. 5, or Lemma 7.6 in Ref. 3).

In any case, it is necessary to investigate the matter further and to study more examples.

In this paper, we investigate an example which for obvious reasons should be particularly important, namely the family $\mathcal{N}$ of Gaussian distribution functions

$$\frac{1}{\sqrt{2\pi}\sigma}\exp\Big\{-\frac{(x-\mu)^2}{2\,\sigma^2}\Big\} \qquad (x \in \mathbb{R}) \tag{1}$$

defined over $\mathbb{R}$. Clearly, $\mathcal{N}$ is a 2-dimensional statistical manifold parameterized by the mean $\mu \in \mathbb{R}$ and the deviation $\sigma > 0$, and it is well-known that it is an exponential family (see Definition 2.31 and (68)). Therefore, $T\mathcal{N}$ is naturally a Kähler manifold of real dimension 4. The objective of this paper is to study the geometry of $T\mathcal{N}$, having in mind quantum mechanics as discussed above. We distinguish two aspects: the *intrinsic geometry* of $T\mathcal{N}$, coming from the fact that $T\mathcal{N}$ is a Kähler manifold by itself, and the *extrinsic geometry*, related to the fact that $T\mathcal{N}$ can be regarded as a submanifold of an infinite-dimensional complex projective space $\mathbb{P}(\mathcal{H})$. Of these two approaches, it is extrinsic geometry which makes the connection between $T\mathcal{N}$ and the quantum formalism most transparent.

Let us now describe our results regarding the geometry of $T\mathcal{N}$.

The intrinsic geometry. As a Kähler manifold, $T\mathcal{N}$ is the *Siegel-Jacobi space* $\mathbb{S}^J$ (see Definition 3.4 and Proposition 3.6). The Siegel-Jacobi space appears in the context of Number Theory, in relation to the so-called *Jacobi forms* (see Refs. 23 and 24). As a complex manifold, it is the product $\mathbb{H}\times\mathbb{C}$, where $\mathbb{H}$ is the Poincaré upper half-plane $\{\tau\in\mathbb{C}\,|\,\mathrm{Im}(\tau)>0\}$, and its Kähler metric is the *Kähler-Berndt metric* g_{KB} (see Definition 3.1). Using the general properties of *Dombrowski's construction* (see Secs. II A and II C), we compute the curvature of $T\mathcal{N}$ and observe that the scalar curvature is constant and negative, albeit not Einstein. The group of holomorphic isometries of $T\mathcal{N}$ is computed in Sec. III C; it is the *affine symplectic group* $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^2$ (see Theorem 3.11). We also describe the whole group of isometries using a result of Kulkarni which characterizes curvature-preserving maps between Riemannian manifolds of dimension ≥ 4 (see Theorem 3.33 and Proposition 3.34). A few geometrical consequences are derived in Proposition 3.31, the most notable being that $T\mathcal{N}$ is a homogeneous Kähler manifold (a result which was already known for $\mathbb{S}^J$). In Sec. III D, we study the space of Kähler functions on $\mathbb{S}^J$. As it turns out, they are conveniently described by means of the *Jacobi group* $G^J(\mathbb{R})$, the semi-direct product $\mathrm{SL}(2,\mathbb{R})\ltimes\mathrm{Heis}(\mathbb{R})$, where $\mathrm{Heis}(\mathbb{R})$ is the Heisenberg group of dimension 3 (see Sec. III A). We show that the Jacobi group acts in a Hamiltonian way on $\mathbb{S}^J$, and compute the corresponding momentum map $\mathbf{J}:\mathbb{S}^J\to(\mathfrak{g}^J)^*$ (here, $\mathfrak{g}^J$ denotes the Lie algebra of $G^J(\mathbb{R})$). We then show that a smooth function $f:\mathbb{S}^J\to\mathbb{R}$ is Kähler if and only if there exists $\xi\in\mathfrak{g}^J$ such that $f(p)=\langle\mathbf{J}(p),\xi\rangle$ for all $p\in\mathbb{S}^J$, where $\langle\,,\,\rangle$ is the natural pairing between $\mathfrak{g}^J$ and $(\mathfrak{g}^J)^*$. From this we deduce that the space of Kähler functions on $\mathbb{S}^J$ is a Poisson algebra of dimension 6, isomorphic in the Lie algebra sense to $\mathfrak{g}^J$. We also use Kostant's Coadjoint Orbit Covering Theorem[25] to deduce that $\mathbb{S}^J$ is a coadjoint orbit of $G^J(\mathbb{R})$ (see Proposition 3.47). Having quantum mechanics in mind, we then study the spectral properties of the Kähler functions of $\mathbb{S}^J$ in the sense discussed above and in Ref. 3. The Kähler functions we consider on $\mathbb{S}^J$ are of the form (see Proposition 3.54) :

$$f(p)=\int_{-\infty}^{\infty}(\alpha x^2+\beta x+\gamma)\big[(\pi\circ\Phi_{g^{-1}})(p)\big](x)dx,\tag{2}$$

where $p\in\mathbb{S}^J$, $\pi:\mathbb{S}^J\cong T\mathcal{N}\to\mathcal{N}$ is the canonical projection, $\Phi_{g^{-1}}$ is a holomorphic isometry of $\mathbb{S}^J$, dx is the Lebesgue measure over $\mathbb{R}$ and where $\alpha x^2+\beta x+\gamma$ is a polynomial with real coefficients in the variable $x\in\mathbb{R}$. We define the spectrum $\mathrm{Spec}(f)$ of a function of this type as the image of the polynomial $\alpha x^2+\beta x+\gamma$. In Lemmas 3.49 and 3.50, we check that this definition is independent of the decomposition in (2). Instances of spectra are given in Example 3.55. Finally, given a point $p\in\mathbb{S}^J$ and a Kähler function f as above, we define a probability measure $P_{f,p}$ on $\mathrm{Spec}(f)$ as the probability distribution of the polynomial $\alpha x^2+\beta x+\gamma$, regarded as a random variable with respect to the probability measure $[(\pi\circ\Phi_{g^{-1}})(p)](x)dx$ (see Lemma 3.56 and Definition 3.57). From a quantum mechanical point of view, the quantity $P_{f,p}(A)$ is interpreted as the probability that the observable f yields upon measurement an "eigenvalue" $\lambda\in A\subseteq\mathrm{Spec}(f)$ while the system is in the state $p\in\mathbb{S}^J$.

The extrinsic geometry. Let $\mathcal{H}:=L^2(\mathbb{R})$ be the Hilbert space of square-integrable functions $f:\mathbb{R}\to\mathbb{C}$ endowed with the Hermitian product $\langle f,g\rangle:=\int_{\mathbb{R}}\bar{f}g\,dx$, where dx is the Lebesgue measure. Associated to $\mathcal{H}$ is the complex projective space $\mathbb{P}(\mathcal{H})$ of complex lines in $\mathcal{H}$, endowed with its natural Kähler structure (Fubini-Study symplectic form and metric). In Sec. IV, we introduce a map $\Psi:\mathbb{S}^J\to\mathcal{H}$ and its companion map $T:=[\Psi]:\mathbb{S}^J\to\mathbb{P}(\mathcal{H})$ having the following properties. The map T is a smooth and symplectic immersion, *but* it is not isometric nor holomorphic (see Proposition 4.1). Moreover, it gives the following characterization (see Proposition 4.5): a smooth function $f:\mathbb{S}^J\to\mathbb{R}$ is Kähler if and only if f can be written as

$$f(p)=\big\langle\Psi(p),H\Psi(p)\big\rangle,\qquad(p\in\mathbb{S}^J),\tag{3}$$

where H is a real linear combination of the following Hermitian operators acting on $C^\infty(\mathbb{R},\mathbb{C})$:

$$-x^2,\ -i\frac{\partial}{\partial x},\ -\frac{\partial^2}{\partial x^2},\ x,\ 2i\Big(x\frac{\partial}{\partial x}+\frac{1}{2}I\Big),\ I\tag{4}$$

(I denotes the identity operator). The precise statement involves a unitary representation of the Lie algebra $\mathfrak{g}^J$ which is essentially the infinitesimal *Schrödinger-Weil representation* (see Ref. 26). Finally, in Sec. IV C, we discuss briefly the Schrödinger equation

$$i\frac{d\psi}{dt} = H\psi, \qquad (\psi \in L^2(\mathbb{R})), \tag{5}$$

where H is a linear combination of the above Hermitian operators. More precisely, given a Kähler function f on $\mathbb{S}^J$ with Hamiltonian vector field X_f, we observe that if $\alpha : I \to \mathbb{S}^J$ is an integral curve of X_f, then there exists a smooth map $\lambda : I \to \mathbb{C} - \{0\}$ such that $\lambda(t)\Psi\big(\alpha(t)\big)$ satisfies the above Schrödinger equation for an appropriate H (see Corollary 4.9). From a physical point of view, the above operators are related to the free quantum particle, the quantum harmonic oscillator, and the forced quantum harmonic oscillator (see Remark 4.3).

Let us comment the above results. Clearly, the main observation of this paper is the connection between the space of Gaussian distributions, the Siegel-Jacobi space and the Jacobi group. Using the terminology introduced in Ref. 3, one may say that *the Kählerification of the space of Gaussian distributions is the Siegel-Jacobi space*.

As we already mentioned, the Siegel-Jacobi space and Jacobi group play an important role in the context of Number Theory, in relation to Jacobi forms.[23,24] The latter are a mixture of modular forms and elliptic functions that generalize classical functions like the *Jacobi theta function* and the Fourier coefficients of the *Siegel modular forms*.[27] Roughly, they are holomorphic functions f on $\mathbb{H} \times \mathbb{C}$ enjoying invariance properties that involve the Jacobi group $G^J(\mathbb{R})$, together with "good" Fourier expansions (see also Remark 3.2 for more details on the role of the Kähler-Berndt metric). In the context of physics, the Jacobi group, also known as the *Schrödinger* or *Hagen group*, is the symmetry group of the one-dimensional Schrödinger equation of a free quantum particle.[28,29] In the context of quantum optics, the Jacobi group is related to the so-called *squeezed coherent states*.[26,30–38]

It is somehow surprising that with so little, the Gaussian distribution, one can arrive at important objects like the Siegel-Jacobi space and the Jacobi group, and discuss a fair amount of their quantum properties without any quantization scheme (especially in view of the intrinsic geometry). This reassures us and lends credence to the idea that the above statistical-Kähler geometry is one of the keys to understand the foundations of quantum physics.

There are however two important questions which are not discussed in this paper: (1) what is the origin of the map $T : \mathbb{S}^J \to \mathbb{P}(\mathcal{H})$ and (2) what are its equivariance properties? In Ref. 3, we observed that the Veronese embedding $S^2 \hookrightarrow \mathbb{P}(\mathbb{C}^{n+1})$, which is a finite-dimensional analogue[39] of T, is essentially the derivative of the inclusion map $\mathcal{B}(n) \hookrightarrow \mathcal{P}_{n+1}^{\times}$ (neglecting completion issues, it is the derivative up to the actions of two discrete groups). In the case of T, such interpretation is not directly available for the following reason. Let $\mathcal{D}$ be the space of smooth density probability functions over $\mathbb{R}$ with respect to the Lebesgue measure. The space $\mathcal{D}$ can be thought of as an infinite-dimensional analogue of $\mathcal{P}_n^{\times}$, but contrary to the latter, its tangent bundle $T\mathcal{D}$ does not have a canonical Kähler structure that could be "compared" with that of $\mathbb{P}(L^2(\mathbb{R}))$. Therefore, the derivative of the inclusion map $\mathcal{N} \hookrightarrow \mathcal{D}$ cannot be interpreted directly as a map $T\mathcal{N} \to \mathbb{P}(L^2(\mathbb{R}))$. To overcome these difficulties, it is necessary to first get a clear idea of what should be the infinite dimensional generalization of the statistical-Kähler geometry discussed above; Refs. 2 and 40–42 might be a good starting point in this respect. Regarding the second question, we observe that T exhibits properties that are usually shared by coherent states (compare for example Proposition 4.5 and Corollary 4.9 with Refs. 43–46). Moreover, T is an infinite-dimensional analogue of the Veronese embedding, which is known to characterize spin coherent states.[3,12] Therefore, it is very likely that T itself is a coherent state in the sense of Perelomov.[44] To prove this, one should establish equivariance properties of the map T, probably by means of the Schrödinger-Weil representation.[23] It is interesting to note, in this respect, that Yang considered in Refs. 47 and 48, a map $\mathbb{S}^J \to L^2(\mathbb{R})$ which is very similar to Ψ, and which enjoys such equivariance properties. It would be very interesting to relate Yang's work to the properties of T, and then make a comparison with the coherent-state approach of Berceanu.[26,30,32–34,43]

For the convenience of the reader, the paper starts with a rather detailed discussion on the relation between Kähler geometry and statistics (see Sec. II). Some of these results are known (Proposition 2.5, Proposition 2.21, Corollary 2.27, Proposition 2.30, Proposition 2.32, Corollary 2.33), others are new (Propositions 2.15, 2.25, 2.26, and 2.28), and others still appear in different contexts and different guises (Propositions 2.10 and 2.12, Corollary 2.13). We shall present the subject in a uniform way by using the concept of *dually flat structure*, with which not all readers may be familiar.[49] The intrinsic and extrinsic geometry of $T\mathcal{N}$ are discussed in Secs. III and IV, respectively.

II. Dually flat structures and Kähler geometry

A. Dombrowski's construction

Let M be a manifold endowed with an affine connection ∇. We denote by $\pi : TM \to M$, the canonical projection and by K, the connector associated to ∇. Recall that K is the unique map $T(TM) \to TM$ satisfying (see Refs. 50–52)

$$\nabla_X Y = K Y_* X \tag{6}$$

for all vector fields X, Y on M (here, Y_*X denotes the derivative of Y in the direction of X).

Given $u_p \in T_pM$, the subspaces

$$\mathrm{Hor}(TM)_{u_p} := \big\{Z \in T_{u_p}(TM)\,\big|\,KZ = 0\big\}, \tag{7}$$

$$\mathrm{Ver}(TM)_{u_p} := \big\{Z \in T_{u_p}(TM)\,\big|\,\pi_{*u_p}Z = 0\big\} \tag{8}$$

are, respectively, called the *space of horizontal tangent vectors* and the *space of vertical tangent vectors* of TM at u_p. They are both isomorphic to T_pM in a natural way, and led to the following decomposition:

$$T_{u_p}(TM) \cong \mathrm{Hor}(TM)_{u_p} \oplus \mathrm{Ver}(TM)_{u_p} \cong T_pM \oplus T_pM. \tag{9}$$

More generally, ∇ determines an isomorphism of vector bundles over M (see Refs. 50 and 51)

$$T(TM) \cong TM \oplus TM \oplus TM, \tag{10}$$

the isomorphism being

$$T_{u_p}(TM) \ni A_{u_p} \mapsto \big(u_p, \pi_{*u_p}A_{u_p}, KA_{u_p}\big). \tag{11}$$

If there is no danger of confusion, we shall thus regard an element of $T_{u_p}(TM)$ as a triple (u_p, v_p, w_p), where $u_p, v_p, w_p \in T_pM$. The second component v_p is usually referred to as the horizontal component (with respect to ∇) and w_p the vertical component.

Let h be a Riemannian metric on M. Together with ∇, the couple (h, ∇) determines an almost Hermitian structure on TM via the following formulas:

$$\begin{aligned}
g_{u_p}\big((u_p, v_p, w_p), (u_p, \overline{v}_p, \overline{w}_p)\big) &:= h_p\big(v_p, \overline{v}_p\big) + h_p\big(w_p, \overline{w}_p\big), && \text{(metric)}\\
\omega_{u_p}\big((u_p, v_p, w_p), (u_p, \overline{v}_p, \overline{w}_p)\big) &:= h_p\big(v_p, \overline{w}_p\big) - h_p\big(w_p, \overline{v}_p\big), && \text{(2-form)}\\
J_{u_p}\big((u_p, v_p, w_p)\big) &:= \big(u_p, -w_p, v_p\big), && \text{(almost complex structure)}
\end{aligned} \tag{12}$$

where $u_p, v_p, w_p, \overline{v}_p, \overline{w}_p \in T_pM$. Clearly, $J^2 = -\mathrm{Id}$ and $g(J\,.\,, J\,.\,) = g(\,.\,,\,.\,)$, which means that (TM, g, J) is an almost Hermitian manifold, and one readily sees that g, J, and ω are compatible, i.e., that $\omega = g\big(J\,.\,,\,.\,\big)$. The 2-form ω is thus the fundamental 2-form of the almost Hermitian manifold (TM, g, J). This is *Dombrowski's construction*.

Remark 2.1. By construction, the map $\pi : (TM, g) \to (M, h)$ is a Riemannian submersion.

Remark 2.2. Let $\gamma(t)$ be a smooth curve in TM. Regarding $\gamma(t)$ as vector field $V(t)$ along $c(t) := (\pi \circ \gamma)(t)$, one has $\pi_*\dot{\gamma} = \dot{c}$ and $K\dot{\gamma} = \nabla_{\dot{c}}V$, where $\dot{\gamma}$ and $\dot{c}$ are the time derivatives of γ and c, respectively, and where $\nabla_{\dot{c}}V$ is the covariant derivative of $V(t)$ along $c(t)$. From this, it follows by inspection of Dombrowski's construction that

$$g_{\gamma(t)}(\dot{\gamma}, \dot{\gamma}) = h_{c(t)}(\dot{c}, \dot{c}) + h_{c(t)}\big(\nabla_{\dot{c}}V, \nabla_{\dot{c}}V\big). \tag{13}$$

We now review the analytical properties of Dombrowski's construction. Let ∇^* be the unique connection on M satisfying

$$X\big(h(Y,Z)\big) = h\big(\nabla_X Y, Z\big) + h\big(Y, \nabla^*_X Z\big) \tag{14}$$

for all vector fields X, Y, Z on M. In the statistical literature, ∇^* is called the *dual connection* of ∇ with respect to h (and vice versa), and the triple (h, ∇, ∇^*) is called a *dualistic structure* (see Ref. 53).

Definition 2.3. The dualistic structure (h, ∇, ∇^*) is dually flat if both ∇ and ∇^* are flat, meaning that their torsions and curvature tensors are zero.

As the literature is not uniform, let us agree that the torsion T and the curvature tensor R of a connection ∇ are defined as

$$\begin{aligned} T(X,Y) &:= \nabla_X Y - \nabla_Y X - [X,Y], \\ R(X,Y)Z &:= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned} \tag{15}$$

where X, Y, Z are vector fields on M.

Remark 2.4. Let R and R^* be the curvature tensors of the dual connections ∇ and ∇^*, respectively. Then,

$$h\big(R(X,Y)Z, W\big) = -h\big(R^*(X,Y)Z, W\big) \tag{16}$$

for all vector fields X, Y, Z, W on M (see Ref. 53*). In particular, R is identically zero if and only if R^* is identically zero.*

Recall that an almost Hermitian structure (g, J, ω) is Kähler when the following two analytical conditions are met: (1) J is integrable; (2) $d\omega = 0$.

Proposition 2.5. Let (h, ∇, ∇^*) be a dualistic structure on M and (g, J, ω) the almost Hermitian structure on TM associated to (h, ∇) via Dombrowski's construction. Then,

$$(TM, g, J, \omega) \text{ is Kähler} \quad \Leftrightarrow \quad (M, h, \nabla, \nabla^*) \text{ is dually flat.} \tag{17}$$

Remark 2.6. Proposition 2.5 is an easy consequence of Remark 2.4 together with the following equivalence which is due to Dombrowski (see Refs. 3 and 50*):*

$$J \text{ is integrable} \quad \Leftrightarrow \quad \nabla \text{ is flat} \tag{18}$$

(here, J is the almost complex structure associated to (h, ∇) via Dombrowski's construction).

B. Local formulas

Let (h, ∇, ∇^*) be a dualistic structure on a manifold M. We denote by (g, J, ω), the almost Hermitian structure of TM associated to (h, ∇) via Dombrowski's construction. We also denote by $\pi : TM \to M$, the canonical projection and by $K : T(TM) \to TM$, the connector associated to ∇.

Let $x = (x_1, \ldots, x_n)$ be system of coordinates on M. If dx_i denotes the differential of x_i (regarded as a local function on TM), then $(x_1 \circ \pi, \ldots, x_n \circ \pi, dx_1, \ldots, dx_n)$ forms a local coordinate system on TM. By repeating, we obtain coordinates on $T(TM)$, say (a_i, b_i, c_i, d_i), $i = 1, \ldots, n$, where

$$a_i = x_i \circ \pi \circ \pi_{TM}, \quad b_i = (dx_i) \circ \pi_{TM}, \quad c_i = d(x_i \circ \pi), \quad d_i = d(dx_i), \tag{19}$$

and where $\pi_{TM} : T(TM) \to TM$ is the canonical projection. Observe that d_i is not zero, for dx_i is regarded as a local function on TM, not as a one form.

Let Γ^k_{ij} be the Christoffel symbols of ∇ in the coordinates $(x_1, \ldots, x_n)$, i.e.,

$$\nabla_{\partial_i} \partial_j = \sum_{k=1}^{n} \Gamma^k_{ij} \partial_k, \tag{20}$$

where $\partial_i = \frac{\partial}{\partial x_i}$. In the coordinates introduced above, one can check that

$$K(a,b,c,d) = \big(a, d + \Gamma_a(b,c)\big), \tag{21}$$

$$\pi_*(a,b,c,d) = (a,c), \tag{22}$$

where Γ_a is the bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ defined by $\big(\Gamma_a(b,c)\big)_k = \sum_{i,j=1}^n \Gamma_{ij}^k(a) b_j c_i, k = 1,\ldots,n$. Observe that if (x_i) is an affine coordinate system[54] with respect to ∇, then K reduces to the projection $(a,b,c,d) \mapsto (a,d)$.

Let us fix a coordinate system (y_i) on M, defined on the same neighborhood as (x_i).

Definition 2.7. The couple $((x_i),(y_i))$ is a pair of dual coordinate systems if:

(i) *(x_i) (respectively, (y_i)) is an affine coordinate system with respect to ∇ (respectively, ∇^*),*
(ii) *$h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = \delta_{ij}$ (Kronecker symbol) for all $i,j \in \{1,\ldots,n\}$.*

The system of coordinates (y_i) is called the dual coordinate system of (x_i) and vice versa.

Remark 2.8. If (x_i) is an affine coordinate system with respect to ∇, then one can find a coordinate system (y_i) dual to (x_i), i.e., such that (y_i) is affine with respect to ∇^* and such that $h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}) = \delta_{ij}$ (see Refs. 53 and 55*).*

Remark 2.9. If $x = (x_i)$ and $y = (y_i)$ are dual to each other, then the $n \times n$ matrices $h_{ij} := h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $h^{ij} := h(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j})$ are inverse to each other, and the following relations hold: $\frac{\partial x_i}{\partial y_j} = h^{ij}$ and $\frac{\partial y_j}{\partial x_i} = h_{ij}$ (see Ref. 53*).*

Throughout this paper, we shall write $(x_1,\ldots,x_n,\dot{x}_1,\ldots,\dot{x}_n) = (x_i,\dot{x}_i)$ instead of $(x_i \circ \pi, dx_i)$ for simplicity. We shall also use the "hybrid" coordinate system $(y_1,\ldots,y_n,\dot{x}_1,\ldots,\dot{x}_n) = (y_i,\dot{x}_i)$. Thus by definition,

$$\begin{cases} (x,\dot{x})(v) := (x_1(p),\ldots,x_n(p),a_1,\ldots,a_n), \\ (y,\dot{x})(v) := (y_1(p),\ldots,y_n(p),a_1,\ldots,a_n), \end{cases} \quad \text{where} \quad v = a_1 \frac{\partial}{\partial x_1}\Big|_p + \cdots + a_n \frac{\partial}{\partial x_n}\Big|_p \in T_pM. \tag{23}$$

Proposition 2.10. Let (h,∇,∇^*) be a dually flat structure on a manifold M and let (g,J,ω) be the Kähler structure on TM associated to (h,∇) via Dombrowski's construction. Let also (x_i) and (y_i) be two coordinate systems on M dual to each other. Then locally,

(i) *in the coordinates $(x_i,\dot{x}_i)$,*

$$g = \begin{bmatrix} h_{ij} & 0 \\ 0 & h_{ij} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 & h_{ij} \\ -h_{ij} & 0 \end{bmatrix}, \tag{24}$$

where $h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, $i,j \in \{1,\ldots,n\}$,
(ii) *in the coordinates $(y_i,\dot{x}_i)$,*

$$g = \begin{bmatrix} h^{ij} & 0 \\ 0 & h_{ij} \end{bmatrix}, \quad J = \begin{bmatrix} 0 & -h_{ij} \\ h^{ij} & 0 \end{bmatrix}, \quad \omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \tag{25}$$

where $h^{ij} := h(\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}), i,j \in \{1,\ldots,n\}$.

Proof of Proposition 2.10. (i) Follows from Dombrowski's construction (see (12)) taking into account: (1) the explicit form of the isomorphism $T(TM) \to TM \oplus TM \oplus TM$ given in (11); (2) the formulas $K(a,b,c,d) = (a,d)$ and $\pi_*(a,b,c,d) = (a,c)$.

(ii) One has $(x,\dot{x})\circ(y,\dot{x})^{-1}=(x\circ y^{-1},\dot{x})$ and $\frac{\partial x_i}{\partial y_j}=h^{ij}$ (see Remark 2.9). Thus, the differential of $(x,\dot{x})\circ(y,\dot{x})^{-1}$ is given by

$$\big[(x,\dot{x})\circ(y,\dot{x})^{-1}\big]_*=\begin{bmatrix}h^{ij} & 0\\ 0 & I\end{bmatrix}. \tag{26}$$

From this together with the formula $h_{ij}h^{ij}=I$, one sees that the matrix representation of g in the coordinates $(y,\dot{x})$ is

$$^t\begin{bmatrix}h^{ij} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}h_{ij} & 0\\ 0 & h_{ij}\end{bmatrix}\begin{bmatrix}h^{ij} & 0\\ 0 & I\end{bmatrix}=\begin{bmatrix}h^{ij} & 0\\ 0 & I\end{bmatrix}\begin{bmatrix}h_{ij}h^{ij} & 0\\ 0 & h_{ij}\end{bmatrix}=\begin{bmatrix}h^{ij} & 0\\ 0 & h_{ij}\end{bmatrix} \tag{27}$$

(the superscript "t" means that we take the transpose of the corresponding matrix). The matrix representations of J and g are obtained similarly. The proposition follows. □

By inspection of (24) and (25), one sees that:

- If ∇ is flat (which means that J is integrable, see Remark 2.6), and if (x_i) is an affine coordinate system with respect to ∇, then
$$(z_1,\ldots,z_n):=(x_1+i\dot{x}_1,\ldots,x_n+i\dot{x}_n) \tag{28}$$
are holomorphic coordinates on the complex manifold (TM,J). To see this, compare (24) with, for example, the first chapter in Ref. 56.
- If (x_i) and (y_i) are dual to each other, then $(y_i,\dot{x}_i)$ are symplectic coordinates on TM, that is, $(y,\dot{x})$ is a Darboux chart for the symplectic manifold (TM,ω).

Remark 2.11. In the context of toric Kähler geometry, Abreu established formulas similar to (24) and (25) in order to get symplectic coordinates on toric manifolds (see Ref. 57). Abreu does not use the language of dually flat manifolds; instead, he focuses on the so-called Guillemin potential and its associated Hessian metric, in a spirit close to Ref. 55.

C. Ricci curvature

Let N be a Kähler manifold with Kähler metric g. We denote by Ric the Ricci tensor of g:

$$\mathrm{Ric}(X,Y):=\mathrm{Trace}\big\{Z\mapsto R(Z,X)Y\big\}, \tag{29}$$

where X,Y,Z are vector fields on N, and where R is the curvature tensor of g.

On the complexified tangent bundle $TN^{\mathbb{C}}=TN\otimes_{\mathbb{R}}\mathbb{C}$, we extend $\mathbb{C}$-linearly every tensor using the superscript "$\mathbb{C}$" to distinguish the corresponding extensions ($g^{\mathbb{C}}$, $\mathrm{Ric}^{\mathbb{C}}$, etc.).

Regarding local computations and indices, Greek indices α,β,γ shall run over $1,\ldots,n$ while capital letters $A,B,C,\ldots$ shall run over $1,\ldots,n,\bar{1},\ldots,\bar{n}$. Let $(z_1,\ldots,z_n)$ be a system of complex coordinates on N. If x_α and y_α are, respectively, the real part and the imaginary part of z_α (i.e., $z_\alpha=x_\alpha+iy_\alpha$), then fiberwise, the vectors

$$\frac{\partial}{\partial z_\alpha}:=\frac{1}{2}\Big\{\frac{\partial}{\partial x_\alpha}-i\frac{\partial}{\partial y_\alpha}\Big\},\qquad \frac{\partial}{\partial \bar{z}_\alpha}:=\frac{1}{2}\Big\{\frac{\partial}{\partial x_\alpha}+i\frac{\partial}{\partial y_\alpha}\Big\} \tag{30}$$

form a basis for $TN^{\mathbb{C}}$. Let $\mathrm{Ric}^{\mathbb{C}}_{AB}$ be the components of $\mathrm{Ric}^{\mathbb{C}}$ in this basis, i.e.,

$$\mathrm{Ric}^{\mathbb{C}}_{AB}:=\mathrm{Ric}^{\mathbb{C}}(Z_A,Z_B),\quad \text{where}\quad Z_\alpha:=\frac{\partial}{\partial z_\alpha}\quad\text{and}\quad Z_{\bar{\alpha}}=\frac{\partial}{\partial \bar{z}_\alpha}. \tag{31}$$

As it is well-known, these components are elegantly expressed via the following formulas (see Refs. 56 and 58):

$$\mathrm{Ric}^{\mathbb{C}}_{\alpha\beta}=\mathrm{Ric}^{\mathbb{C}}_{\bar{\alpha}\bar{\beta}}=0,\qquad \mathrm{Ric}^{\mathbb{C}}_{\bar{\alpha}\beta}=\overline{\mathrm{Ric}^{\mathbb{C}}_{\alpha\bar{\beta}}},\qquad \mathrm{Ric}^{\mathbb{C}}_{\alpha\bar{\beta}}=-\frac{\partial^2\ln d}{\partial z_\alpha\partial z_{\bar{\beta}}}, \tag{32}$$

where d is the determinant of the matrix $g^{\mathbb{C}}_{\alpha\bar{\beta}}=g^{\mathbb{C}}(Z_\alpha,Z_{\bar{\beta}})$.

We now specialize to the case $N = TM$, assuming that g is the Kähler metric associated to a dually flat structure (h, ∇, ∇^*) on an M via Dombrowski's construction.

Fix an affine coordinate system $(x_1, \ldots, x_n)$ with respect to ∇, and denote by $(x_\alpha, \dot{x}_\alpha)$ the corresponding coordinates on TM, as defined in Sec. II B. If $z_\alpha := x_\alpha + i\dot{x}_\alpha$, then $(z_1, \ldots, z_n)$ is a system of complex coordinates on TM, and one can apply (32). One obtains

$$g^{\mathbb{C}}_{\alpha\bar{\beta}} = \frac{1}{2} h_{\alpha\beta} \circ \pi \quad \text{and} \quad \mathrm{Ric}^{\mathbb{C}}_{\alpha\bar{\beta}} = -\frac{1}{4}\Big(\frac{\partial^2 \ln\, d}{\partial x_\alpha \partial x_\beta}\Big) \circ \pi, \tag{33}$$

where d is the determinant of the matrix $h_{\alpha\beta} = h(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta})$. The second formula in (33) is the local expression for the Ricci tensor in the basis $\{Z_\alpha, Z_{\bar{\alpha}}\}$. Returning to the coordinates $(x, \dot{x})$, a direct calculation using

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial z_\alpha} + \frac{\partial}{\partial \bar{z}_\alpha} \qquad \text{and} \qquad \frac{\partial}{\partial \dot{x}_\alpha} = i\Big(\frac{\partial}{\partial z_\alpha} - \frac{\partial}{\partial \bar{z}_\alpha}\Big) \tag{34}$$

shows the following result.

Proposition 2.12. Let (h, ∇, ∇^*) be a dually flat structure on a manifold M and g the Kähler metric on TM associated to (h, ∇) via Dombrowski's construction. If $x = (x_1, \ldots, x_n)$ is an affine coordinate system on M, then in the coordinates $(x, \dot{x})$, the matrix representation of the Ricci tensor of g is

$$\mathrm{Ric}(x, \dot{x}) = \begin{bmatrix} \beta_{\alpha\beta}(x) & 0 \\ 0 & \beta_{\alpha\beta}(x) \end{bmatrix}, \quad \text{where} \quad \beta_{\alpha\beta} = -\frac{1}{2}\frac{\partial^2 \ln d}{\partial x_\alpha \partial x_\beta}, \tag{35}$$

and where d is the determinant of the matrix $h_{\alpha\beta} = h(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta})$.

Recall that the scalar curvature is by definition, the trace of the Ricci tensor.

Corollary 2.13. In the coordinates $(x, \dot{x})$, the scalar curvature of g is given by

$$\mathrm{Scal}(x, \dot{x}) = -\sum_{\alpha,\beta=1}^{n} h^{\alpha\beta}(x) \frac{\partial^2 \ln d}{\partial x_\alpha \partial x_\beta}(x), \tag{36}$$

where d is the determinant of the matrix $h_{\alpha\beta}$, and where $h^{\alpha\beta}$ are the coefficients of the inverse matrix of $h_{\alpha\beta}$.

Remark 2.14. Observe that the scalar curvature on TM can be written $\mathrm{Scal} = S \circ \pi$, *where $S : M \to \mathbb{R}$ is a globally defined function whose local expression is given by the right hand side of (36) (see also Ref. 55).*

D. Completeness

Let (h, ∇, ∇^*) be a dually flat structure on a manifold M. We denote by g the Riemannian metric on TM associated to (h, ∇) via Dombrowski's construction. The corresponding Riemannian distances on M and TM are, respectively, denoted by d and ρ.

Proposition 2.15. In this situation, we have

$$(TM, \rho) \ \text{ is complete} \quad \Leftrightarrow \quad (M, d) \ \text{ is complete.} \tag{37}$$

The rest of this section is devoted to the proof of Proposition 2.15.

Lemma 2.16. If (TM, ρ) is complete, then (M, d) is complete.

Proof. This is a direct consequence of the fact that $\pi : (TM, g) \to (M, h)$ is a Riemannian submersion (take horizontal geodesics in TM and project them on M). □

From now on we assume (M,d) is complete. Let us fix a Cauchy sequence $(v_n)_{n\in\mathbb{N}}$ in (TM,ρ). Since π is a Riemannian submersion,

$$d(\pi(u),\pi(v)) \leq \rho(u,v) \quad \text{for all } u,v \in TM. \tag{38}$$

In particular, if $p_n := \pi(v_n)$, then $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (M,d), and there exists $p \in M$ such that $p_n \to p$ when $n \to \infty$. Take an affine coordinate system $x : U \to \mathbb{R}^n$ around p. We denote by h_{eu} the Euclidean metric pulled-back on U via the coordinate system $x : U \to \mathbb{R}^n$. By restricting U if necessary, we can assume that there exists $C > 0$ such that (by local compactness):

$$(h_{eu})_q(u,u) \leq C\, h_q(u,u) \quad \text{for all } \ q \in U \ \text{ and all } \ u \in T_qM. \tag{39}$$

We also choose $\varepsilon > 0$ and $N \in \mathbb{N}$ such that

$$\begin{aligned}
&B(p,3\,\varepsilon) := \{q \in M \mid d(q,p) < 3\,\varepsilon\} \subseteq U,\\
&B(p,3\,\varepsilon) \quad \text{is a normal ball},\\
&n,m \geq N \qquad \Rightarrow \qquad \rho(v_n,v_m) < \varepsilon,\\
&\quad n \geq N \qquad \Rightarrow \qquad v_n \in \pi^{-1}\big(B(p,\varepsilon)\big).
\end{aligned}$$

Lemma 2.17. Let $\gamma(t)$ be a piecewise smooth curve in TM joining v_n and v_m $(n,m \geq N)$. If the length $l(\gamma)$ of γ is less than $2\,\varepsilon$, then $c(t) := (\pi \circ \gamma)(t)$ lies in $B(p,3\,\varepsilon)$ for all t. In particular, $\gamma(t) \in \pi^{-1}(U)$ for all t.

Proof. By hypothesis, $l(\gamma) < 2\,\varepsilon$, and since π is a Riemannian submersion, $l(c) \leq l(\gamma)$. Thus, $l(c) < 2\,\varepsilon$. Therefore, $c(t)$ is a curve in M whose extremities p_n and p_m lie in $B(p,\varepsilon)$ and such that $l(c) < 2\,\varepsilon$. Since $B(p,3\,\varepsilon)$ is a normal ball, this implies $c(t) \in B(p,3\,\varepsilon)$ for all t (otherwise, we would have $l(c) \geq 2\,\varepsilon$ by application of the Gauss Lemma). The lemma follows. □

Let $\gamma(t)$ be a curve in TM as in Lemma 2.17, with $l(\gamma) < 2\,\varepsilon$. Since $\gamma(t) \in \pi^{-1}(U)$ for all t, one can represent γ in the coordinates $(x_i,\dot{x}_i)$:

$$\tilde{\gamma}(t) := (x_i,\dot{x}_i)(\gamma(t)) = \big(c_1(t),\ldots,c_n(t),V_1(t),\ldots,V_n(t)\big). \tag{40}$$

If $\gamma(t)$ is regarded as a vector field $V(t)$ along the curve $c(t) = (\pi \circ \gamma)(t)$, then $(c_1(t),\ldots,c_n(t))$ and $(V_1(t),\ldots,V_n(t))$ are just the local expressions for $c(t)$ and $V(t)$ in the coordinates (x_i). Observe also that the local expression for the covariant derivative $\nabla_{\dot{c}}V$ is exactly $(\dot{V}_1,\ldots,\dot{V}_n)$ since (x_i) are affine coordinates.

Similarly, we denote by $\tilde{v}_n$ the local representation of v_n in the coordinates $(x_i,\dot{x}_i)$ $(n \geq N)$. This defines a sequence $(\tilde{v}_n)_{n\in\mathbb{N}}$ in $W \subseteq \mathbb{R}^{2n}$, where

$$W := x(\overline{B(p,\varepsilon)}) \times \mathbb{R}^n, \tag{41}$$

and where $\overline{B(p,\varepsilon)}$ is the closure of $B(p,\varepsilon)$ in M.

Lemma 2.18. $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in W with respect to the Euclidean distance.

Proof. Let $\gamma(t)$ be a curve in TM joining v_n and v_m $(n,m \geq N)$, whose length is less than $2\,\varepsilon$. If γ is smooth at t, then

$$\left\|\frac{d\tilde{\gamma}}{dt}\right\|^2 = \sum_{i=1}^{n}\Big(|\dot{c}_i(t)|^2 + |\dot{V}_i|^2\Big) = (h_{eu})_{c(t)}(\dot{c},\dot{c}) + (h_{eu})_{c(t)}\big(\nabla_{\dot{c}}V,\nabla_{\dot{c}}V\big). \tag{42}$$

Let $l_{eu}(\tilde{\gamma})$ be the length of $\tilde{\gamma}$ with respect to the Euclidean metric and $l(\gamma)$ the length of γ with respect to g. Taking into account (39), (42) as well as Remark 2.2, we see that $l_{eu}(\tilde{\gamma}) \leq \sqrt{C}\, l(\gamma)$, from which we get,

$$\|\tilde{v}_n - \tilde{v}_m\| = \|\tilde{\gamma}(0) - \tilde{\gamma}(1)\| \leq l_{eu}(\tilde{\gamma}) \leq \sqrt{C}\, l(\gamma). \tag{43}$$

Hence, $\|\tilde{v}_n - \tilde{v}_m\| \leq \sqrt{C}\, l(\gamma)$ for all curves γ joining v_n and v_m with $l(\gamma) < 2\varepsilon$. In particular, using a sequence $(\gamma_k)_{k\in\mathbb{N}}$ of curves joining v_n and v_m and such that $l(\gamma_k) \to \rho(v_n,v_m)$, we deduce that

$$\|\tilde{v}_n - \tilde{v}_m\| \leq \sqrt{C}\, \rho(v_n,v_m). \tag{44}$$

Since $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in (TM,ρ), we conclude that $(\tilde{v}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in W. The lemma follows. □

Since W is complete (it is a closed subspace of the Euclidean space $\mathbb{R}^{2n}$), $(\tilde{v}_n)_{n\in\mathbb{N}}$ converges in W, and consequently, $(v_n)_{n\in\mathbb{N}}$ converges in $\pi^{-1}(U)\subseteq TM$. This achieves the proof of Proposition 2.15.

Remark 2.19. The above proof is inspired by a paper of Ebin where the following similar result is shown (see Ref. 59). Let M be a Hilbert manifold endowed with a Riemannian metric h and Levi-Civita connection ∇, not necessarily flat. Let also g be the Riemannian metric on TM associated to (h,∇) via Dombrowski's construction. In this situation, if M is complete, then TM is complete.

E. Kähler functions

Let N be a Kähler manifold with Kähler structure (g,J,ω).

Definition 2.20. A smooth function $f : N\to\mathbb{R}$ is called a Kähler function if it satisfies

$$\mathscr{L}_{X_f}g = 0, \tag{45}$$

where X_f is the Hamiltonian vector field associated to f (i.e., $\omega(X_f,\,.\,)=df(.)$) and where $\mathscr{L}_{X_f}$ is the Lie derivative in the direction of X_f.

Following Ref. 5, we shall denote by $\mathscr{K}(N)$ the space of Kähler functions on N. When N has a finite number of connected components, then $\mathscr{K}(N)$ is a finite dimensional[60,97] Lie algebra for the Poisson bracket $\{f,g\}:=\omega(X_f,X_g)$.

Given a smooth function $f : N\to\mathbb{R}$, we denote by $\mathrm{Hess}(f)$ the Riemannian Hessian of f with respect to g. If D denotes the Levi-Civita connection with respect to g, then by definition

$$\mathrm{Hess}(f)(u,v) = g\big(D_u\mathrm{grad}(f),v\big), \tag{46}$$

where $u,v\in TM$, and where $\mathrm{grad}(f)$ is the Riemannian gradient of f with respect to g, i.e., $g(\mathrm{grad}(f),\,.\,)=df(.)$. It can be shown that $\mathrm{Hess}(f)$ is a symmetric tensor (see Ref. 61).

Proposition 2.21 (Cirelli-Manià-Pizzocchero[5]). A smooth function $f : N\to\mathbb{R}$ is Kähler if and only if

$$\mathrm{Hess}(f)(JX,JY) = \mathrm{Hess}(f)(X,Y) \tag{47}$$

for all vector fields X,Y on N.

We now specialize to the case $N=TM$, assuming that g is the Kähler metric associated to a dually flat structure (h,∇,∇^*) on an M via Dombrowski's construction. We denote by $\pi : TM\to M$ the canonical projection.

Let $x : U\to\mathbb{R}^n$ be an affine coordinate system on M with associated coordinates $(x_i,\dot{x}_i)$ on $\pi^{-1}(U)\subseteq TM$. For $i\in\{1,\ldots,n\}$, set

$$\xi_i := \frac{\partial}{\partial x_i}, \qquad \xi_{\overline{i}} := \frac{\partial}{\partial \dot{x}_i}, \tag{48}$$

and denote by $(\Gamma^g)^C_{AB}$ the Christoffel symbols of g in the basis $\{\xi_1,\ldots,\xi_n,\xi_{\overline{1}},\ldots,\xi_{\overline{n}}\}$,

$$D_{\xi_A}\xi_B = \sum_C (\Gamma^g)^C_{AB}\,\xi_C, \tag{49}$$

where $A,B,C\in\{1,\ldots,n,\overline{1},\ldots,\overline{n}\}$. We also denote by $(\Gamma^h)^k_{ij}$ the Christoffel symbols of h in the coordinates $(x_1,\ldots,x_n)$.

Lemma 2.22. For $i, j, k \in \{1, \ldots, n\}$, we have

$$(\Gamma^g)^k_{ij} = (\Gamma^h)^k_{ij} \circ \pi, \qquad (\Gamma^g)^k_{\bar{i}j} = 0, \qquad (\Gamma^g)^k_{\bar{i}\bar{j}} = -(\Gamma^h)^k_{ij} \circ \pi,$$
$$(\Gamma^g)^{\bar{k}}_{ij} = 0, \qquad (\Gamma^g)^{\bar{k}}_{\bar{i}j} = (\Gamma^h)^k_{ij} \circ \pi, \qquad (\Gamma^g)^{\bar{k}}_{\bar{i}\bar{j}} = 0. \tag{50}$$

Proof. By a direct calculation. □

Remark 2.23. Similar formulas can be obtained in relation to the curvature. Indeed, if R^g and R^h are the curvature tensors of g and h, respectively, then one can show that

$$(R^g)^a_{\bar{i}\bar{j}k} = \Big(-(R^h)^a_{ijk} + \frac{\partial(\Gamma^h)^a_{ik}}{\partial x_j} - \frac{\partial(\Gamma^h)^a_{jk}}{\partial x_i}\Big) \circ \pi, \tag{51}$$

and similar for $(R^g)^a_{ijk}, (R^g)^{\bar{a}}_{\bar{i}jk}$, etc. In particular, one can prove the Ricci curvature formula given in Proposition 2.12 without using the classical formulas (32).[62]

Lemma 2.24. Let $f : TM \to \mathbb{R}$ be a smooth function. Then, on $\pi^{-1}(U)$,

$$\mathrm{Hess}(f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{b=1}^n (\Gamma^h_{ij})^b \circ \pi \, \frac{\partial f}{\partial x_b}, \qquad \mathrm{Hess}(f)_{i\bar{j}} = \frac{\partial^2 f}{\partial x_i \partial \dot{x}_j} - \sum_{b=1}^n (\Gamma^h_{ij})^b \circ \pi \, \frac{\partial f}{\partial \dot{x}_b}, \tag{52}$$

$$\mathrm{Hess}(f)_{\bar{i}\bar{j}} = \frac{\partial^2 f}{\partial \dot{x}_i \partial \dot{x}_j} + \sum_{b=1}^n (\Gamma^h_{ij})^b \circ \pi \, \frac{\partial f}{\partial x_b}. \tag{53}$$

Proof. By a direct calculation using Lemma 2.22 and the definition of Hess(f). □

Proposition 2.25. Let (h, ∇, ∇^*) be a dually flat structure on a manifold M and let (g, J, ω) be the Kähler structure on TM associated to (h, ∇) via Dombrowski's construction. Let $f : TM \to \mathbb{R}$ be a smooth function. Given an affine coordinate system $x : U \to \mathbb{R}^n$ with respect to ∇ on M, we have the following equivalence: f is Kähler on $\pi^{-1}(U)$ if and only if

$$\begin{cases} \dfrac{\partial^2 f}{\partial x_i \partial x_j} - \dfrac{\partial^2 f}{\partial \dot{x}_i \partial \dot{x}_j} = 2\displaystyle\sum_{b=1}^n (\Gamma^h_{ij})^b \circ \pi \, \frac{\partial f}{\partial x_b}, \\ \dfrac{\partial^2 f}{\partial x_i \partial \dot{x}_j} + \dfrac{\partial^2 f}{\partial x_j \partial \dot{x}_i} = 2\displaystyle\sum_{b=1}^n (\Gamma^h_{ij})^b \circ \pi \, \frac{\partial f}{\partial \dot{x}_b}, \end{cases} \tag{54}$$

for all $i, j = 1, \ldots, n$.

Proof. According to Proposition 2.21, f is Kähler if and only if $\mathrm{Hess}(f)(JX, JY) = \mathrm{Hess}(f)(X, Y)$ for all vector fields X, Y. If $\mathrm{Hess}(f) = \left[\begin{smallmatrix} A & B \\ {}^tB & C \end{smallmatrix}\right]$ is the matrix representation of Hess(f) in the coordinates $(x_i, \dot{x}_i)$, then this condition reads

$${}^t\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ {}^tB & C \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ {}^tB & C \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} C & -{}^tB \\ -B & A \end{bmatrix} = \begin{bmatrix} A & B \\ {}^tB & C \end{bmatrix}, \tag{55}$$

that is, $A = C$ and ${}^tB = -B$. Writing explicitly these equations using Lemma 2.24 exactly yields (54). The proposition follows. □

Using the complex coordinates $(z_1, \ldots, z_n) := (x_1 + i\dot{x}_1, \ldots, x_n + i\dot{x}_n)$, one can rewrite Proposition 2.25 more compactly as follows.

Proposition 2.26. In the same situation as above,

$$f \text{ is Kähler on } \pi^{-1}(U) \quad \Leftrightarrow \quad \frac{\partial^2 f}{\partial z_i \partial z_j} = \sum_{b=1}^{n} (\Gamma^h)_{ij}^b \circ \pi \, \frac{\partial f}{\partial z_b} \quad \text{for all } i,j = 1,\ldots,n \tag{56}$$

(here $\frac{\partial}{\partial z_k} := \frac{1}{2}\{\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial \dot{x}_k}\}$*).*

Recall that a vector field X on a manifold M is ∇-parallel with respect to a connection ∇ if $\nabla_Y X = 0$ for all vector fields Y on M.

Corollary 2.27 (Molitor[3]). Let (h,∇,∇^*) *be a dually flat structure on a manifold* M *and let* (g,J,ω) *be the Kähler structure on* TM *associated to* (h,∇) *via Dombrowski's construction. Let* $f : M \to \mathbb{R}$ *be a smooth function. Then,*

$$f \circ \pi \quad \text{is Kähler} \quad \Leftrightarrow \quad \text{grad}(f) \quad \text{is } \nabla\text{-parallel}, \tag{57}$$

where grad(f) *is the Riemannian gradient of* f *with respect to* h*.*

Proof. We use Einstein summation convention and the notation $\partial_i = \frac{\partial}{\partial x_i}$, $h_{ij} = h(\partial_i,\partial_j)$. The coefficients of the inverse matrix of h_{ij} are denoted by h^{ij}. Let $f : M \to \mathbb{R}$ be a smooth function. In the affine coordinate system $(x_1,\ldots,x_n)$, we have:

$$\begin{aligned} h\big(\nabla_{\partial_i}\text{grad}(f),\partial_j\big) &= h\big(\nabla_{\partial_i}\big(h^{ab}\partial_b(f)\partial_a,\partial_j\big)\big) = h\big(\partial_i(h^{ab})\partial_b(f)\partial_a + h^{ab}\partial_i\partial_b(f)\partial_a,\partial_j\big) \\ &= h^{ab}h_{aj}\partial_i\partial_b(f) + \partial_i(h^{ab})\partial_b(f)h_{aj} = \partial_i\partial_j(f) + \partial_i\big(h^{ab}h_{aj}\partial_b(f)\big) \\ &\quad -h^{ab}\partial_i(h_{aj})\partial_b(f) - h^{ab}h_{aj}\partial_i\partial_b(f) \\ &= \partial_i\partial_j(f) + \partial_i\partial_j(f) - \partial_i\partial_j(f) - 2\frac{1}{2}h^{ab}\partial_a(h_{ij})\partial_b(f) \\ &= \partial_i\partial_j(f) - 2(\Gamma^h)_{ij}^b\partial_b(f), \end{aligned} \tag{58}$$

where we have used the formula $(\Gamma^h)_{ij}^b = \frac{1}{2}h^{ab}\partial_a(h_{ij})$ which comes from the fact that h is a Hessian metric (see Refs. 53 and 55). From this, it is clear that grad(f) is ∇-parallel if and only if locally $\partial_i\partial_j(f) - 2(\Gamma^h)_{ij}^b\partial_b(f) = 0$ for all $i,j = 1,\ldots,n$. But these are exactly the equations characterizing locally a Kähler function of the form $f \circ \pi$ (compare with Proposition 2.25). □

Let (x_i) (respectively, (y_i)) be an affine coordinate system with respect to a flat connection ∇ (respectively, ∇^*) on a Riemannian manifold (M,h). Assume that (h,∇,∇^*) is dually flat and that (x_i) and (y_i) are dual to each other (in particular, TM is a Kähler manifold for the Kähler structure associated to (h,∇) via Dombrowski's construction). Taking into account Remark 2.9, it is not difficult to see that grad$(y_i) = \frac{\partial}{\partial x_i}$, and since $\frac{\partial}{\partial x_i}$ is obviously ∇-parallel, we deduce the following result.

Proposition 2.28. In this situation, the function $y_i \circ \pi : \pi^{-1}(U) \to \mathbb{R}$ *is Kähler for all* $i = 1,\ldots,n$*.*

F. Application: Information geometry

Definition 2.29. A statistical manifold (or statistical model) is a couple (S,j)*, where* S *is a manifold and where* j *is an injective map from* S *to the space of all probability density functions* p *defined on a fixed measure space* (Ω,dx)*:*

$$j : S \hookrightarrow \left\{p : \Omega \to \mathbb{R} \,\middle|\, p \text{ is measurable}, \ p \geq 0 \ \text{ and } \int_\Omega p(x)\,dx = 1\right\}. \tag{59}$$

If $\xi = (\xi_1,\ldots,\xi_n)$ is a coordinate system on a statistical manifold S, then we shall indistinctly write $p(x;\xi)$ or $p_\xi(x)$ for the probability density function determined by ξ.

Given a "reasonable" statistical manifold S, it is possible to define a metric h_F and a family of connections $\nabla^{(\alpha)}$ on S ($\alpha \in \mathbb{R}$) in the following way: for a chart $\xi = (\xi_1, \ldots, \xi_n)$ of S, define

$$(h_F)_\xi(\partial_i, \partial_j) \quad := \mathbb{E}_{p_\xi}\big(\partial_i \ln(p_\xi) \cdot \partial_j \ln(p_\xi)\big), \tag{60}$$

$$\Gamma^{(\alpha)}_{ij,k}(\xi) \quad := \mathbb{E}_{p_\xi}\Big[\Big(\partial_i\partial_j \ln(p_\xi) + \frac{1-\alpha}{2}\partial_i \ln(p_\xi) \cdot \partial_j \ln(p_\xi)\Big)\,\partial_k \ln(p_\xi)\Big], \tag{61}$$

where $\mathbb{E}_{p_\xi}$ denotes the mean, or expectation, with respect to the probability $p_\xi\,dx$, and where ∂_i is a shorthand for $\frac{\partial}{\partial \xi_i}$. It can be shown that if the above expressions are defined and smooth for every chart of S (this is not always the case), then h_F is a well defined metric on S called the *Fisher metric*, and that the $\Gamma^{(\alpha)}_{ij,k}$'s define a connection $\nabla^{(\alpha)}$ via the formula $\Gamma^{(\alpha)}_{ij,k}(\xi) = (h_F)_\xi\big(\nabla^{(\alpha)}_{\partial_i}\partial_i, \partial_k\big)$ which is called the *α-connection*.

Among the α-connections, the (± 1)-connections are particularly important; the 1-connection is usually referred to as the *exponential connection*, also denoted $\nabla^{(e)}$, while the (-1)-connection is referred to as the *mixture connection*, denoted $\nabla^{(m)}$.

In this paper, we will only consider statistical manifolds S for which the Fisher metric and α-connections are well defined.

Proposition 2.30. Let S be a statistical manifold. Then, $(h_F, \nabla^{(\alpha)}, \nabla^{(-\alpha)})$ is a dualistic structure on S. In particular, $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$.

Proof. See Ref. 53. □

We now introduce an important class of statistical manifolds.

Definition 2.31. An exponential family $\mathcal{E}$ on a measure space (Ω, dx) is a set of probability density functions $p(x;\theta)$ of the form

$$p(x;\theta) = \exp\Big\{C(x) + \sum_{i=1}^{n} \theta_i F_i(x) - \psi(\theta)\Big\}, \tag{62}$$

where $C, F_1, \ldots, F_n$ are measurable functions on Ω, $\theta = (\theta_1, \ldots, \theta_n)$ is a vector varying in an open subset Θ of $\mathbb{R}^n$ and where ψ is a function defined on Θ.

In the above definition, it is assumed that the family $\{1, F_1, \ldots, F_n\}$ is linearly independent, so that the map $p(x,\theta) \mapsto \theta \in \Theta$ becomes a bijection, hence defining a global chart of $\mathcal{E}$. The parameters $\theta_1, \ldots, \theta_n$ are called the *natural* or *canonical parameters* of the exponential family $\mathcal{E}$.

Besides the natural parameters $\theta_1, \ldots, \theta_n$, an exponential family $\mathcal{E}$ possesses another particularly important parametrization which is given by the *expectation* or *dual parameters* $\eta_1, \ldots, \eta_n$:

$$\eta_i(p_\theta) := \mathbb{E}_{p_\theta}(F_i) = \int_\Omega F_i(x)\, p_\theta(x)\, dx. \tag{63}$$

It is not difficult, assuming ψ to be smooth, to show that $\eta_i(p_\theta) = \partial_{\theta_i}\psi$. The map $\eta = (\eta_1, \ldots, \eta_n)$ is thus a global chart of $\mathcal{E}$ provided that $(\partial_{\theta_1}\psi, \ldots, \partial_{\theta_n}\psi) : \Theta \to \mathbb{R}^n$ is a diffeomorphism onto its image, condition that we will always assume.

Proposition 2.32. Let $\mathcal{E}$ be an exponential family such as in (62). Then, $(\mathcal{E}, h_F, \nabla^{(e)}, \nabla^{(m)})$ is dually flat and $\theta = (\theta_1, \ldots, \theta_n)$ is an affine coordinate system with respect to $\nabla^{(e)}$ while $\eta = (\eta_1, \ldots, \eta_n)$ is an affine coordinate system with respect to $\nabla^{(m)}$. Moreover, the following relation holds:

$$h_F\Big(\frac{\partial}{\partial\theta_i}, \frac{\partial}{\partial\eta_j}\Big) = \delta_{ij}, \tag{64}$$

that is, θ and η are mutually dual coordinate systems.

Proof. See Ref. 53. □

Corollary 2.33. The tangent bundle $T\mathcal{E}$ of an exponential family $\mathcal{E}$ is a Kähler manifold for the Kähler structure (g, J, ω) associated to $(h_F, \nabla^{(e)})$ via Dombrowski's construction.

In the sequel, by the Kähler structure of $T\mathcal{E}$, we shall implicitly refer to the Kähler structure of $T\mathcal{E}$ described in Corollary 2.33.

Corollary 2.34 (Molitor[3]). Let $\mathcal{E}$ be an exponential family defined over a measure space (Ω, dx) (as in Definition 2.31), and let $\mathcal{A}_{\mathcal{E}}$ be the real vector space generated by the random variables $1, F_1, \ldots, F_n : \Omega \to \mathbb{R}$. In this situation, if $\Phi : T\mathcal{E} \to T\mathcal{E}$ is a holomorphic isometry and if $X \in \mathcal{A}_{\mathcal{E}}$, then the function

$$T\mathcal{E} \to \mathbb{R}, \quad p \mapsto \int_\Omega X(x)[(\pi \circ \Phi)(p)](x)dx \tag{65}$$

is Kähler (here, $\pi : T\mathcal{E} \to \mathcal{E}$ is the canonical projection).

Proof. Assume that $X = \lambda_0 + \lambda_1 F_1 + \cdots + \lambda_n F_n$, $\lambda_i \in \mathbb{R}$. Clearly, the above function is Kähler if and only if $T\mathcal{E} \ni p \mapsto \int_\Omega X(x)\pi(p)(x)dx$ is Kähler, which is the case since it is a linear combination of Kähler functions. Indeed, taking into account the definition of the expectation parameters η_i, one has

$$\int_\Omega X(x)\pi(p)(x)dx = \lambda_0 + \sum_{i=1}^n \lambda_i \int_\Omega F_i(x)\pi(p)(x)dx = \lambda_0 + \sum_{i=1}^n (\eta_i \circ \pi)(p), \tag{66}$$

and since θ and η are affine coordinate systems dual to each other (see Proposition 2.32), it follows from Proposition 2.28 that $\eta_i \circ \pi$ is Kähler for all $i = 1, \ldots, n$. The corollary follows. □

III. GAUSSIAN DISTRIBUTIONS: INTRINSIC GEOMETRY

Let $\mathcal{N}$ be the set of all Gaussian distributions of mean μ and deviation σ over $\mathbb{R}$, that is, $\mathcal{N}$ is the set of all $p(x; \mu, \sigma)$, where

$$p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{-\frac{(x-\mu)^2}{2\sigma^2}\Big\}. \tag{67}$$

It is a 2-dimensional statistical manifold parameterized by $\mu \in \mathbb{R}$ and $\sigma > 0$, and since $p(x; \mu, \sigma) = \exp\big\{F_1(x)\theta_1 + F_2(x)\theta_2 - \psi(\theta)\big\}$, where

$$\theta_1 = \frac{\mu}{\sigma^2},\ \ \theta_2 = -\frac{1}{2\sigma^2},\ \ C(x) = 0,\ \ F_1(x) = x,\ \ F_2(x) = x^2,\ \ \psi(\theta) = -\frac{(\theta_1)^2}{4\theta_2} + \frac{1}{2}\ln\Big(-\frac{\pi}{\theta_2}\Big), \tag{68}$$

it is also an exponential family (see Definition 2.31). Observe that $\theta_1 \in \mathbb{R}$ and $\theta_2 < 0$, and that the expectation parameters are (see Ref. 53)

$$\eta_1 = \mu = -\frac{\theta_1}{2\theta_2}, \qquad \eta_2 = \mu^2 + \sigma^2 = \frac{(\theta_1)^2 - 2\theta_2}{4(\theta_2)^2}. \tag{69}$$

We denote by $h_F, \nabla^{(e)}$, and $\nabla^{(m)}$ the Fisher metric, exponential connection, and mixture connection on $\mathcal{N}$, respectively. According to Proposition 2.32, $(h_F, \nabla^{(e)}, \nabla^{(m)})$ is a dually flat structure, and consequently, the almost Hermitian structure (g, J, ω) on $T\mathcal{N}$ associated to $(h_F, \nabla^{(e)})$ via Dombrowski's construction is Kähler.

In this section, we study the geometrical properties of $T\mathcal{N}$, regarded as a Kähler manifold.

A. Preliminaries: Siegel-Jacobi space and Jacobi group

Let Heis($\mathbb{R}$) and SL($2,\mathbb{R}$) denote, respectively, the *Heisenberg group* and the *special linear group* of dimension 3. Recall that Heis($\mathbb{R}$) can be identified with $\mathbb{R}^2 \times \mathbb{R}$ endowed with the multiplication

$$(X_1, \kappa_1) \cdot (X_2, \kappa_2) := \big(X_1 + X_2, \kappa_1 + \kappa_2 + \Omega(X_1, X_2)\big), \tag{70}$$

where Ω is the symplectic form on $\mathbb{R}^2$ whose matrix representation in the canonical basis of $\mathbb{R}^2$ is $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, i.e., $\Omega(X_1, X_2) := \lambda_1\mu_2 - \lambda_2\mu_1$, where $X_1 = (\lambda_1, \mu_1)$ and $X_2 = (\lambda_2, \mu_2)$. Recall also that

$$\mathrm{SL}(2,\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}(2,\mathbb{R}) \,\middle|\, ad - bc = 1 \right\} \tag{71}$$

(here, Mat($n,\mathbb{R}$) denote the space of $n \times n$ real matrices), and that we have the identification SL($2,\mathbb{R}$) = Sp($2,\mathbb{R}$), where

$$\mathrm{Sp}(2,\mathbb{R}) := \big\{ M \in \mathrm{Mat}(2,\mathbb{R}) \,\big|\, {}^t M \Omega M = \Omega \big\}, \quad \text{where} \quad \Omega = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]. \tag{72}$$

Let Aut$\big($Heis($\mathbb{R}$)$\big)$ denote the group of automorphisms of Heis($\mathbb{R}$), that is, the group of diffeomorphisms of Heis($\mathbb{R}$) that are also homeomorphisms. Consider the following map:

$$\tau \,:\, \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}\big(\mathrm{Heis}(\mathbb{R})\big), \quad \tau(M)(X, \kappa) := (XM, \kappa), \tag{73}$$

where $M \in$ SL($2,\mathbb{R}$), $(X, \kappa) \in$ Heis($\mathbb{R}$), and where XM has to be understood has the multiplication of a row vector with a 2×2 matrix. The fact that $\tau(M)$ is an automorphism of Heis($\mathbb{R}$) is a simple consequence of the identity SL($2,\mathbb{R}$) = Sp($2,\mathbb{R}$), and clearly, τ is an anti-homomorphism of groups, i.e., $\tau(M_1 M_2) = \tau(M_2) \circ \tau(M_2)$. Therefore, one can form the semi-direct product SL($2,\mathbb{R}$) $\ltimes$ Heis($\mathbb{R}$). By definition,[63] it is the Cartesian product SL($2,\mathbb{R}$) $\times$ Heis($\mathbb{R}$) endowed with the multiplication

$$(M_1, X_1, \kappa_1) \cdot (M_2, X_2, \kappa_2) := \big(M_1 M_2, X_1 M_2 + X_2, \kappa_1 + \kappa_2 + \Omega(X_1 M_2, X_2)\big), \tag{74}$$

where $M_1, M_2 \in$ SL($2,\mathbb{R}$), and $(X_1, \kappa_1), (X_2, \kappa_2) \in$ Heis($\mathbb{R}$). Following Refs. 23 and 24, we call SL($2,\mathbb{R}$) $\ltimes$ Heis($\mathbb{R}$) the *Jacobi group*, and denote it by $G^J(\mathbb{R})$, that is,

$$G^J(\mathbb{R}) := \mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{Heis}(\mathbb{R}). \tag{75}$$

We shall also consider the *affine symplectic group*,

$$\mathrm{ASp}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2, \tag{76}$$

which is by definition the semi-direct product of SL($2,\mathbb{R}$) with the abelian group $\mathbb{R}^2$ relative to the following anti-homomorphism of groups:

$$\tau \,:\, \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(\mathbb{R}^2), \quad \tau(M)X := XM \quad (\text{row vector } \times \text{ square matrix}), \tag{77}$$

where $M \in$ SL($2,\mathbb{R}$) and $X \in \mathbb{R}^2$. By definition, the group multiplication on ASp($2,\mathbb{R}$) is $(M_1, X_1) \cdot (M_2, X_2) = (M_1 M_2, X_1 M_2 + X_2)$, where $M_1, M_2 \in$ SL($2,\mathbb{R}$), and $X_1, X_2 \in \mathbb{R}^2$. Beware that ASp($2,\mathbb{R}$) is *not* a subgroup of $G^J(\mathbb{R})$, but the latter is a central extension of the former for, there is a short exact sequence of Lie groups,

$$\{e\} \longrightarrow \mathbb{R} \xrightarrow{i} G^J(\mathbb{R}) \xrightarrow{\pi} \mathrm{ASp}(2,\mathbb{R}) \to \{e\}, \tag{78}$$

where $i(\kappa) := \big(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], 0, \kappa\big)$ and $\pi\big(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], X, \kappa\big) := \big(\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right], X\big)$, and where obviously the image of i lies in the center of $G^J(\mathbb{R})$.

Let $\mathbb{H} = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ denote the upper half-plane. We define a left action of the Jacobi group $G^J(\mathbb{R})$ on $\mathbb{H} \times \mathbb{C}$ as follows:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu, \kappa) \right) \cdot (\tau, z) := \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d} \right), \tag{79}$$

where $(\tau, z) \in \mathbb{H} \times \mathbb{C}$. It is not an effective action, but by "forgetting" κ in the above formula, one obtains a left action of $\mathrm{ASp}(2,\mathbb{R})$ on $\mathbb{H} \times \mathbb{C}$ which is effective. In particular, one can regard $\mathrm{ASp}(2,\mathbb{R})$ as a subgroup of the group $\mathrm{Diff}(\mathbb{H} \times \mathbb{C})$ of diffeomorphisms of $\mathbb{H} \times \mathbb{C}$.

Definition 3.1 (Kähler-Berndt metric). Let $A, B > 0$ be arbitrary. The Kähler-Berndt metric is the metric $g_{A,B}$ on $\mathbb{H} \times \mathbb{C}$ whose matrix representation in the coordinates (u, v, x, y) is

$$g_{A,B}(\tau, z) := \begin{bmatrix} \frac{Av+By^2}{v^3} & 0 & -\frac{By}{v^2} & 0 \\ 0 & \frac{Av+By^2}{v^3} & 0 & -\frac{By}{v^2} \\ -\frac{By}{v^2} & 0 & \frac{B}{v} & 0 \\ 0 & -\frac{By}{v^2} & 0 & \frac{B}{v} \end{bmatrix}, \tag{80}$$

where $\tau = u + iv \in \mathbb{H}$ and $z = x + iy \in \mathbb{C}$.

Remark 3.2. The Kähler-Berndt metric is a Kähler metric with respect to the natural complex structure of $\mathbb{H} \times \mathbb{C}$, invariant under the action of the Jacobi group $G^J(\mathbb{R})$ (see for example Refs. 64 and 65 *and below). It was introduced independently by Kähler and Berndt in the 80's for the following reasons. Berndt was apparently looking for an invariant Riemannian metric on $\mathbb{H} \times \mathbb{C}$ whose Laplacian could be used to impose good analytical conditions (like being an eigenfunction) on complex functions defined on $\mathbb{H} \times \mathbb{C}$, the objective being to define "Jacobi-like" functions;[66] this was just before Eichler and Zagier introduced and systematically studied Jacobi forms in their classic book.[24] Kähler, on the other hand, was apparently motivated by totally different reasons related to physics (see* Refs. 67 and 68*).*

Remark 3.3. Berceanu showed that the Kähler-Berndt metric can be understood within the group-theoretical framework of Perelomov's coherent states.[26,30,32–34]

In a series of papers, Yang introduced the terminology "Siegel-Jacobi space" (or "Siegel-Jacobi disk") for the complex space $\mathbb{H} \times \mathbb{C}$ together with a choice of one of the Kähler metrics $g_{A,B}$ above (see Refs. 64, 65, and 69–71). In this paper, we shall adopt the following definition.

Definition 3.4 (Siegel-Jacobi space $\mathbb{S}^J$). The Siegel-Jacobi space is the Kähler manifold

$$\mathbb{S}^J := \big(\mathbb{H} \times \mathbb{C}, \tfrac{1}{2} g_{1,1}\big). \tag{81}$$

In the sequel, we shall denote by g_{KB} and ω_{KB} the metric and symplectic form of $\mathbb{S}^J$, that is, $g_{KB} := \frac{1}{2} g_{1,1}$. From now on, we shall refer to this metric as the *Kähler-Berndt metric*.

B. Kähler structure

In this section, we return to the study of the Kähler structure (g, J, ω) of $T\mathcal{N}$. We start by recalling the following result (see Ref. 53).

Proposition 3.5.

(i) *In the natural coordinates $\theta = (\theta_1, \theta_2)$, the Fisher metric reads*

$$h_F(\theta) = \frac{1}{2(\theta_2)^2} \begin{bmatrix} -\theta_2 & \theta_1 \\ \theta_1 & \frac{\theta_2 - (\theta_1)^2}{\theta_2} \end{bmatrix}, \tag{82}$$

(ii) *in the coordinates (θ_1, θ_2), the Christoffel symbols Γ^k_{ij} of h_F are*

$$\Gamma^1_{11}(\theta) = \frac{\theta_1}{2\theta_2}, \qquad \Gamma^1_{12}(\theta) = -\frac{(\theta_1)^2 + \theta_2}{2(\theta_2)^2}, \qquad \Gamma^1_{22}(\theta) = \frac{1}{2}\Big(\frac{\theta_1}{\theta_2}\Big)^3, \tag{83}$$

$$\Gamma^2_{11}(\theta) = \frac{1}{2}, \qquad \Gamma^2_{12}(\theta) = -\frac{\theta_1}{2\theta_2}, \qquad \Gamma^2_{22}(\theta) = \frac{(\theta_1)^2 - 2\theta_2}{2(\theta_2)^2}, \tag{84}$$

(iii) *$(\mathcal{N}, h_F)$ is a complete Riemannian manifold with constant sectional curvature $-\frac{1}{2}$.*

Proposition 3.6. As a Kähler manifold, $T\mathcal{N}$ is the Siegel-Jacobi space $\mathbb{S}^J$ (see Definition 3.4), that is,

$$T\mathcal{N} \cong \mathbb{S}^J. \tag{85}$$

Proof. According to Proposition 2.32, (θ_1, θ_2) are affine coordinates with respect to $\nabla^{(e)}$. Consequently, one can apply Proposition 2.10 and conclude that in the coordinates $(\theta, \dot{\theta}) = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$ the matrix representations of g, J, ω are

$$g(\theta,\dot{\theta}) = \begin{bmatrix} h_F(\theta) & 0 \\ 0 & h_F(\theta) \end{bmatrix}, \quad J(\theta,\dot{\theta}) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}, \quad \omega(\theta,\dot{\theta}) = \begin{bmatrix} 0 & h_F(\theta) \\ -h_F(\theta) & 0 \end{bmatrix}, \tag{86}$$

where $h_F(\theta)$ is given in (82), and where I is the 2×2 identity matrix (recall that $\dot{\theta}_k$ is just the differential of θ_k, regarded as a function $T\mathcal{N} \to \mathbb{R}$). From a complex point of view, we know that $(z_1, z_2) := (\theta_1 + i\dot{\theta}_1, \theta_2 + i\dot{\theta}_2)$ are global holomorphic coordinates on the complex manifold $(T\mathcal{N}, J)$ (see (28)). Consequently, one has an identification of complex manifolds $T\mathcal{N} \cong \mathbb{C} \times i\mathbb{H}$ (observe that $i\mathbb{H} = \{z \in \mathbb{C} \,|\, \mathrm{Real}(z) < 0\}$). Let f be the map

$$T\mathcal{N} \cong \mathbb{C} \times i\mathbb{H} \to \mathbb{H} \times \mathbb{C}, \;\; (z_1, z_2) \mapsto (-iz_2, iz_1). \tag{87}$$

Clearly, f is biholomorphic, and in the coordinates $(\theta, \dot{\theta})$ on $T\mathcal{N}$ and (u, v, x, y) on $\mathbb{H} \times \mathbb{C}$ (see Definition 3.1), it reads $f(\theta, \dot{\theta}) = (\dot{\theta}_2, -\theta_2, -\dot{\theta}_1, \theta_1)$. Now, using (86) together with the explicit description of $g_{1,1}$ given in Definition 3.1, a straightforward computation shows that $f^* g_{KB} = g$. The proposition follows. □

Proposition 3.7. $(T\mathcal{N}, g)$ is complete.

Proof. There are two ways to prove it. The first is to use Proposition 2.15 and the fact that $(\mathcal{N}, h_F)$ is complete (see Proposition 3.5). The second is to observe that the Siegel-Jacobi space $\mathbb{S}^J$ is a homogeneous Riemannian manifold (see Remark 3.2 and Proposition 3.31). □

Proposition 3.8. In the coordinates $(\theta, \dot{\theta})$, the matrix representation of the Ricci tensor of g is

$$\mathrm{Ric}(\theta,\dot{\theta}) = \begin{bmatrix} \beta(\theta) & 0 \\ 0 & \beta(\theta) \end{bmatrix}, \quad \text{where} \quad \beta(\theta) = -\frac{3}{2} \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{(\theta_2)^2} \end{bmatrix}. \tag{88}$$

Proof. Follows from Propositions 2.12 and 3.5. □

From Proposition 3.8, one easily deduces the following corollary.

Corollary 3.9.

(i) *$Ric(X, X) \leq 0$ for all $X \in T(T\mathcal{N})$.*
(ii) *$(T\mathcal{N}, g)$ is not Einstein.*[72,98] *In particular, the holomorphic sectional curvature*[73,99] *of $T\mathcal{N}$ is not constant.*
(iii) *The scalar curvature of $(T\mathcal{N}, g)$ is constant and equal to -6.*

Remark 3.10. Since $T\mathcal{N} \cong \mathbb{S}^J$, one has the analogues of Proposition 3.7, Proposition 3.8, and Corollary 3.9 for the Siegel-Jacobi space $\mathbb{S}^J$. The analogue of Corollary 3.9 for $\mathbb{S}^J$ was established by Yang in Ref. 69*, and later on generalized by Berceanu*[34] *and Yang*[71] *for the metric $g_{A,B}$. They showed, in particular, that the scalar curvature of $g_{A,B}$ is constant and equal to $-\frac{3}{A}$.*

C. The group of holomorphic isometries

Recall that the affine symplectic group $\mathrm{ASp}(2,\mathbb{R})$ acts effectively on the Siegel-Jacobi space $\mathbb{S}^J \cong T\mathcal{N}$. Therefore, $\mathrm{ASp}(2,\mathbb{R})$ can be regarded as a subgroup of the group $\mathrm{Diff}(T\mathcal{N})$ of diffeomorphisms of $T\mathcal{N}$. Recall also that the group of holomorphic isometries of $T\mathcal{N}$ is the subgroup of $\mathrm{Diff}(T\mathcal{N})$ whose elements satisfy $\varphi^* g = g$ and $\varphi_* J = J\varphi_*$.

Theorem 3.11. *The group of holomorphic isometries of $T\mathcal{N}$ is the affine symplectic group* $\mathrm{ASp}(2,\mathbb{R})$.

As explained below, our proof relies on the resolution of the following system of partial differential equations

$$\begin{cases} \Big(\dfrac{\partial u}{\partial x}\Big)^2 + \Big(\dfrac{\partial u}{\partial y}\Big)^2 = \Big(\dfrac{u}{x}\Big)^2, \\ \Delta u \equiv 0, \end{cases} \tag{89}$$

where $u(x, y)$ is a smooth function defined on $U := \big\{(x, y) \in \mathbb{R}^2 \,\big|\, x < 0\big\}$, and where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplace operator.

Remark 3.12. If a solution u of the first equation in (89) satisfies $u(x, y) < 0$ for all $(x, y) \in U$, then $v := \ln(-u)$ is a solution of the 2-dimensional Eikonal equation

$$\Big(\frac{\partial v}{\partial x}\Big)^2 + \Big(\frac{\partial v}{\partial y}\Big)^2 = \frac{1}{\big(f(x, y)\big)^2}, \tag{90}$$

with $f(x, y) = x$. In geometrical optics, the Eikonal equation describes the wave fronts of light in an inhomogeneous medium with a variable index of refraction $\frac{1}{f^2}$ (see for example Refs. 74 and 75*). Mathematically, only a few explicit solutions are known (see Refs.* 76 and 77*).*

Remark 3.13. Every solution of (89) is real analytic (since it is harmonic). In particular, if u, v are two solutions of (89) which coincide on an open subset of U, then they coincide on U (see Ref. 78*).*

Let us fix a smooth solution u of (89) satisfying $u(x, y) < 0$ for all $(x, y) \in U$ (this last condition will be justified below). Set

$$U_0 := \Big\{(x, y) \in U \,\Big|\, \frac{\partial u}{\partial y}(x, y) = 0\Big\}. \tag{91}$$

Lemma 3.14. If $U_0 = U$ (i.e., $\frac{\partial u}{\partial y} \equiv 0$ on U), then there exists $a \in \mathbb{R}, a \neq 0$, such that for all $(x, y) \in U$,

$$u(x, y) = a^2 x. \tag{92}$$

Proof. By a direct calculation. □

Let us now assume $U_0 \neq U$. This means that there exists $p = (p_1, p_2) \in U$ such that $\frac{\partial u}{\partial y}(p) \neq 0$. Without loss of generality, we can assume $\frac{\partial u}{\partial y}(p) > 0$ (the case < 0 is completely analog). Fix $\varepsilon > 0$ such that

$$\frac{\partial u}{\partial y}(q) > 0 \quad \text{for all} \quad q \in \,]\,p_1 - \varepsilon, p_1 + \varepsilon\,[\times]\,p_2 - \varepsilon, p_2 + \varepsilon\,[=: C. \tag{93}$$

On C, there exists a smooth function $\alpha : C \to \mathbb{R}$ which satisfies (see the first equation in (89))

$$\frac{x}{u}\frac{\partial u}{\partial x} = \cos(\alpha(x, y)) \qquad \text{and} \qquad \frac{x}{u}\frac{\partial u}{\partial y} = \sin(\alpha(x, y)) \tag{94}$$

for all $(x, y) \in C$. By specifying the image of α, such a function is unique. We choose $0 < \alpha < \pi$.

Lemma 3.15. We have

$$\begin{cases} \dfrac{\partial \alpha}{\partial x} &= \dfrac{\sin(\alpha)}{x}, \\ \dfrac{\partial \alpha}{\partial y} &= \dfrac{1-\cos(\alpha)}{x}. \end{cases} \tag{95}$$

Proof. Observe that (94) can be rewritten

$$\frac{\partial}{\partial x}\big(\ln(-u)\big) = \frac{\cos(\alpha)}{x} \qquad \text{and} \qquad \frac{\partial}{\partial y}\big(\ln(-u)\big) = \frac{\sin(\alpha)}{x}. \tag{96}$$

Taking the partial derivative with respect to y of the first equation and the partial derivative with respect to x of the second equation immediately yields the equality

$$\frac{\partial}{\partial y}\Big(\frac{\cos(\alpha)}{x}\Big) = \frac{\partial}{\partial x}\Big(\frac{\sin(\alpha)}{x}\Big) \tag{97}$$

which can be rewritten

$$\cos(\alpha)\frac{\partial \alpha}{\partial x} + \sin(\alpha)\frac{\partial \alpha}{\partial y} = \frac{\sin(\alpha)}{x}. \tag{98}$$

On the other hand, the equation $\Delta u \equiv 0$ together with (94) yields

$$\frac{\partial}{\partial x}\Big(\frac{u}{x}\cos(\alpha)\Big) + \frac{\partial}{\partial y}\Big(\frac{u}{x}\sin(\alpha)\Big) = 0, \tag{99}$$

which is equivalent to

$$-\sin(\alpha)\frac{\partial \alpha}{\partial x} + \cos(\alpha)\frac{\partial \alpha}{\partial y} = \frac{\cos(\alpha)-1}{x}. \tag{100}$$

Multiplying (98) by $\sin(\alpha)$ (respectively, $\cos(\alpha)$) and (100) by $\cos(\alpha)$ (respectively, $\sin(\alpha)$), then summing (respectively, subtracting) exactly yields (95). The lemma follows. □

Lemma 3.16. There exists $b \in \mathbb{R}$ such that on C,

$$\begin{cases} \cos(\alpha(x,y)) &= \dfrac{(y+b)^2 - x^2}{(y+b)^2 + x^2}, \\ \sin(\alpha(x,y)) &= -\dfrac{2x(y+b)}{(y+b)^2 + x^2}. \end{cases} \tag{101}$$

Proof. According to Lemma 3.15, we have

$$\frac{1}{\sin(\alpha)}\frac{\partial \alpha}{\partial x} = \frac{1}{x} \quad \Rightarrow \quad \ln(\tan(\alpha/2)) = \ln(-x) + g(y) \quad \Rightarrow \quad \tan(\alpha/2) = -x\, e^{g(y)}, \tag{102}$$

$$\frac{1}{1-\cos(\alpha)}\frac{\partial \alpha}{\partial y} = \frac{1}{x} \quad \Rightarrow \quad -\frac{1}{\tan(\alpha/2)} = \frac{y}{x} + h(x) \quad \Rightarrow \quad \tan(\alpha/2) = -\frac{1}{\frac{y}{x} + h(x)}, \tag{103}$$

where g and h are smooth functions of the variables y and x, respectively. Thus,

$$-x\, e^{g(y)} = -\frac{1}{\frac{y}{x} + h(x)} \quad \Rightarrow \quad x h(x) = e^{g(y)} - y, \tag{104}$$

from which we deduce the existence of a constant $E \in \mathbb{R}$ such that $xh(x) = E$ and $e^{g(y)} - y = E$ for all $x \in\,]\, p_1 - \varepsilon, p_1 + \varepsilon\,[$ and all $y \in\,]\, p_2 - \varepsilon, p_2 + \varepsilon\,[$. Thus,

$$g(y) = -\ln(E + y), \qquad h(x) = \frac{E}{x}. \tag{105}$$

Taking into account the last equation in (102) (or (103)), we thus have

$$\alpha(x,y) = -2\arctan\Big(\frac{x}{y+E}\Big). \tag{106}$$

The lemma is now a simple consequence of (106) together with the following formulas: $\cos\big(2\arctan(r)\big) = \frac{1-r^2}{1+r^2}$ and $\sin\big(2\arctan(r)\big) = \frac{2r}{1+r^2}, r \in \mathbb{R}$. □

Lemma 3.17. There exists $a \in \mathbb{R}, a \neq 0$, and $b \in \mathbb{R}$ such that on U,

$$u(x,y) = \frac{a^2 x}{(y+b)^2 + x^2}. \tag{107}$$

Proof. Since $\frac{x}{u}\frac{\partial u}{\partial y} = \sin(\alpha)$, Lemma 3.16 implies that on C,

$$\frac{\partial}{\partial y}\big(\ln(-u)\big) = -\frac{2(y+b)}{(y+b)^2+x^2} \quad \Leftrightarrow \quad \ln(-u) = -\ln\big((y+b)^2 + x^2\big) + f(x) \tag{108}$$

$$\Leftrightarrow \quad u = -\frac{e^{f(x)}}{(y+b)^2+x^2}, \tag{109}$$

where f is a smooth function depending on the variable $x \in \,]p_1 - \varepsilon, p_1 + \varepsilon[$. In order to find f, we differentiate the right hand side of the equivalence in (108) and use $\frac{\partial}{\partial x}\big(\ln(-u)\big) = \frac{\cos(x)}{x}$. We obtain,

$$f'(x) - \frac{2x}{(y+b)^2+x^2} = \frac{1}{x}\frac{(y+b)^2 - x^2}{(y+b)^2+x^2}, \tag{110}$$

which leads to $xf'(x) = 1$, i.e., $f(x) = \ln(-x)$ (+ constant). Hence, (107) holds on C. Using the fact that u is analytic (see Remark 3.13), it also holds on U. The lemma follows. □

Collecting our results, we deduce the following.

Proposition 3.18. Let u be a solution of (89) satisfying $u(x,y) < 0$ for all $(x,y) \in U$. Then, u has the following form (two possibilities) :

(1) $u(x,y) = \frac{a^2 x}{(y+b)^2+x^2}, \ \ a, b \in \mathbb{R}, \ \ a \neq 0,$
(2) $u(x,y) = a^2 x, \ \ a \in \mathbb{R}, \ \ a \neq 0.$

Remark 3.19. A variant of Proposition 3.18 is as follows. Consider the system of partial differential equations

$$\begin{cases} \Big(\dfrac{\partial u}{\partial x}\Big)^2 + \Big(\dfrac{\partial u}{\partial y}\Big)^2 = \lambda^2, \\ \Delta u \equiv 0, \end{cases} \tag{111}$$

where $u(x,y)$ is a smooth function defined on $\mathbb{R}^2$, and where $\lambda \in \mathbb{R}, \lambda \neq 0$. If u is a smooth solution of (111), then there exist $a, b, c \in \mathbb{R}$ such that $a^2 + b^2 = \lambda^2$, and such that for all $(x,y) \in \mathbb{R}^2$,

$$u(x,y) = ax + by + c. \tag{112}$$

This can be shown using arguments similar to the ones we already used.

We now return to the group of holomorphic isometries of $T\mathcal{N}$. Let $\varphi : T\mathcal{N} \to T\mathcal{N}$ be a diffeomorphism. In the coordinates $(\theta, \dot{\theta})$, φ can be written

$$\varphi(\theta,\dot{\theta}) = \big(\varphi^1(\theta,\dot{\theta}), \varphi^2(\theta,\dot{\theta}), \varphi^3(\theta,\dot{\theta}), \varphi^4(\theta,\dot{\theta})\big), \tag{113}$$

with $\varphi^2 < 0$, and its derivative can be decomposed into blocks of 2×2 real matrices

$$\varphi_{*_{(\theta,\dot{\theta})}} = \begin{bmatrix} A(\theta,\dot{\theta}) & B(\theta,\dot{\theta}) \\ C(\theta,\dot{\theta}) & D(\theta,\dot{\theta}) \end{bmatrix}. \tag{114}$$

The entries of the matrices A, B, C, D are denoted by $a_{ij}, b_{ij}, c_{ij}, d_{ij}$, respectively. Hence, $a_{11} = \frac{\partial \varphi^1}{\partial \theta_1}, b_{22} = \frac{\partial \varphi^4}{\partial \theta_2}$, etc.

From a complex point of view, recall that $(z_1, z_2) = (\theta_1 + i\dot{\theta}_1, \theta_2 + i\dot{\theta}_2)$ are global complex coordinates on $T\mathcal{N}$. Therefore, $T\mathcal{N} \cong \mathbb{C} \times i\mathbb{H}$, and we have

$$\begin{aligned} \varphi \text{ is holomorphic} \quad &\Leftrightarrow \quad \varphi^1 + i\varphi^3 \text{ and } \varphi^2 + i\varphi^4 \text{ are holomorphic functions} \\ &\Leftrightarrow \quad \frac{\partial}{\partial \bar{z}_k}(\varphi^1 + i\varphi^3) = \frac{\partial}{\partial \bar{z}_k}(\varphi^2 + i\varphi^4) = 0, \quad k = 1, 2, \end{aligned} \tag{115}$$

where $\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left\{\frac{\partial}{\partial \theta_k} + i\frac{\partial}{\partial \dot{\theta}_k}\right\}$. Equivalently, φ is holomorphic if and only if $A = D$ and $B = -C$ (Cauchy-Riemann equations).

Lemma 3.20. Assume that φ is holomorphic. In this situation, φ is an isometry if and only if φ^1 and φ^2 are solutions of the following system of partial differential equations:

$$\begin{aligned} h_{11}(\theta) &= h_{11}(\varphi)[(a_{11})^2 + (b_{11})^2] + 2\,h_{12}(\varphi)[a_{11}a_{21} + b_{11}b_{21}] + h_{22}(\varphi)[(a_{21})^2 + (b_{21})^2], \\ h_{12}(\theta) &= h_{11}(\varphi)[a_{11}a_{12} + b_{11}b_{12}] + h_{12}(\varphi)[a_{11}a_{22} + a_{21}a_{12} + b_{11}b_{22} + b_{21}b_{12}] + h_{22}(\varphi)[a_{21}a_{22} + b_{21}b_{22}], \\ h_{22}(\theta) &= h_{11}(\varphi)[(a_{12})^2 + (b_{12})^2] + 2\,h_{12}(\varphi)[a_{12}a_{22} + b_{12}b_{22}] + h_{22}(\varphi)[(a_{22})^2 + (b_{22})^2], \\ 0 &= h_{11}(\varphi)[a_{11}b_{12} - a_{12}b_{11}] + h_{12}(\varphi)[a_{11}b_{22} + a_{21}b_{12} - b_{11}a_{22} - b_{21}a_{12}] + h_{22}(\varphi)[a_{21}b_{22} - b_{21}a_{22}], \end{aligned}$$

where $h_{ij}(\theta) := h_F(\theta)\left(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j}\right)$.

Remark 3.21. Observe that $h_{ij}(\varphi) = h_{ij} \circ \varphi$ only depends on φ^1 and φ^2 (see item (i) in Proposition 3.5).

Proof of Lemma 3.20. By hypothesis, φ is holomorphic, which means that $A = D$ and $B = -C$. Consequently, the matrix representation of the equation $\varphi^* g = g$ reads

$$\begin{aligned} &{}^t\begin{bmatrix} A(\theta,\dot{\theta}) & B(\theta,\dot{\theta}) \\ -B(\theta,\dot{\theta}) & A(\theta,\dot{\theta}) \end{bmatrix} \begin{bmatrix} h_F(\varphi) & 0 \\ 0 & h_F(\varphi) \end{bmatrix} \begin{bmatrix} A(\theta,\dot{\theta}) & B(\theta,\dot{\theta}) \\ -B(\theta,\dot{\theta}) & A(\theta,\dot{\theta}) \end{bmatrix} = \begin{bmatrix} h_F(\theta) & 0 \\ 0 & h_F(\theta) \end{bmatrix} \\ &\Leftrightarrow \quad \begin{cases} {}^tA(\theta,\dot{\theta})\,(h_F(\varphi))\,A(\theta,\dot{\theta}) + {}^tB(\theta,\dot{\theta})\,(h_F(\varphi))\,B(\theta,\dot{\theta}) = h_F(\theta), \\ {}^tA(\theta,\dot{\theta})\,(h_F\varphi))\,B(\theta,\dot{\theta}) - {}^tB(\theta,\dot{\theta})\,(h_F(\varphi))\,A(\theta,\dot{\theta}) = 0. \end{cases} \end{aligned} \tag{116}$$

The first equation in (116) is an equality of symmetric matrices, and thus produces three equations which are after a direct calculation the first three equations of the lemma. The second equation in (116) is an equality of anti-symmetric matrices, thus it yields only one equation which is the last equation of the lemma, as a simple calculation shows. The lemma follows. □

Instead of trying to solve directly the system of equations in Lemma 3.20, our strategy will be to use the fact that the Ricci tensor is a Riemannian invariant, that is, $\varphi^*\mathrm{Ric} = \mathrm{Ric}$ for every isometry φ.

Lemma 3.22. If φ is an isometry, then

$$\frac{\partial \varphi^2}{\partial \theta_1} = \frac{\partial \varphi^2}{\partial \dot{\theta}_1} = 0, \qquad \left(\frac{\partial \varphi^2}{\partial \theta_2}\right)^2 + \left(\frac{\partial \varphi^4}{\partial \theta_2}\right)^2 = \left(\frac{\varphi^2}{\theta_2}\right)^2, \tag{117}$$

$$\frac{\partial \varphi^4}{\partial \theta_1} = \frac{\partial \varphi^4}{\partial \dot{\theta}_1} = 0, \qquad \left(\frac{\partial \varphi^2}{\partial \dot{\theta}_2}\right)^2 + \left(\frac{\partial \varphi^4}{\partial \dot{\theta}_2}\right)^2 = \left(\frac{\varphi^2}{\theta_2}\right)^2. \tag{118}$$

Proof. In the coordinates $(\theta, \dot{\theta})$, we have (see Proposition 3.8)

$$\mathrm{Ric}(\theta,\dot{\theta}) = \begin{bmatrix} \beta(\theta) & 0 \\ 0 & \beta(\theta) \end{bmatrix}, \quad \text{where} \quad \beta(\theta) = -\frac{3}{2}\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{(\theta_2)^2} \end{bmatrix}. \tag{119}$$

Using the bloc decomposition of φ_* given in (114), the equation $\varphi^*\mathrm{Ric} = \mathrm{Ric}$ reads

$$
{}^t\begin{bmatrix} A(\theta,\dot\theta) & B(\theta,\dot\theta) \\ C(\theta,\dot\theta) & D(\theta,\dot\theta) \end{bmatrix} \begin{bmatrix} -\beta(\varphi) & 0 \\ 0 & -\beta(\varphi) \end{bmatrix} \begin{bmatrix} A(\theta,\dot\theta) & B(\theta,\dot\theta) \\ C(\theta,\dot\theta) & D(\theta,\dot\theta) \end{bmatrix} = \begin{bmatrix} -\beta(\theta) & 0 \\ 0 & -\beta(\theta) \end{bmatrix}
$$

$$
\Leftrightarrow \quad \begin{cases} {}^tA(\theta,\dot\theta)\,(\beta(\varphi))\,A(\theta,\dot\theta) + {}^tC(\theta,\dot\theta)\,(\beta(\varphi))\,C(\theta,\dot\theta) &= \beta(\theta), \\ {}^tA(\theta,\dot\theta)\,(\beta(\varphi))\,B(\theta,\dot\theta) + {}^tC(\theta,\dot\theta)\,(\beta(\varphi))\,D(\theta,\dot\theta) &= 0, \\ {}^tB(\theta,\dot\theta)\,(\beta(\varphi))\,B(\theta,\dot\theta) + {}^tD(\theta,\dot\theta)\,(\beta(\varphi))\,D(\theta,\dot\theta) &= \beta(\theta). \end{cases} \tag{120}
$$

Taking into account the explicit form of β in (119), the first equation in (120) yields

$$
\begin{bmatrix} (a_{21})^2 + (c_{21})^2 & a_{21}a_{22} + c_{21}c_{22} \\ a_{22}a_{21} + c_{22}c_{21} & (a_{22})^2 + (c_{22})^2 \end{bmatrix} = \Big(\frac{\varphi^2}{\theta_2}\Big) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{121}
$$

This implies $a_{21} = c_{21} = 0$ and $(a_{22})^2 + (c_{22})^2 = \big(\frac{\varphi^2}{\theta_2}\big)^2$ which corresponds exactly to the first two equations of the proposition (see (117)). The other two equations are obtained similarly using the third equation in (120). The lemma follows. □

Combining the Cauchy-Riemann equation $\frac{\partial \varphi^2}{\partial \dot\theta_2} = -\frac{\partial \varphi^4}{\partial \theta_2}$ together with the second equation in (117) immediately yields the following lemma.

Lemma 3.23. If φ is a holomorphic isometry, then φ^2 is a solution of the system of partial differential equations (89). In particular, it has to be of the form (two possibilities)

(1) $\varphi^2(\theta,\dot\theta) = \frac{a^2\theta_2}{(\dot\theta_2+b)^2+(\theta_2)^2}$, $\quad a, b \in \mathbb{R}, a \neq 0,$
(2) $\varphi^2(\theta,\dot\theta) = a^2\theta_2$, $\quad a \in \mathbb{R}$, $a \neq 0$.

From now on, we will assume that φ is a holomorphic isometry (in particular, φ^2 is given by Lemma 3.23).

For convenience, let us rewrite explicitly the system of equations in Lemma 3.20, taking into account Lemmas 3.22 and 3.23.

Lemma 3.24. We have

$$
\Big(\frac{\partial \varphi^1}{\partial \theta_1}\Big)^2 + \Big(\frac{\partial \varphi^1}{\partial \dot\theta_1}\Big)^2 = \frac{\varphi^2}{\theta_2}, \tag{122}
$$

$$
\varphi^1\Big[\frac{\partial \varphi^1}{\partial \theta_1}\frac{\partial \varphi^2}{\partial \theta_2} + \frac{\partial \varphi^1}{\partial \dot\theta_1}\frac{\partial \varphi^2}{\partial \dot\theta_2}\Big] - \varphi^2\Big[\frac{\partial \varphi^1}{\partial \theta_1}\frac{\partial \varphi^1}{\partial \theta_2} + \frac{\partial \varphi^1}{\partial \dot\theta_1}\frac{\partial \varphi^1}{\partial \dot\theta_2}\Big] = \frac{\theta_1}{(\theta_2)^2}(\varphi^2)^2, \tag{123}
$$

$$
2\varphi^1\Big[\frac{\partial \varphi^1}{\partial \theta_2}\frac{\partial \varphi^2}{\partial \theta_2} + \frac{\partial \varphi^1}{\partial \dot\theta_2}\frac{\partial \varphi^2}{\partial \dot\theta_2}\Big] - \varphi^2\Big[\Big(\frac{\partial \varphi^1}{\partial \theta_2}\Big)^2 + \Big(\frac{\partial \varphi^1}{\partial \dot\theta_2}\Big)^2\Big] + \frac{\varphi^2 - (\varphi^1)^2}{\varphi^2}\Big(\frac{\varphi^2}{\theta_2}\Big)^2 = \frac{\theta_2 - (\theta_1)^2}{(\theta_2)^3}(\varphi^2)^2, \tag{124}
$$

$$
\varphi^1\Big[\frac{\partial \varphi^1}{\partial \theta_1}\frac{\partial \varphi^2}{\partial \dot\theta_2} - \frac{\partial \varphi^1}{\partial \dot\theta_1}\frac{\partial \varphi^2}{\partial \theta_2}\Big] + \varphi^2\Big[\frac{\partial \varphi^1}{\partial \dot\theta_1}\frac{\partial \varphi^1}{\partial \theta_2} - \frac{\partial \varphi^1}{\partial \theta_1}\frac{\partial \varphi^1}{\partial \dot\theta_2}\Big] = 0. \tag{125}
$$

Since φ^2 does not depend on θ_1 and $\dot\theta_1$, it follows from Remark 3.19 together with (122) that

$$
\varphi^1(\theta,\dot\theta) = r(\theta_2,\dot\theta_2)\theta_1 + s(\theta_2,\dot\theta_2)\dot\theta_1 + t(\theta_2,\dot\theta_2), \tag{126}
$$

where r, s, t are smooth functions depending on $\theta_2, \dot\theta_2$, and such that $r(\theta_2,\dot\theta_2)^2 + s(\theta_2,\dot\theta_2)^2 = \frac{\varphi^2(\theta_2,\dot\theta_2)}{\theta_2}$.

Lemma 3.25. We have

$$
\Big(\frac{\partial r}{\partial \theta_2}\Big)^2 + \Big(\frac{\partial r}{\partial \dot\theta_2}\Big)^2 + \Big(\frac{\partial s}{\partial \theta_2}\Big)^2 + \Big(\frac{\partial s}{\partial \dot\theta_2}\Big)^2 = \frac{1}{(\theta_2)^2}\Big[\frac{1}{\theta_2}\varphi^2 - \frac{\partial \varphi^2}{\partial \theta_2}\Big]. \tag{127}
$$

If $\varphi^2(\theta_2,\dot\theta_2) = a^2\theta_2$, then the right hand side of (127) is zero. If $\varphi^2(\theta_2,\dot\theta_2) = \frac{a^2\theta_2}{(\dot\theta_2+b)^2+(\theta_2)^2}$, then the right hand side is $\frac{2a^2}{[(\dot\theta_2+b)^2+(\theta_2)^2]^2}$.

Proof. First, observe that r and s are harmonic. Indeed, if $\Delta = \frac{\partial^2}{\partial \theta_2} + \frac{\partial^2}{\partial \dot{\theta}_2}$, then,

$$0 = \Delta \varphi^1 = \theta_1 \Delta r + \dot{\theta}_1 \Delta s + \Delta t \tag{128}$$

for all $\theta_1, \dot{\theta}_1 \in \mathbb{R}$, which is only possible if $\Delta r = \Delta s = \Delta t = 0$. Now, taking the Laplacian of both side of the equation, $r^2 + s^2 = \frac{\varphi^2}{\theta_2}$ yields

$$2A + 2r\Delta r + 2s\Delta s = \Delta\Big(\frac{\varphi^2}{\theta_2}\Big), \tag{129}$$

where A is the left hand side of (127). From this together with the harmonicity of r, s and φ^2, one easily obtains (127). □

Lemma 3.26. If $\varphi^2(\theta_2, \dot{\theta}_2) = a^2\theta_2, a \neq 0$, then there exist $b, c, d \in \mathbb{R}$, and $\varepsilon \in \{+1, -1\}$ such that

$$\varphi(\theta, \dot{\theta}) = \big(\epsilon a\, \theta_1 + b\, \theta_2, a^2\theta_2, \epsilon a\, \dot{\theta}_1 + b\, \dot{\theta}_2 + c, a^2\dot{\theta}_2 + d\big). \tag{130}$$

Moreover, every transformation of this form is a holomorphic isometry of $T\mathcal{N}$.

Proof. Lemma 3.25 implies that $\varphi^1(\theta, \dot{\theta}) = r\theta_1 + s\dot{\theta}_1 + t(\theta_2, \dot{\theta}_2)$, where $r, s \in \mathbb{R}$ are such that $r^2 + s^2 = a^2$. Using (125), one easily obtains $s = 0$, $r = \pm a$ and $\frac{\partial t}{\partial \dot{\theta}_2} = 0$. From (123), one also get $t(\theta_2) = b\, \theta_2$ for some constant $b \in \mathbb{R}$. Hence, $\varphi^1(\theta, \dot{\theta}) = (\pm a)\theta_1 + b\, \theta_2$. The other components of φ are obtained using the Cauchy-Riemann equations. The lemma follows. □

Remark 3.27. By changing the sign of a if necessary, one may assume $\varepsilon a = a$ in the above lemma.

Remark 3.28. Written in the complex coordinates $(z_1, z_2) \in \mathbb{C} \times i\mathbb{H}$, the transformation in (130) reads $\varphi(z_1, z_2) = \big((\epsilon a)z_1 + bz_2 + ic, (\epsilon a)^2 z_2 + id\big)$.

Let us now consider the case $\varphi^2(\theta_2, \dot{\theta}_2) = \frac{a^2\theta_2}{(\dot{\theta}_2 + b)^2 + (\theta_2)^2}$. In order to find φ^1, we will use the following facts:

(1) the map $-2\,\eta_1 : T\mathcal{N} \to \mathbb{R}$, $(\theta, \dot{\theta}) \mapsto \frac{\theta_1}{\theta_2}$ is a Kähler function (see Proposition 2.28, Proposition 2.32 and (69)),
(2) the composition of a Kähler function with a holomorphic isometry is a Kähler function (obvious).

It follows from these two facts that $\frac{\varphi^1}{\varphi^2} = \frac{r}{\varphi^2}\theta_1 + \frac{s}{\varphi^2}\dot{\theta}_1 + \frac{t}{\varphi^2}$ is a Kähler function on $T\mathcal{N}$.

Lemma 3.29. A function on $T\mathcal{N}$ of the form $R(\theta_2, \dot{\theta}_2)\theta_1 + S(\theta_2, \dot{\theta}_2)\dot{\theta}_1 + T(\theta_2, \dot{\theta}_2)$, where R, S, T are smooth functions, is Kähler if and only if there exist $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$R = \frac{C_1 - C_2\dot{\theta}_2}{\theta_2}, \qquad S = C_2, \qquad T = C_3. \tag{131}$$

Proof. Taking into account Proposition 2.25 together with Proposition 3.5, one obtains after a direct calculation that: $R(\theta_2, \dot{\theta}_2)\theta_1 + S(\theta_2, \dot{\theta}_2)\dot{\theta}_1 + T(\theta_2, \dot{\theta}_2)$ is Kähler if and only if

$$\frac{\partial S}{\partial \theta_2} = \frac{\partial S}{\partial \dot{\theta}_2} = \frac{\partial T}{\partial \theta_2} = \frac{\partial T}{\partial \dot{\theta}_2} = \frac{R}{\theta_2} + \frac{\partial R}{\partial \theta_2} = \frac{S}{\theta_2} + \frac{\partial R}{\partial \dot{\theta}_2} = 0. \tag{132}$$

Solving these equations exactly yields the lemma. □

From Lemma 3.29, it follows that there exist $C_1, C_2, C_3 \in \mathbb{R}$ such that

$$\frac{r}{\varphi^2} = \frac{C_1 - C_2\dot{\theta}_2}{\theta_2}, \qquad \frac{s}{\varphi^2} = C_2, \qquad \frac{t}{\varphi^2} = C_3. \tag{133}$$

Now, rewriting the equation $r^2 + s^2 = \frac{\varphi^2}{\theta_2}$ using (133) leads to an equality of two polynomials in θ_2 and $\dot{\theta}_2$

$$(C_1)^2 - 2C_1C_2\dot{\theta}_2 + (C_2)^2(\dot{\theta}_2)^2 + (C_2)^2(\theta_2)^2 = \frac{b^2 + 2b\dot{\theta}_2 + (\dot{\theta}_2)^2 + (\theta_2)^2}{a^2}, \tag{134}$$

from which we get a system of equations which is equivalent to $C_1 = -bC_2$ and $(C_2)^2 = \frac{1}{a^2}$. Since there is no constraints on the sign of a, we can assume $C_2 = \frac{1}{a}$ and $C_1 = -\frac{b}{a}$. Returning to (126), and setting $c := aC_3$ for convenience, a direct calculation gives

$$\varphi^1(\theta, \dot{\theta}) = a\,\frac{-(\dot{\theta}_2 + b)\theta_1 + (\dot{\theta}_1 + c)\theta_2}{(\dot{\theta}_2 + b)^2 + (\theta_2)^2}, \qquad \varphi^2(\theta, \dot{\theta}) = \frac{a^2\theta_2}{(\dot{\theta}_2 + b)^2 + (\theta_2)^2}, \tag{135}$$

where $a, b, c \in \mathbb{R}, a \neq 0$. Finally, solving the Cauchy-Riemann equations corresponding to the holomorphic functions $\varphi^1 + i\varphi^3$ and $\varphi^2 + i\varphi^4$ gives

$$\varphi^3(\theta, \dot{\theta}) = -a\,\frac{(\dot{\theta}_1 + c)(\dot{\theta}_2 + b) + \theta_1\theta_2}{(\dot{\theta}_2 + b)^2 + (\theta_2)^2} + d, \qquad \varphi^4(\theta, \dot{\theta}) = -\frac{a^2(\dot{\theta}_2 + b)}{(\dot{\theta}_2 + b)^2 + (\theta_2)^2} + e, \tag{136}$$

where $d, e \in \mathbb{R}$. In terms of the complex variables $z_k = \theta_k + i\dot{\theta}_k$, this can be rewritten

$$(\varphi^1 + i\varphi^3)(z_1, z_2) = -ia\,\frac{z_1 + ic}{z_2 + ib} + id, \qquad (\varphi^2 + i\varphi^4)(z_1, z_2) = \frac{a^2}{z_2 + ib} + ie. \tag{137}$$

Collecting our results, we obtain the following lemma.

Lemma 3.30. Let φ be a diffeomorphism of $T\mathcal{N} \cong \mathbb{C} \times i\mathbb{H}$. Then, φ is a holomorphic isometry if and only if it has the following form (two possibilities):

$$\varphi_1(z_1, z_2) = \Big(-ia\,\frac{z_1 + ic}{z_2 + ib} + id,\ \frac{a^2}{z_2 + ib} + ie\Big), \qquad a, b, c, d, e \in \mathbb{R},\ a \neq 0, \tag{138}$$

$$\varphi_2(z_1, z_2) = \big(az_1 + bz_2 + ic,\ a^2 z_2 + id\big), \qquad a, b, c, d \in \mathbb{R},\ a \neq 0. \tag{139}$$

To conclude the proof of Theorem 3.11, recall that the map $f : \mathbb{C} \times i\mathbb{H} \to \mathbb{H} \times \mathbb{C}$, $(z_1, z_2) \mapsto (-iz_2, iz_1)$ is a biholomorphic isometry (see (87)) and that the action of $\mathrm{ASp}(2, \mathbb{R})$ on $\mathbb{H} \times \mathbb{C}$ is given by

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu)\right) \cdot (\tau, z) = \Big(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\Big). \tag{140}$$

Having this in mind, we observe after a direct calculation that for $(\tau, z) \in \mathbb{H} \times \mathbb{C}$,

$$(f \circ \varphi_1 \circ f^{-1})(\tau, z) = \left(-\tfrac{1}{a}\begin{bmatrix} e & eb - a^2 \\ 1 & b \end{bmatrix}, \big(\tfrac{d}{a}, -c + \tfrac{bd}{a}, 0\big)\right) \cdot (\tau, z), \tag{141}$$

$$(f \circ \varphi_2 \circ f^{-1})(\tau, z) = \left(\tfrac{1}{a}\begin{bmatrix} a^2 & d \\ 0 & 1 \end{bmatrix}, \big(-\tfrac{b}{a}, -\tfrac{c}{a}, 0\big)\right) \cdot (\tau, z), \tag{142}$$

where φ_1 and φ_2 are defined in (138) and (139), respectively. From this it follows that $f \circ \varphi \circ f^{-1} \in \mathrm{ASp}(2, \mathbb{R})$ for all holomorphic isometries φ of $T\mathcal{N}$, which shows that the group of holomorphic isometries of $T\mathcal{N}$ is included in $\mathrm{ASp}(2, \mathbb{R})$. The converse inclusion being obviously true (by inspection of (141) and (142)), the equality holds.

Let us now derive a few consequences. Consider the following subgroup of $\mathrm{SL}(2, \mathbb{R})$:

$$K := \left\{\begin{bmatrix} a & b \\ 0 & \frac{1}{a} \end{bmatrix} \in \mathrm{Mat}(2, \mathbb{R})\ \Big|\ a, b \in \mathbb{R},\ a \neq 0\right\}. \tag{143}$$

Clearly, K is a 2-dimensional Lie group having two connected components (according to the sign of a). We denote by K_0 the connected component of K containing the identity. Since K_0 is a subgroup of $\mathrm{SL}(2,\mathbb{R})$, one can form the semi-direct product $K_0 \ltimes \mathbb{R}^2$; it is naturally a subgroup of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2 = \mathrm{ASp}(2,\mathbb{R})$.

Proposition 3.31. In this situation,

(i) *The actions of $G^J(\mathbb{R})$, $\mathrm{ASp}(2,\mathbb{R})$ and $K_0 \ltimes \mathbb{R}^2$ on $T\mathcal{N}$ are transitive,*
(ii) *The isotropy subgroups of $o := (i, 0) \in \mathbb{H} \times \mathbb{C}$ relative to the actions of $G^J(\mathbb{R})$, $\mathrm{ASp}(2,\mathbb{R})$ and $K_0 \ltimes \mathbb{R}^2$ are isomorphic to $\mathrm{SO}(2) \times \mathbb{R}$, $\mathrm{SO}(2)$ and $\{0\}$, respectively.*

Therefore, $T\mathcal{N}$ is a homogeneous Kähler manifold and we have the identifications

$$T\mathcal{N} \cong G^J(\mathbb{R})/\mathrm{SO}(2) \times \mathbb{R} \cong \mathrm{ASp}(2,\mathbb{R})/\mathrm{SO}(2) \cong K_0 \ltimes \mathbb{R}^2. \tag{144}$$

Proof. By a direct calculation. □

Corollary 3.32. $T\mathcal{N}$ itself is a Lie group (isomorphic to $K_0 \ltimes \mathbb{R}^2$) whose Kähler structure is left-invariant.

Let us now discuss the whole group of isometries of $T\mathcal{N}$. To this end, we introduce the following group:

$$\mathrm{SL}^{\pm}(2,\mathbb{R}) := \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Mat}(2,\mathbb{R}) \,\middle|\, ad - bc = \pm 1 \right\}. \tag{145}$$

Since $\mathrm{SL}^{\pm}(2,\mathbb{R})$ acts linearly on the right on $\mathbb{R}^2$, one has the semi-direct product $\mathrm{SL}^{\pm}(2,\mathbb{R}) \ltimes \mathbb{R}^2$, with multiplication $(M_1, X_1) \cdot (M_2, X_1) = (M_1 M_2, X_2 + X_1 \cdot M_2)$. We define an action of $\mathrm{SL}^{\pm}(2,\mathbb{R})$ on $\mathbb{H} \times \mathbb{C}$ as follows:

$$\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, (\lambda, \mu) \right) \cdot (\tau, z) := \begin{cases} \left(\dfrac{a\tau + b}{c\tau + d}, \dfrac{z + \lambda\tau + \mu}{c\tau + d} \right) & \text{if} \quad ad - bc = \ \ 1, \\ \left(\dfrac{a\bar{\tau} + b}{c\bar{\tau} + d}, \dfrac{\bar{z} + \lambda\bar{\tau} + \mu}{c\bar{\tau} + d} \right) & \text{if} \quad ad - bc = -1, \end{cases} \tag{146}$$

where $\bar{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Since this action is effective, one can regard $\mathrm{SL}^{\pm}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ as a subgroup of $\mathrm{Diff}(\mathbb{H} \times \mathbb{C}) \cong \mathrm{Diff}(T\mathcal{N})$.

Theorem 3.33. *The group of isometries of $T\mathcal{N}$ (not necessarily holomorphic) is the semi-direct product $\mathrm{SL}^{\pm}(2,\mathbb{R}) \ltimes \mathbb{R}^2$.*

The proof of Theorem 3.33 is based on the following result which is due to Kulkarni.

Proposition 3.34 (Kulkarni[79]). Let N_1 and N_2 be two connected Kähler manifolds with corresponding holomorphic sectional curvature functions[80] H_1 and H_2. Suppose that the real dimension of N_1 is greater than 4 and that there exists a diffeomorphism $f : N_1 \to N_2$ such that $f^* H_2 = H_1$. Then, either $H_1 = H_2 =$ constant or f is a holomorphic or an anti-holomorphic isometry.

Corollary 3.35 (of Proposition 3.34). Let N be a connected Kähler manifold whose holomorphic sectional curvature is not constant, and whose real dimension is greater than 4. Then, every isometry of N is either holomorphic or anti-holomorphic.

Proof of Theorem 3.33. In terms of the variables $(z_1, z_2) \in \mathbb{C} \times i\mathbb{H}$, it not difficult to see that the map $T\mathcal{N} \to T\mathcal{N}$, $(z_1, z_2) \mapsto (\bar{z}_1, \bar{z}_2)$ is an anti-holomorphic isometry of $T\mathcal{N}$ (this is actually a general feature of Dombrowski's construction). In terms of the variables $(\tau, z) = (-iz_2, iz_1) \in \mathbb{H} \times \mathbb{C}$, this means that the map $(\tau, z) \mapsto (-\bar{\tau}, -\bar{z})$ is an anti-holomorphic isometry of $\mathbb{H} \times \mathbb{C}$. Therefore, there is a 1-to-1 correspondence between the set of holomorphic isometries and the set of anti-holomorphic isometries of $T\mathcal{N}$ which is given by $\varphi(\tau, z) \mapsto \varphi(-\bar{\tau}, -\bar{z})$. From this, it is easy to see that (146) exhausts all the possible holomorphic and anti-holomorphic isometries of $T\mathcal{N}$ (and nothing else). But according to Corollary 3.35, this is already the whole isometry group of $T\mathcal{N}$. The proposition follows. □

Let us conclude this section with a discussion on the Lie group structure of the group of isometries of $T\mathcal{N}$. To this end, we recall the following result which is due to Myers and Steenrod[81] (see also Ref. 82 or Ref. 83 for a modern proof).

Proposition 3.36 (Myers-Steenrod[81]). Let M be a connected Riemannian manifold. Then, the group $\mathrm{Isom}(M)$ *of isometries of M is a Lie group with respect to the compact-open topology*[84] *in M. Moreover, the natural action of* $\mathrm{Isom}(M)$ *on M is smooth.*

Let M be a manifold acted upon by a Lie group G with Lie algebra $\mathfrak{g}$. Given $\xi \in \mathfrak{g}$, the *fundamental vector field* ξ_M is the vector field on M which is defined, for $p \in M$, by

$$(\xi_M)_p := \frac{d}{dt}\bigg|_0 \exp(t\xi)\cdot p, \tag{147}$$

where $\exp : \mathfrak{g} \to G$ is the standard exponential map. Observe that fundamental vector fields only depend on the action of G_0 on M, where G_0 is the connected component of G containing the identity. If G acts via isometries on a Riemannian manifold M, then every fundamental vector field ξ_M is a Killing vector field. We denote by $\mathfrak{i}(M)$ the space of Killing vector fields of a Riemannian manifold M; it is a Lie algebra for the Lie bracket of vector fields.

Proposition 3.37 (Complement of Proposition 3.36). Let M be a connected Riemannian manifold with isometry group $\mathrm{Isom}(M)$ *and Lie algebra* $\mathfrak{g}$. *If M is complete, then the map* $\phi : \mathfrak{g} \to \mathfrak{i}(M),\ \xi \mapsto \xi_M$ *is an anti-isomorphism of Lie algebras, that is, it is an isomorphism of vector spaces satisfying*

$$\phi([\xi,\eta]) = -[\phi(\xi),\phi(\eta)] \tag{148}$$

for all $\xi,\eta \in \mathfrak{g}$.

If a Lie group G acts effectively on a manifold M, then there are a priori two topologies on G : the intrinsic topology of G, and the compact-open topology coming from the injection $G \to \mathrm{Diff}(M)$. If the image of G coincides with $\mathrm{Isom}(M)$ in $\mathrm{Diff}(M)$, like in Theorem 3.33, then we have the following result.

Lemma 3.38. Let $\Phi : G\times M \to M$ *be an action of a Lie group G on a connected and complete Riemannian manifold M. Suppose that this action is smooth, effective and that* $\mathrm{Isom}(M) = \{\Phi_g \,|\, g \in G\}$, *where* $\Phi_g : M \to M,\ p \mapsto \Phi(g,p)$. *Then, the map* $G \to \mathrm{Isom}(M),\ g \mapsto \Phi_g$ *is an isomorphism of Lie groups (here,* $\mathrm{Isom}(M)$ *is endowed with the Lie group structure described in Proposition 3.36).*

Proof. It is based on the following result: if $(\varphi_n)_{n\in\mathbb{N}}$ is a sequence of isometries of M such that $\varphi_n(p)$ converges to $\varphi(p)$ for all $p \in M$, where φ is a fixed isometry, then φ_n converges to φ for the compact-open topology (see Ref. 83, Lemma 5, Chapter 1 and Theorem 3.10, Chapter 4). From this together with the continuity of $\Phi : G\times M \to M$, one sees that $G \to \mathrm{Isom}(M),\ g \mapsto \Phi_g$ is a continuous and bijective homomorphism of topological groups. Since continuous homomorphisms of Lie groups are automatically smooth, the map $g \mapsto \Phi_g$ is smooth. By the inverse function theorem, its inverse is also smooth. The lemma follows. □

Combining Theorem 3.33, Proposition 3.37, Lemma 3.38, and the fact that $(\mathrm{SL}^{\pm}(2,\mathbb{R}) \ltimes \mathbb{R}^2)_0 = \mathrm{ASp}(2,\mathbb{R})$, we obtain the following result.

Proposition 3.39. Let $\mathfrak{asp}(2,\mathbb{R})$ *be the Lie algebra of* $\mathrm{ASp}(2,\mathbb{R})$. *Then, the map* $\mathfrak{asp}(2,\mathbb{R}) \to \mathfrak{i}(T\mathcal{N}),\ \xi \mapsto \xi_{T\mathcal{N}}$, *is an anti-isomorphism of Lie algebras.*

D. Kähler functions and momentum map

Let $\mathfrak{g}^J$, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{h}$ denote, respectively, the Lie algebras of $G^J(\mathbb{R}), \mathrm{SL}(2,\mathbb{R})$, and $\mathrm{Heis}(\mathbb{R})$. We recall that $\mathfrak{sl}(2,\mathbb{R})$ is the space of 2×2 real matrices of trace 0,

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{Mat}(2,\mathbb{R}) \,\Big|\, \alpha + \delta = 0 \right\}, \tag{149}$$

and that $\mathfrak{h}$ can be identified with $\mathbb{R}^2 \times \mathbb{R}$ endowed with the Lie bracket

$$\big[(\xi,r),(\eta,s)\big] = \big(0, 2\,\Omega(\xi,\eta)\big), \tag{150}$$

where $\xi,\eta \in \mathbb{R}^2$, $r,s \in \mathbb{R}$ and where $\Omega(\xi,\eta) = \xi_1\eta_2 - \xi_2\eta_1$. In the sequel, we shall use the following basis for $\mathfrak{sl}(2,\mathbb{R})$:

$$F := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad G := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad H := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{151}$$

and denote by $\{P,Q,R\}$ the canonical basis of $\mathfrak{h} \cong \mathbb{R}^2 \times \mathbb{R} \cong \mathbb{R}^3$,

$$P := (1,0,0), \qquad Q := (0,1,0), \qquad R := (0,0,1). \tag{152}$$

The Lie algebra $\mathfrak{g}^J$ of the Jacobi group $G^J(\mathbb{R})$ is the semi-direct product $\mathfrak{g}^J = \mathfrak{sl}(2,\mathbb{R}) \ltimes \mathfrak{h}$, that is, it is the Cartesian product $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{h}$ endowed with the Lie bracket

$$\big[(A,\xi,r),(B,\eta,s)\big] = \big([A,B], \xi B - \eta A, 2\,\Omega(\xi,\eta)\big), \tag{153}$$

where $A,B \in \mathfrak{sl}(2,\mathbb{R})$, $\xi,\eta \in \mathbb{R}^2$, $r,s \in \mathbb{R}$, and where $[A,B] = AB - BA$ is the usual commutator of matrices. By construction, $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g}^J$, therefore, $\{F,G,H,P,Q,R,\}$ can be regarded as a basis for $\mathfrak{g}^J$. A direct calculation using (153) gives the following commutation relations (see also Ref. 23):

$$\begin{aligned}
&[F,G] = H, && [F,Q] = 0, && [G,Q] = -P, && [P,Q] = 2R, && (154)\\
&[F,H] = -2F, && [G,H] = 2G, && [H,P] = -P, && [R,.\,] = 0, && (155)\\
&[F,P] = -Q, && [G,P] = 0, && [H,Q] = Q. && && (156)
\end{aligned}$$

Let us now recall a few basic definitions related to Lie group actions (see Ref. 85). Let (M,ω) be a symplectic manifold acted upon by a Lie group G with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the dual of the Lie algebra $\mathfrak{g}$. A *momentum map* is a smooth map $\mathbf{J} : M \to \mathfrak{g}^*$ satisfying $\xi_M = X_{\mathbf{J}^\xi}$ for all $\xi \in \mathfrak{g}$, where ξ_M is the fundamental vector field of ξ and where $\mathbf{J}^\xi$ is the function $M \to \mathbb{R}$ defined by $\mathbf{J}^\xi(p) := \mathbf{J}(p)(\xi)$ (here, $X_{\mathbf{J}^\xi}$ denotes the Hamiltonian vector field associated to $\mathbf{J}^\xi$). Let us denote explicitly the action of G on M by $\Phi : G \times M \to M$. Given $g \in G$, we also denote by Φ_g the diffeomorphism $M \to M$, $p \mapsto \Phi(g,p)$. In this situation, a momentum map is said to be *equivariant* if it satisfies

$$\mathrm{Ad}^*(g) \circ \mathbf{J} = \mathbf{J} \circ \Phi_g \tag{157}$$

for all $g \in G$, where Ad^* is the coadjoint representation[86] of G. Equivalently, $\mathbf{J}$ is equivariant if $\mathbf{J}^\xi \circ \Phi_g = \mathbf{J}^{\mathrm{Ad}(g^{-1})\xi}$ for all $g \in G$ and all $\xi \in \mathfrak{g}$.

Having this in mind, let $C^\infty(\mathbb{S}^J)$ denote the space of smooth functions on the Siegel-Jacobi space $\mathbb{S}^J$. Using the symplectic coordinates $(\eta,\dot{\theta})$ on $\mathbb{S}^J \cong T\mathcal{N}$ (see Propositions 2.10 and 2.32), we define a linear map $\psi : \mathfrak{g}^J \to C^\infty(\mathbb{S}^J)$ as follows:

$$\begin{aligned}
&F \mapsto -\eta_2, && P \mapsto \tfrac{1}{2}\dot{\theta}_1 + \eta_1\dot{\theta}_2, && (158)\\
&G \mapsto \tfrac{1}{4}(\dot{\theta}_1)^2 + \eta_2(\dot{\theta}_2)^2 + \eta_1\dot{\theta}_1\dot{\theta}_2 - \tfrac{1}{4((\eta_1)^2-\eta_2)}, && Q \mapsto \eta_1, && (159)\\
&H \mapsto -\eta_1\dot{\theta}_1 - 2\,\eta_2\dot{\theta}_2, && R \mapsto -\tfrac{1}{4}. && (160)
\end{aligned}$$

Remark 3.40. Observe that the last term of $\psi(G)$ can be rewritten $\frac{1}{4((\eta_1)^2-\eta_2)} = \frac{\theta_2}{2}$.

Proposition 3.41. For every $L \in \mathfrak{g}^J$, the Hamiltonian vector field of $\psi(L)$ coincide with the fundamental vector field generated by L, that is, $X_{\psi(L)} = L_{\mathbb{S}^J}$. Therefore, the map $\mathbf{J} : \mathbb{S}^J \to (\mathfrak{g}^J)^*$ defined by

$$\mathbf{J}(p)(L) := \psi(L)(p), \quad (p \in \mathbb{S}^J,\ L \in \mathfrak{g}^J) \tag{161}$$

is a momentum map.

Proof. Using the relations $\eta_1 = -\frac{\theta_1}{2\theta_2}$ and $\eta_2 = \frac{(\theta_1)^2 - 2\theta_2}{4(\theta_2)^2}$, one can rewrite the functions $\psi(L)$ in terms of the coordinates $(\theta, \dot{\theta})$, and compute their Hamiltonian vector fields $X_{\psi(L)}$ via the formula $(X_f)_{(\theta,\dot{\theta})} = h^{ij}\frac{\partial f}{\partial \dot{\theta}_i}\frac{\partial}{\partial \theta_j} - h^{ij}\frac{\partial f}{\partial \theta_i}\frac{\partial}{\partial \dot{\theta}_j}$. One obtains

$$\begin{aligned}
(X_{\psi(F)})_{(\theta,\dot{\theta})} &= (0,0,0,1), & (X_{\psi(P)})_{(\theta,\dot{\theta})} &= (-\theta_2, 0, -\dot{\theta}_2, 0),\\
(X_{\psi(G)})_{(\theta,\dot{\theta})} &= (-\dot{\theta}_1\theta_2 - \theta_1\dot{\theta}_2, -2\theta_2\dot{\theta}_2, -\dot{\theta}_1\dot{\theta}_2 + \theta_1\theta_2, (\theta_2)^2 - (\dot{\theta}_2)^2), & (X_{\psi(Q)})_{(\theta,\dot{\theta})} &= (0,0,-1,0),\\
(X_{\psi(H)})_{(\theta,\dot{\theta})} &= (\theta_1, 2\theta_2, \dot{\theta}_1, 2\dot{\theta}_2), & (X_{\psi(R)})_{(\theta,\dot{\theta})} &= (0,0,0,0).
\end{aligned}$$

On the other hand, the fundamental vector fields associated to F, G, H, P, Q, R can be computed in the $(\theta, \dot{\theta})$-coordinates using (140) and the relation $(\tau, z) = (-iz_2, iz_1) = (-i(\theta_2 + i\dot{\theta}_2), i(\theta_1 + i\dot{\theta}_1)) = (\dot{\theta}_2 - i\theta_2, -\dot{\theta}_1 + i\theta_1)$. By comparing the results, one sees that $X_{\psi(L)} = L_{\mathbb{S}^J}$ for all $L \in \mathfrak{g}^J$. The proposition follows. □

Since $G^J(\mathbb{R})$ acts via isometries on $\mathbb{S}^J$, it follows from the relation $X_{\psi(L)} = L_{\mathbb{S}^J}$ that $X_{\psi(L)}$ is a Killing vector field for all $L \in \mathfrak{g}^J$, which means that $\psi(L)$ is a Kähler function for all $L \in \mathfrak{g}^J$. One can thus regard ψ as a map $\psi : \mathfrak{g}^J \to \mathscr{K}(\mathbb{S}^J)$, where $\mathscr{K}(\mathbb{S}^J)$ is the Lie algebra of Kähler functions on $\mathbb{S}^J$ (see Sec. II E).

Proposition 3.42. The map $\psi : \mathfrak{g}^J \to \mathscr{K}(\mathbb{S}^J)$ is a Lie algebra isomorphism.

Proof. The fact that $\psi : \mathfrak{g}^J \to \mathscr{K}(\mathbb{S}^J)$ is an injective homomorphism of Lie algebras follows from a direct calculation. For dimensional reasons, it is also surjective. Indeed, one has $\dim(\mathfrak{i}(\mathbb{S}^J)) = 5$ (see Proposition 3.39), and the kernel of the linear map $\phi : \mathscr{K}(\mathbb{S}^J) \to \mathfrak{i}(\mathbb{S}^J),\ f \mapsto X_f$ is isomorphic to $\mathbb{R}$ (by connectedness of $\mathbb{S}^J$). Thus,

$$\dim(\mathscr{K}(\mathbb{S}^J)) - 1 = \dim(\phi(\mathscr{K}(\mathbb{S}^J))) \leq \dim(\mathfrak{i}(\mathbb{S}^J)) = 5. \tag{162}$$

Therefore, $\dim(\mathscr{K}(\mathbb{S}^J)) \leq 6$. Since $\psi(\mathfrak{g}^J)$ is a 6-dimensional subspace of $\mathscr{K}(\mathbb{S}^J)$, this implies $\psi(\mathfrak{g}^J) = \mathscr{K}(\mathbb{S}^J)$. The proposition follows. □

Corollary 3.43. A smooth function $f : \mathbb{S}^J \to \mathbb{R}$ is a Kähler function if and only if there exists $L \in \mathfrak{g}^J$ such that $f = \mathbf{J}^L$.

Corollary 3.44. The momentum map $\mathbf{J} : \mathbb{S}^J \to (\mathfrak{g}^J)^*$ is equivariant.

Proof. It is a consequence of the connectedness of $G^J(\mathbb{R})$ and the fact that $\psi : \mathfrak{g}^J \to \mathscr{K}(\mathbb{S}^J)$ is a Lie algebra homomorphism (see Ref. 85, Chapter 12). □

Remark 3.45. If we denote explicitly the action of $G^J(\mathbb{R})$ on $\mathbb{S}^J$ by Φ, then the equivariance of $\mathbf{J}$ *can be reformulated in terms of the map $\psi : \mathfrak{g}^J \to \mathscr{K}(\mathbb{S}^J)$ as follows:*

$$\psi(\mathrm{Ad}(g^{-1})L) = \psi(L) \circ \Phi_g, \tag{163}$$

where $g \in G^J(\mathbb{R})$ and $L \in \mathfrak{g}^J$.

One of the *raison d'être* of the momentum map is the classification of all homogeneous symplectic manifolds in terms of coadjoint orbits (up to coverings); this is *Kostant's Coadjoint Orbit Covering Theorem* (stated below). For the convenience of the reader, we recall the main ingredients of this classification (see Ref. 85).

Let M be a manifold acted upon by a Lie group G with Lie algebra $\mathfrak{g}$. Given $\mu \in \mathfrak{g}^*$, the coadjoint orbit of G through μ is the subset

$$\mathrm{Orb}(\mu) := \{\mathrm{Ad}^*(g)(\mu) \in \mathfrak{g}^* \,|\, g \in G\}, \tag{164}$$

where $\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation of G. Being an orbit, $\mathrm{Orb}(\mu)$ is automatically an immersed submanifold of $\mathfrak{g}^*$, and its tangent bundle at $\alpha \in \mathrm{Orb}(\mu)$ can be identified with $\{\mathrm{ad}^*(\xi)(\alpha) \in \mathfrak{g}^* \,|\, \xi \in \mathfrak{g}\}$, where $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is defined by $\langle \mathrm{ad}^*(\xi)(\alpha), \eta\rangle = \langle \alpha, [\xi, \eta]\rangle, \xi, \eta \in \mathfrak{g}$. Using this identification, one defines a symplectic form on $\mathcal{O} := \mathrm{Orb}(\mu)$ as follows:

$$(\omega_{\mathcal{O}})_\alpha\big(\mathrm{ad}^*(\xi)(\alpha), \mathrm{ad}^*(\eta)(\alpha)\big) := \langle \alpha, [\xi, \eta]\rangle, \tag{165}$$

where $\alpha \in \mathcal{O}$ and $\xi, \eta \in \mathfrak{g}$. The symplectic form $\omega_{\mathcal{O}}$ is known as the *Kirillov-Kostant-Souriau symplectic form*.

Theorem 3.46 (Kostant's Coadjoint Orbit Covering Theorem[25]). *Let (M, ω) be a symplectic manifold and let $\Phi : G \times M \to M$ be a left and transitive action having an equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Then, $\mathbf{J}$ is a local diffeomorphism onto a coadjoint orbit $\mathcal{O}$, and it satisfies $\mathbf{J}^*\omega_{\mathcal{O}} = \omega$.*

Returning to the Siegel-Jacobi space $\mathbb{S}^J$, we have the following result which is a complement of Corollary 3.44.

Proposition 3.47. The momentum map $\mathbf{J} : \mathbb{S}^J \to (\mathfrak{g}^J)^*$ is a diffeomorphism onto a coadjoint orbit $\mathcal{O}$, and it satisfies $\mathbf{J}^*\omega_{\mathcal{O}} = \omega_{KB}$, where ω_{KB} is the Kähler-Berndt symplectic form. In other words, the Siegel-Jacobi space $\mathbb{S}^J$ (regarded as a symplectic manifold) is a coadjoint orbit of the Jacobi group $G^J(\mathbb{R})$.

Proof. By application of Theorem 3.46, it suffices to show that $\mathbf{J} : \mathbb{S}^J \to (\mathfrak{g}^J)^*$ is injective, or equivalently, to show that given two points $p, q \in \mathbb{S}^J$,

$$f(p) = f(q) \quad \text{for all} \quad f \in \mathscr{K}(\mathbb{S}^J) \quad \Rightarrow \quad p = q. \tag{166}$$

This can be seen using (158)-(160). □

Remark 3.48. In Ref. 3, we defined the Kählerification of an exponential family $\mathcal{E}$ as the quotient $\mathcal{E}^{\mathbb{C}} := T\mathcal{E}/\Gamma(\mathcal{E})$, where $\Gamma(\mathcal{E})$ is the subgroup of $\mathrm{Diff}(T\mathcal{E})$ *defined by*

$$\Gamma(\mathcal{E}) := \big\{\phi \in \mathrm{Diff}(T\mathcal{E}) \,\big|\, \phi^* g = g,\ \phi_* J = J\phi_* \ \text{ and } \ f \circ \phi = f \ \text{ for all } \ f \in \mathscr{K}(T\mathcal{E})\big\}, \tag{167}$$

where (g, J) is the natural Kähler structure of $T\mathcal{E}$, as described in Sec. II F. If $\Gamma(\mathcal{E})$ is discrete and if its natural action on $T\mathcal{E}$ is free and proper, then $\mathcal{E}^{\mathbb{C}}$ is a Kähler manifold in a natural way. In the case $\mathcal{E} = \mathcal{N}$, it follows from (166) that $\Gamma(\mathcal{N})$ is trivial. Therefore, the Kählerification of $\mathcal{N}$ is the Siegel-Jacobi space $\mathbb{S}^J$, that is, $\mathcal{N}^{\mathbb{C}} \cong \mathbb{S}^J$.

We now discuss the spectral theory of the Kähler functions of $\mathbb{S}^J$ (in a sense to be discussed below). Let $\mathfrak{a}$ be the abelian Lie subalgebra of $\mathfrak{g}^J$ generated by F, Q, R, i.e.,

$$\mathfrak{a} := \mathrm{Vect}_{\mathbb{R}}\{F, Q, R\}. \tag{168}$$

In what follows, we shall identify $\mathfrak{a}$ with the space $\mathscr{P}_2(\mathbb{R})$ of polynomials in one variable of degree ≤ 2 with real coefficients, via the isomorphism

$$F \mapsto -x^2, \quad Q \mapsto x, \quad R \mapsto -\tfrac{1}{4}. \tag{169}$$

Thus, an arbitrary element of $\mathfrak{a} \cong \mathscr{P}_2(\mathbb{R})$ can be written as $k(x) = \alpha x^2 + \beta x + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{R}$. We also introduce the following subgroup of $G^J(\mathbb{R})$:

$$B := \left\{\left(\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}, (\lambda, \mu, \kappa)\right) \,\middle|\, a, b, \lambda, \mu, \kappa \in \mathbb{R},\ a \neq 0\right\}. \tag{170}$$

The group B is a maximal closed, connected, and solvable subgroup of $G^J(\mathbb{R})$, i.e., it is a Borel subgroup of $G^J(\mathbb{R})$ (see Ref. 23). For $b = \big(\big[\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\big], (\lambda, \mu, \kappa)\big) \in B$ and $x \in \mathbb{R}$, the formula

$$x \cdot \left(\begin{bmatrix} a & b \\ 0 & a^{-1}\end{bmatrix}, (\lambda, \mu, \kappa)\right) := ax - \frac{\lambda}{2} \tag{171}$$

defines a right action of B on $\mathbb{R}$. Therefore, B also acts on the left on $\mathscr{P}_2(\mathbb{R})$ via the formula $b\cdot k(x) := k(x\cdot b)$, where $b\in B$.

Lemma 3.49. (*i*) *Let* $\mathrm{Ad} : G^J(\mathbb{R})\times\mathfrak{g}^J \to \mathfrak{g}^J$ *be the adjoint representation of* $G^J(\mathbb{R})$*. Then,*

$$\mathrm{Ad}(M,X,\kappa)\cdot(A,\xi,r) = \big(MAM^{-1}, XAM^{-1}+\xi M^{-1}, r-2\Omega(\xi,X)-\Omega(XA,X)\big), \tag{172}$$

where $M\in \mathrm{SL}(2,\mathbb{R})$*,* $A\in\mathfrak{sl}(2,\mathbb{R})$*,* $X,\xi\in\mathbb{R}^2$ *and* $\kappa,r\in\mathbb{R}$*.*

(*ii*) *For* $k(x)\in\mathfrak{a}$ *and* $g\in G^J(\mathbb{R})$*, we have*

$$\mathrm{Ad}(g)\,k(x)\in\mathfrak{a} \quad\Leftrightarrow\quad g\in B \quad\text{or}\quad k(x)\ \text{is a constant polynomial.} \tag{173}$$

In particular, $\mathrm{Ad}(b)\mathfrak{a}\subseteq\mathfrak{a}$ *for all* $b\in B$*. Moreover, if* $k(x)$ *is a constant polynomial, then* $\mathrm{Ad}(g)k(x)=k(x)$ *for all* $g\in G^J(\mathbb{R})$*.*

(*iii*) *For* $b\in B$ *and* $k(x)\in\mathfrak{a}$*, we have:*

$$\mathrm{Ad}(b)k(x) = k(x\cdot b) \tag{174}$$

(here Ad *is the adjoint representation of* $G^J(\mathbb{R})$*).*

Proof. The first item follows from a direct calculation while (*ii*) and (*iii*) are easily obtained from the matrix representation of the restriction of $\mathrm{Ad}(M,X,\kappa)$ to $\mathfrak{a}$ relative to the basis $\{F,Q,R\}$ and $\{F,G,H,P,Q,R\}$. As a simple calculation shows, this matrix is:

$$\begin{bmatrix} a^2 & 0 & 0\\ -c^2 & 0 & 0\\ -ac & 0 & 0\\ -c\lambda & -c & 0\\ a\lambda & a & 0\\ \lambda^2 & 2\lambda & 1\end{bmatrix},\quad\text{where}\quad M=\begin{bmatrix} a & b\\ c & d\end{bmatrix}\in\mathrm{SL}(2,\mathbb{R}),\quad\text{and}\quad X=(\lambda,\mu)\in\mathbb{R}^2. \tag{175}$$

From this, one easily concludes the proof. □

Lemma 3.50. For $g_1,g_2\in G^J(\mathbb{R})$ *and* $k_1(x),k_2(x)\in\mathfrak{a}$*, we have:*

$$\mathrm{Ad}(g_1)k_1(x)=\mathrm{Ad}(g_2)k_2(x)\quad\Rightarrow\quad \mathrm{Im}(k_1)=\mathrm{Im}(k_2), \tag{176}$$

where $\mathrm{Im}(k_i)$ *is the image of the polynomial* $k_i(x)$ *(regarded as a function* $k_i:\mathbb{R}\to\mathbb{R}$*).*

Proof. If $\mathrm{Ad}(g_1)k_1(x)=\mathrm{Ad}(g_2)k_2(x)$, then $\mathrm{Ad}((g_2)^{-1}g_1)k_1(x)=k_2(x)$ and according to Lemma 3.49, $(g_2)^{-1}g_1\in B$ or $k_1(x)=\text{constant}$. If $(g_2)^{-1}g_1\in B$, then there exists $b\in B$ such that $g_1=g_2b$, and we have, taking into account Lemma 3.49,

$$\begin{aligned}\mathrm{Ad}(g_1)k_1(x)=\mathrm{Ad}(g_2)k_2(x) \quad&\Rightarrow\quad \mathrm{Ad}(g_2)\mathrm{Ad}(b)k_1(x)=\mathrm{Ad}(g_2)k_2(x),\\ &\Rightarrow\quad \mathrm{Ad}(b)k_1(x)=k_2(x),\\ &\Rightarrow\quad k_1(x\cdot b)=k_2(x),\\ &\Rightarrow\quad \mathrm{Im}(k_1)=\mathrm{Im}(k_2).\end{aligned} \tag{177}$$

In the case $k_1(x)=\text{constant}$, Lemma 3.49 implies that $k_1(x)=\mathrm{Ad}(g)k_1(x)$ for all $g\in G^J(\mathbb{R})$. Consequently, $k_1(x)=\mathrm{Ad}((g_2)^{-1}g_1)k_1(x)=k_2(x)$, that is, $k_1(x)=k_2(x)$. The lemma follows. □

Definition 3.51 (Spectrum of a Kähler function). The spectrum of a Kähler function $f\in\mathscr{K}(\mathbb{S}^J)$ *of the form* $f=\mathbf{J}^{\mathrm{Ad}(g)k(x)}$*, where* $g\in G^J(\mathbb{R})$ *and* $k(x)\in\mathfrak{a}$*, is the following subset of* $\mathbb{R}$*:*

$$\mathrm{Spec}(f) := \mathrm{Im}(k), \tag{178}$$

where $\mathrm{Im}(k)$ *is the image of the polynomial* $k(x)$ *(regarded as a function* $k:\mathbb{R}\to\mathbb{R}$*).*

Remark 3.52. Not every Kähler function $f\in\mathscr{K}(\mathbb{S}^J)$ *can be written as* $f=\mathbf{J}^{\mathrm{Ad}(g)k(x)}$ *(consider* $\mathbf{J}^H$ *for example). Therefore, not every Kähler function* $f=\mathbf{J}^L$ *possesses a spectrum. But if it does, Lemma 3.50 guaranties that its spectrum is independent of the decomposition* $L=\mathrm{Ad}(g)k(x)$ *(such decomposition is not unique in general).*

Remark 3.53. Due to the equivariance of the momentum map $\mathbf{J} : \mathbb{S}^J \to (\mathfrak{g}^J)^*$*, one easily sees that* $\mathrm{Spec}(f \circ \Phi_g) = \mathrm{Spec}(f)$ *for all* $g \in G^J(\mathbb{R})$ *(provided that* $f \in \mathcal{K}(\mathbb{S}^J)$ *possesses a spectrum).*

In order to give a statistical meaning to the spectrum of a Kähler function $f \in \mathcal{K}(\mathbb{S}^J)$, let us recall the following facts:

- We have an identification of Kähler manifolds $\mathbb{S}^J \cong T\mathcal{N}$ (see Proposition 3.6), and consequently, the canonical projection $T\mathcal{N} \to \mathcal{N}$ gives a projection $\mathbb{S}^J \to \mathcal{N}$ that we shall also denote by π. Thus, for every $p \in \mathbb{S}^J$, $\pi(p)$ is a Gaussian distribution function over $\mathbb{R}$. If dx denotes the Lebesgue measure, then $\pi(p)(x)dx$ is the associated probability measure (here, we denote by x the variable living in the measure space $(\mathbb{R}, dx)$).
- The expectation parameters $\eta_1, \eta_2 : \mathcal{N} \to \mathbb{R}$ are by definition the expectations (in the probabilistic sense) of the random variables x and x^2 over $\mathbb{R}$ with respect to the probability measures $p(x)dx$ ($p \in \mathcal{N}$), that is, $\eta_1(p) := \int_{-\infty}^{\infty} xp(x)dx$ and $\eta_2(p) = \int_{-\infty}^{\infty} x^2 p(x)dx$ (see (63) and (68)).
- We have identified the vectors $F, Q, R \in \mathfrak{g}^J$ with the polynomials $-x^2, x$ and $-\frac{1}{4}$, respectively (see (169)), and we have $\mathbf{J}^F = -\eta_2 \circ \pi$, $\mathbf{J}^Q = \eta_1 \circ \pi$ and $\mathbf{J}^R = -\frac{1}{4}$ (see (158)).

Let us denote by Φ the action of $G^J(\mathbb{R})$ on $\mathbb{S}^J$, and let f be a Kähler function of the form $f = \mathbf{J}^{\mathrm{Ad}(g)k(x)}$, where $k(x) = \alpha x^2 + \beta x + \gamma \in \mathfrak{a}$ and $g \in G^J(\mathbb{R})$. Using the equivariance of $\mathbf{J} : \mathbb{S}^J \to (\mathfrak{g}^J)^*$, one sees that

$$\begin{aligned} f(p) &= \mathbf{J}^{\mathrm{Ad}(g)k(x)}(p) = (\mathbf{J}^{k(x)} \circ \Phi_{g^{-1}})(p) = (\mathbf{J}^{-\alpha F + \beta Q - 4\gamma R} \circ \Phi_{g^{-1}})(p) \\ &= \big[(\alpha\eta_2 + \beta\eta_1 + \gamma) \circ \pi \circ \Phi_{g^{-1}}\big](p) = \int_{-\infty}^{\infty} (\alpha x^2 + \beta x + \gamma)\big[(\pi \circ \Phi_{g^{-1}})(p)\big](x)dx, \end{aligned} \tag{179}$$

where $p \in \mathbb{S}^J$. We thus have proved the following "spectral decomposition" result.

Proposition 3.54. Let $f \in \mathcal{K}(\mathbb{S}^J)$ *be a Kähler function of the form* $f = \mathbf{J}^{\mathrm{Ad}(g)k(x)}$*, where* $g \in G^J(\mathbb{R})$ *and* $k(x) = \alpha x^2 + \beta x + \gamma \in \mathfrak{a}$*. Then,*

$$f(p) = \int_{-\infty}^{\infty} (\alpha x^2 + \beta x + \gamma)\big[(\pi \circ \Phi_{g^{-1}})(p)\big](x)dx \tag{180}$$

for all $p \in \mathbb{S}^J$*.*

Therefore, a Kähler function of the form $\mathbf{J}^{\mathrm{Ad}(g)k(x)}$ is simply the expectation of the polynomial $k(x) = \alpha x^2 + \beta x + \gamma$ with respect to the probability measure $\big[(\pi \circ \Phi_{g^{-1}})(p)\big](x)dx$, and its spectrum is the set of all possible expectations.

Example 3.55. Using the matrix representation of $\mathrm{Ad}(g)$ *given in (175) together with the invariance property of* Spec *(see Remark 3.53), it is not difficult to see that*

$$\mathrm{Spec}(\mathbf{J}^F) = (-\infty, 0], \qquad \mathrm{Spec}(\mathbf{J}^G) = [0, \infty), \qquad \mathrm{Spec}(\mathbf{J}^P) = (-\infty, \infty), \tag{181}$$

$$\mathrm{Spec}(\mathbf{J}^Q) = (-\infty, \infty), \qquad \mathrm{Spec}(\mathbf{J}^R) = \{-\tfrac{1}{4}\}. \tag{182}$$

As we already mentioned, $\mathbf{J}^H$ *does not have a spectrum in the sense of Definition 3.51.*

Following Ref. 3, we want to associate to a Kähler function $f = \mathbf{J}^{\mathrm{Ad}(g)k(x)}$ and a point $p \in \mathbb{S}^J$, a probability measure $P_{f,p}$ on $\mathrm{Spec}(f)$. To this end, recall that the subgroup B acts on the right on $\mathbb{R}$ as follows (see (171)) : $\Psi_g(x) := x \cdot g = ax - \frac{\lambda}{2}$, where $g = \big(\big[\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\big], (\lambda, \mu, \kappa)\big) \in B$ and $x \in \mathbb{R}$. With this notation, we have the following lemma.

Lemma 3.56. Let $p \in \mathbb{S}^J$ *be such that* $\pi(p)$ *is the Gaussian distribution function of mean* μ *and deviation* σ*, that is,* $\pi(p)(x) = \frac{1}{(2\pi)^{1/2}\sigma}\exp\big\{-\frac{(x-\mu)^2}{2\sigma^2}\big\}$*,* $x \in \mathbb{R}$*. Let* $g = \big(\big[\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\big], (\lambda, \mu, \kappa)\big) \in B$ *be arbitrary. Then,*

(*i*) $(\pi \circ \Phi_g)(p)$ *is the Gaussian distribution function of mean* $\mu' = (\frac{\lambda}{2} + \mu)/a$ *and deviation* $\sigma' = \frac{\sigma}{|a|}$*.*

(*ii*) *If dx is regarded as the Riemannian volume form of the Euclidean metric on $\mathbb{R}$, then,*

$$\Psi_g^*(\pi(p)dx) = \varepsilon(g)\cdot(\pi\circ\Phi_g)(p)dx, \tag{183}$$

where Ψ_g^* is the pull-back operator on differential forms, and where $\varepsilon(g) = 1$ if Ψ_g is orientation preserving and -1 otherwise.

Proof. The first item can be easily obtained by remembering the various identifications and changes of variables me made

- $\theta_1 = \frac{\mu}{\sigma^2}$, $\theta_2 = -\frac{1}{2\sigma^2}$ (see (68)),
- $T\mathcal{N} \cong \mathbb{C}\times i\mathbb{H}$ by means of the complex coordinates $z_1 = \theta_1 + i\dot{\theta}_1$ and $z_2 = \theta_2 + i\dot{\theta}_2$,
- $\mathbb{S}^J = \mathbb{H}\times\mathbb{C}$, and we have the identification $\mathbb{C}\times i\mathbb{H} \cong \mathbb{H}\times\mathbb{C}$ via the map $(z_1, z_2) \mapsto (-iz_2, iz_1)$,
- the action of B on $\mathbb{H}\times\mathbb{C}$ is explicitly given by $\big(\big[\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\big], (\lambda,\mu,\kappa)\big)\cdot(\tau, z) = \big(a(a\tau+b), a(z+\lambda\tau + \mu)\big)$.

The second item is an easy consequence of (*i*) together with the fact that $\Psi_g^*(\pi(p)dx) = (\pi(p)\circ\Psi_g)\Psi^*dx = (\pi(p)\circ\Psi_g)(adx)$. The lemma follows. □

A direct consequence of Lemmas 3.49 and 3.56 is that if $\mathbf{J}^{\mathrm{Ad}(g_1)k_1(x)} = \mathbf{J}^{\mathrm{Ad}(g_2)k_2(x)}$, where $g_1, g_2 \in G^J(\mathbb{R})$, and $k_1(x), k_2(x) \in \mathfrak{a} \cong \mathscr{P}_2(\mathbb{R})$, then the probability distribution functions of $k_1(x)$ and $k_2(x)$ with respect to $\big[(\pi\circ\Phi_{g_1^{-1}})(p)\big](x)dx$ and $\big[(\pi\circ\Phi_{g_2^{-1}})(p)\big](x)dx$ are equal.

Definition 3.57 (Spectral measure). Let $f \in \mathscr{K}(\mathbb{S}^J)$ be a Kähler function of the form $f(p) = \int_{-\infty}^{\infty} k(x)\big[(\pi\circ\Phi_{g^{-1}})(p)\big](x)dx$, where $k(x) \in \mathscr{P}_2(\mathbb{R})$ and $g \in G^J(\mathbb{R})$. For $p \in \mathbb{S}^J$, the spectral measure $P_{f,p}$ is the probability distribution functions of $k(x)$ with respect to $\big[(\pi\circ\Phi_{g_1^{-1}})(p)\big](x)dx$, that is,

$$P_{f,p}(A) := \int_{k^{-1}(A)} [(\pi\circ\Phi_{g^{-1}})(p)](x)dx, \tag{184}$$

where $A \subseteq \mathrm{Spec}(f)$ is a measurable subset.

From a quantum mechanical point of view, the quantity $P_{f,p}(A)$ is interpreted as the probability that the observable $f \in \mathscr{K}(\mathbb{S}^J)$ yields upon measurement an eigenvalue $\lambda \in A \subseteq \mathrm{Spec}(f)$ while the system is in the state $p \in \mathbb{S}^J$.

IV. GAUSSIAN DISTRIBUTIONS: EXTRINSIC GEOMETRY

Let $\mathcal{H} := L^2(\mathbb{R})$ be the Hilbert space of square integrable functions $f : \mathbb{R} \to \mathbb{C}$ endowed with the Hermitian product $\langle f, g\rangle := \int_{\mathbb{R}} \bar{f} g dx$, where dx is the Lebesgue measure. Associated to it is the complex projective space $\mathbb{P}(\mathcal{H}) := (\mathcal{H} - \{0\})/\sim$, where the equivalence relation is defined by

$$f \sim g \quad \Leftrightarrow \quad \exists\,\lambda \in \mathbb{C} - \{0\} \;:\; f = \lambda g. \tag{185}$$

We denote by $[f]$ the equivalence class of $f \in \mathcal{H} - \{0\}$, that is, $[f] = \mathbb{C}\cdot f$. In this section, we shall regard the Siegel-Jacobi space $\mathbb{S}^J$ as a subspace of $\mathbb{P}(\mathcal{H})$ via the injection

$$T : \mathbb{S}^J \hookrightarrow \mathbb{P}(\mathcal{H}), \quad T(\tau, z) := \Big[e^{\frac{i}{2}(\tau x^2 - zx)}\Big], \tag{186}$$

where $(\tau, z) \in \mathbb{H}\times\mathbb{C} \cong \mathbb{S}^J$, and where $x \in \mathbb{R}$.

A. Symplectic immersion

Let us recall a few facts related to the Kähler structure of $\mathbb{P}(\mathcal{H})$. Given $f \in \mathcal{H}$ such that $\|f\|^2 = \langle f, f\rangle = 1$, we can define a chart (U_f, ϕ_f) of $\mathbb{P}(\mathcal{H})$ by letting

$$\begin{cases} U_f := \Big\{[g] \in \mathbb{P}(\mathcal{H}) \,\Big|\, [f] \cap [g] = \{0\}\Big\}, \\ \phi_f : U_f \to [f]^{\perp} \subseteq \mathcal{H}, \ [g] \mapsto \dfrac{1}{\langle f, g \rangle} \cdot g - f, \end{cases} \tag{187}$$

where $[f]^{\perp} := \{g \in \mathcal{H} \,|\, \langle f, g\rangle = 0\}$. If f varies among all the unit vectors in $\mathcal{H}$, then the corresponding charts (U_f, ϕ_f) form an atlas for $\mathbb{P}(\mathcal{H})$ which becomes an infinite dimensional manifold.

The Fubini-Study metric g_{FS} and the Fubini-Study symplectic form ω_{FS} are now characterized as follows. Let $B := \{f \in \mathcal{H} \,|\, \langle f, f\rangle = 1\}$ be the unit ball with inclusion map $j : B \hookrightarrow \mathcal{H}$. We denote by $\pi : B \to \mathbb{P}(\mathcal{H})$ the projection induced by the action of the circle $S^1 := \{e^{i\theta} | \theta \in \mathbb{R}\}$ on B (the action being $e^{i\theta} \cdot f := f\, e^{i\theta}$). Regarded as a real vector space, it is known that $\mathcal{H}$ is a Kähler manifold whose symplectic form (respectively, metric) is the imaginary part (respectively, real part) of the Hermitian inner product $\langle\, ,\, \rangle$, and we have (see Ref. 87)

$$\pi^*\omega_{FS} = j^*\,\mathrm{Im}(\langle\, ,\, \rangle), \quad \pi^* g_{FS} = j^*\,\mathrm{Real}(\langle\, ,\, \rangle). \tag{188}$$

Since π is a submersion, these formulas characterize the Fubini-Study symplectic form and the Fubini-Study metric.[88]

Having this in mind, let us return to the properties of the map $T(\tau, z) = \Big[\mathrm{e}^{\frac{i}{2}(\tau x^2 - z x)}\Big]$.

Proposition 4.1. The map $T : \mathbb{S}^J \hookrightarrow \mathbb{P}(\mathcal{H})$ is a smooth immersion satisfying

$$T^*\omega_{FS} = \tfrac{1}{4}\omega_{KB} \qquad \text{and} \qquad T^* g_{FS} = \tfrac{1}{4} g_{KB} + \tfrac{1}{4} S, \tag{189}$$

where S is the tensor field of symmetric bilinear forms on $\mathbb{S}^J$ whose matrix representation in the coordinates $(\theta, \dot{\theta})$ is

$$S(\theta, \dot{\theta}) := \begin{bmatrix} 0 & 0 \\ 0 & \eta_i \eta_j \end{bmatrix} \tag{190}$$

(here, η_i, $i = 1, 2$, are the expectation parameters of $\mathcal{N}$).

Remark 4.2. It follows from (189) that T is a symplectic map,[89] but it not isometric nor holomorphic.

Before showing Proposition 4.1, let us make a few remarks. The map T has been defined above in terms of the variables $(\tau, z) \in \mathbb{H} \times \mathbb{C}$, but in terms of the variables $(z_1, z_2) = (-iz, i\tau) \in \mathbb{C} \times i\mathbb{H}$, it reads

$$T(z_1, z_2) = \Big[e^{\frac{1}{2}(z_1 x + z_2 x^2)}\Big] = \Big[e^{\frac{1}{2}(\theta_1 x + \theta_2 x^2) + \frac{i}{2}(\dot{\theta}_1 x + \dot{\theta}_2 x^2)}\Big], \tag{191}$$

where θ_k are the natural parameters of $\mathcal{N}$ (in particular, $z_k = \theta_k + i\dot{\theta}_k$, see (87) and Definition 3.1). In order to use the unit ball in $\mathcal{H} = L^2(\mathbb{R})$, we want to normalize the function within bracket in (191). To this end, we introduce the following map:

$$\begin{aligned} \Psi : \mathbb{S}^J \to \mathcal{H}, \quad \Psi(z_1, z_2)(x) &:= e^{\frac{1}{2}(\theta_1 x + \theta_2 x^2 - \psi(\theta)) + \frac{i}{2}(\dot{\theta}_1 x + \dot{\theta}_2 x^2)} \\ &= e^{\frac{1}{2}(z_1 x + z_2 x^2 - \psi(\theta))}, \end{aligned} \tag{192}$$

where $\psi(\theta) = -\frac{(\theta_1)^2}{4\theta_2} + \frac{1}{2}\ln\big(-\frac{\pi}{\theta_2}\big)$. By comparing (192) with the exponential-family form of the Gaussian distribution in (68), one sees that $\Psi(z_1, z_2)$ is normalized, that is, $\langle \Psi(z_1, z_2), \Psi(z_1, z_2)\rangle = 1$ for all $(z_1, z_2) \in \mathbb{C} \times i\mathbb{H}$. Therefore, Ψ can be regarded as a smooth map $\mathbb{S}^J \to B \subseteq \mathcal{H}$, and we have $T(z_1, z_2) = \big[\Psi(z_1, z_2)\big]$.

Proof of Proposition 4.1. Taking into account Ref. 89 together with the characterization of the Fubini-Study metric and symplectic form given above (in terms of the unit ball $B \in \mathcal{H}$, see (188)), it suffices to show that

$$\big\langle \Psi_{*_p} A, \Psi_{*_p} B \big\rangle = 4\big\{g_{KB}(A, B) + i\omega_{KB}(A, B) + S(A, B)\big\} \tag{193}$$

for all $p \in \mathbb{S}^J$ and all $A, B \in T_p\mathbb{S}^J$ (in the above formula it is understood that $T_{\Psi(p)}\mathcal{H} \cong \mathcal{H}$). We work in the coordinates $(\theta, \dot{\theta})$. Take $p = (\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \in \mathbb{S}^J$ and choose $A = (A_1, A_2, A_3, A_4)$ and $B = (B_1, B_2, B_3, B_4)$ in $T_p\mathbb{S}^J$. Using the notation

$$X_1 := A_1 + iA_3, \qquad X_2 := A_2 + iA_4, \qquad Y_1 := B_1 + iB_3, \qquad Y_2 := B_2 + iB_4, \tag{194}$$

we see that

$$\begin{aligned}
\Psi_{*_p}A &= \frac{d}{dt}\Big|_0 \Psi\big(\theta_1 + tA_1, \theta_2 + tA_2, \dot{\theta}_1 + tA_3, \dot{\theta}_2 + tA_4\big) \\
&= \frac{d}{dt}\Big|_0 \mathrm{e}^{\frac{1}{2}\left[(z_1 + tX_1)x + (z_2 + tX_2)x^2 - \psi(\theta + tA)\right]} \\
&= \frac{1}{2}\Big(X_1 x + X_2 x^2 - \frac{\partial \psi}{\partial \theta_1} A_1 - \frac{\partial \psi}{\partial \theta_2} A_2\Big) \cdot \Psi.
\end{aligned} \tag{195}$$

As a direct calculation shows, $\frac{\partial \psi}{\partial \theta_1} = \eta_1$ and $\frac{\partial \psi}{\partial \theta_2} = \eta_2$ (see (68) and (69)), and thus,

$$\Psi_{*_p}A = \tfrac{1}{2}\big(X_1 x + X_2 x^2 - \eta_1 A_1 - \eta_2 A_2\big) \cdot \Psi, \tag{196}$$

from which it follows that

$$\begin{aligned}
&\big\langle \Psi_{*_p}A, \Psi_{*_p}B \big\rangle \\
&= \frac{1}{4}\Big\langle \big(X_1 x + X_2 x^2 - \eta_1 A_1 - \eta_2 A_2\big) \cdot \Psi, \big(Y_1 x + Y_2 x^2 - \eta_1 B_1 - \eta_2 B_2\big) \cdot \Psi \Big\rangle \\
&= \frac{1}{4}\int_{-\infty}^{\infty} (\overline{X}_1 x + \overline{X}_2 x^2 - \eta_1 A_1 - \eta_2 A_2)(Y_1 x + Y_2 x^2 - \eta_1 B_1 - \eta_2 B_2)\, p(x;\theta) dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} \Big[\overline{X}_1 Y_1 x^2 + \overline{X}_1 Y_2 x^3 - \overline{X}_1 B_1 x \eta_1 - \overline{X}_1 B_2 x \eta_2 + \overline{X}_2 Y_1 x^3 + \overline{X}_2 Y_2 x^4 - \overline{X}_2 B_1 x^2 \eta_1 \\
&\quad - \overline{X}_2 B_2 x^2 \eta_2 - A_1 Y_1 x \eta_1 - A_1 Y_2 x^2 \eta_1 + A_1 B_1 (\eta_1)^2 + A_1 B_2 \eta_1 \eta_2 - A_2 Y_1 x \eta_2 - A_2 Y_2 x^2 \eta_2 \\
&\quad + A_2 B_1 \eta_1 \eta_2 + A_2 B_2 (\eta_2)^2\Big]\, p(x;\theta) dx,
\end{aligned} \tag{197}$$

where $p(x;\theta) := e^{x\theta_1 + x^2\theta_2 - \psi(\theta)}$. To compute the above integral, we use the following well-known result (see Ref. 53) : if $\mathcal{E}$ is an exponential family whose elements can be written $p(x;\theta) = \exp\{C(x) + \sum_{i=1}^n \theta_i F_i(x) - \psi(\theta)\}$ (as in Definition 2.31), then the components of the Fisher metric are $(h_F)_{ij}(\theta) = \mathbb{E}((F_i - \eta_i)(F_j - \eta_i))$, where η_i are the expectation parameters, and where the expectation is taking with respect to the probability determined by $p(x;\theta)$. In our case, $F_1(x) = x$ and $F_2(x) = x^2$, and thus, we easily see that for $i, j \in \{1, 2\}$,

$$\int_{-\infty}^{\infty} x^{i+j} p(x;\theta) dx = (h_F)_{ij} + \eta_i \eta_j. \tag{198}$$

By separating the real and imaginary parts in (197), and taking into account (86), (198) together with the fact that $\eta_1(\theta) = \int_{-\infty}^{\infty} x p(x;\theta) dx$, one exactly finds (193). The proposition follows. □

B. Schrödinger-Weil representation and quantum observables

Let $\mathrm{End}\big(C^\infty(\mathbb{R}, \mathbb{C})\big)$ denotes the space of $\mathbb{C}$-linear endomorphisms of $C^\infty(\mathbb{R}, \mathbb{C})$, and let $\mathbf{Q} : \mathfrak{g}^J \to \mathrm{End}\big(C^\infty(\mathbb{R}, \mathbb{C})\big)$ be the linear map

$$F \mapsto -x^2, \qquad P \mapsto -i\frac{\partial}{\partial x}, \tag{199}$$

$$G \mapsto -\frac{\partial^2}{\partial x^2}, \qquad Q \mapsto x, \tag{200}$$

$$H \mapsto 2i\Big(x\frac{\partial}{\partial x} + \frac{1}{2}I\Big), \qquad R \mapsto -\frac{1}{4}I. \tag{201}$$

(I denotes the identity operator). In the above formulas, it is understood that $-x^2$ and x act by multiplication. Regarded as unbounded operators acting on $L^2(\mathbb{R})$ with appropriate domains, these operators are Hermitian.

Remark 4.3. From a physical point of view, the operators

$$-\frac{\partial^2}{\partial x^2} = \mathbf{Q}(G), \quad -\frac{\partial^2}{\partial x^2} + ax^2 = \mathbf{Q}(G - aF), \quad -\frac{\partial^2}{\partial x^2} + ax^2 - bx = \mathbf{Q}(G - aF - bQ), \tag{202}$$

where $a > 0$ and $b \in \mathbb{R}$, are, respectively, the Hamiltonians of the free quantum particle, the quantum harmonic oscillator and the (time-independent) quantum forced oscillator. The operators $\mathbf{Q}(Q) = x$ and $\mathbf{Q}(P) = -i\frac{\partial}{\partial x}$ are the usual position and momentum operators.

Proposition 4.4. We have

$$[\mathbf{Q}(A), \mathbf{Q}(B)] := 2i\mathbf{Q}([A, B]) \tag{203}$$

for all $A, B \in \mathfrak{g}^J$. In particular, $-\frac{i}{2}\mathbf{Q}$ is a unitary representation of the Lie algebra $\mathfrak{g}^J$.

Proof. By a direct calculation using the commutation relations (154)-(156). □

In the literature, the representation $-\frac{i}{2}\mathbf{Q}$ is essentially known as the *infinitesimal Schrödinger-Weil representation* (see Refs. 23 and 26).

Proposition 4.5. For every $L \in \mathfrak{g}^J$ and every $p \in \mathbb{S}^J$, we have

$$\big\langle \Psi(p), \mathbf{Q}(L)\Psi(p)\big\rangle = \mathbf{J}^L(p), \tag{204}$$

where $\Psi : \mathbb{S}^J \to L^2(\mathbb{R})$ is the map introduced in (192).

Proof. By a direct verification using (158) and (199)–(201). □

Remark 4.6. Given an arbitrary Hilbert space $\mathcal{H}$ and a bounded[90] *self-adjoint operator H, it is known that the function $f_H([\psi]) := \frac{\langle\psi, H\psi\rangle}{\langle\psi,\psi\rangle}$ is a Kähler function on the complex projective space $\mathbb{P}(\mathcal{H})$ (see* Refs. 5 and 10*). Therefore, one can reformulate Proposition 4.5 heuristically as follows: every Kähler function on $\mathbb{S}^J$ extends as a Kähler function on $\mathbb{P}(\mathcal{H})$ via the map $T = [\Psi]$.*

Remark 4.7. Given $L \in \mathfrak{g}^J$, it would be interesting to compare the spectrum of the operator $\mathbf{Q}(L)$ with that of $\mathbf{J}^L$ (in the sense of Definition 3.51). In this paper, we do not treat this question, but the reader can easily see that $\mathrm{Spec}\big(\mathbf{Q}(L)\big) = \mathrm{Spec}\big(\mathbf{J}^L\big)$ *for all $L \in \{P, Q, R, F, G\}$ (see Example 3.55). It is also interesting to note, in relation to the quantum harmonic oscillator, that the spectrum of the operator $\mathbf{Q}(G - aF)$ (see (202)) is discrete*[91,100] *and that $\mathbf{J}^{G-aF}$ does not have a spectrum in the sense of Definition 3.51.*

C. Dynamics and the Schrödinger equation

Given $L \in \mathfrak{g}^J$, we denote by $X_{\mathbf{J}L}$ the Hamiltonian vector field of the Kähler function $\mathbf{J}^L : \mathbb{S}^J \to \mathbb{R}$ with respect to the Kähler-Berndt symplectic form ω_{KB}.

Proposition 4.8. There exists a smooth map $\kappa : \mathbb{S}^J \times \mathfrak{g}^J \to \mathbb{C}$, linear in the second entry, with the following property: if $\alpha : I \to \mathbb{S}^J$ is an integral curve of the Hamiltonian vector field $X_{\mathbf{J}L}$, then $\psi(t) := \Psi\big(\alpha(t)\big)$ satisfies

$$i\frac{d\psi}{dt} = \frac{1}{2}\mathbf{Q}(L)\psi + \frac{1}{2}\kappa_L(t)\psi, \tag{205}$$

where $\kappa_L(t) := \kappa(\alpha(t), L)$ and where $\Psi : \mathbb{S}^J \to L^2(\mathbb{R})$ is the map introduced in (192).

Proof. Given $p = (\theta, \dot\theta) = (\eta, \dot\theta) \in \mathbb{S}^J$, we define a linear map $\mathfrak{g}^J \to \mathbb{C}$ as follows:

$$F \mapsto 0, \qquad P \mapsto i\big(\tfrac{\theta_1 + i\dot\theta_1}{2} + \theta_2\eta_1\big), \tag{206}$$

$$G \mapsto i\eta_1(\dot\theta_1\theta_2 + \theta_1\dot\theta_2) + 2i\eta_2\theta_2\dot\theta_2 - \tfrac{1}{4}(\dot\theta_1)^2 + \theta_2 + i\dot\theta_2 + \tfrac{1}{4}(\theta_1)^2 + \tfrac{1}{2}i\theta_1\dot\theta_1, \qquad Q \mapsto 0, \tag{207}$$

$$H \mapsto -i(\eta_1\theta_1 + 2\eta_2\theta_2 + 1), \qquad R \mapsto \frac{1}{4}. \tag{208}$$

In this way, one obtains a map $\kappa : \mathbb{S}^J \times \mathfrak{g}^J \to \mathbb{C}$ which is linear in the second entry. Now, by a direct calculation using the proof of Proposition 3.41, (196) and the definition of $\mathbf{Q}$, one sees that (205) holds. The proposition follows. □

Corollary 4.9. Let $\alpha : I \to \mathbb{S}^J$ be an integral curve of the Hamiltonian vector field $X_{\mathbf{J}L}$, and let $F(t)$ be a primitive of $\kappa(\alpha(t), L)$ on I. Then, $\psi(t) := e^{\frac{i}{2}F(t)}\Psi(\alpha(t))$ satisfies the Schrödinger equation

$$i\frac{d\psi}{dt} = H\psi, \tag{209}$$

where $H := \frac{1}{2}\mathbf{Q}(L)$.

Proof. Again by a direct verification using Proposition 4.8. □

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$$\frac{1}{2}\Big\{\frac{\partial(\Gamma^h)^a_{jk}}{\partial x_i} - \frac{\partial(\Gamma^h)^a_{ik}}{\partial x_j}\Big\} = \sum_{b=1}^{n}\Big\{(\Gamma^h)^b_{ik}(\Gamma^h)^a_{jb} - (\Gamma^h)^b_{jk}(\Gamma^h)^a_{ib}\Big\} \text{ and } \sum_{k=1}^{n}(\Gamma^h)^k_{ik} = \frac{1}{2}\frac{\partial \ln d}{\partial x_i},$$

where d is the determinant of the matrix $h_{ij} = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. We caution that these two formulas are only valid for affine coordinate systems. For similar computations, see Ref. 55.
[63] Let G, H be two groups and let $\tau : H \to \mathrm{Aut}(G)$ be an anti-homomorphism of groups. By definition, the semi-direct product $H \ltimes G$ is the Cartesian product $H \times G$ endowed with the multiplication $(h_1, g_1) \cdot (h_2, g_2) := (h_1h_2, (\tau(h_2)g_1) \cdot g_2)$. One can check that $H \ltimes G$ is a group and that $(h, g)^{-1} = (h^{-1}, \tau(h^{-1})g^{-1})$.
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[90] To some extent, this is also true for unbounded self-adjoint operators (see Ref. 4).
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