Gravitational field of a rotating global monopole

R. M. Teixeira Filho*
Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-970 João Pessoa, Pb, Brazil and Instituto de Física, Universidade Federal da Bahia, Campus de Ondina, 40210-340 Salvador, Ba, Brazil

V. B. Bezerra†
Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58059-970 João Pessoa, Pb, Brazil

(Received 27 April 2001; published 18 September 2001)

By applying the method of complex coordinate transformation, the rotating global monopole solution is obtained from the nonrotating counterpart solution that corresponds to the global monopole of Barriola and Vilenkin.

DOI: 10.1103/PhysRevD.64.084009 PACS number(s): 04.20.Cv, 04.20.Jb

I. INTRODUCTION

Different types of topological objects may have been formed during phase transitions in the early universe. These include domain walls, cosmic strings, and monopoles [1] and have attracted much attention because of their peculiar physical properties, spacetime geometries, and astrophysical and cosmological implications. These topological defects appear as a consequence of a breakdown of local or global gauge symmetries of a system composed by self-coupling isoscalar Higgs fields Φ^a. In the case of global monopoles [2], the model that gives rise to these defects is described by a triplet of scalar fields. The model has a global O(3) symmetry spontaneously broken down to U(1).

There is a curious relationship between the nonrotating and rotating spacetime solutions of Einstein theory discovered by Newman and Janis [3], obtained by applying a complex coordinate transformation. Using this method it was possible to construct [3] the Kerr solution from the Schwarzschild solution and also construct [4] from the Reissner-Nordström solution its rotating counterpart that corresponds to the Kerr-Newman solution. More recently, the rotating Bañados-Teitelboim-Zanelli (BTZ) black hole solution was derived [5] from its nonrotating counterpart [6] using the complex coordinate transformation method introduced by Newman and Janis [3].

In this paper we use the method of complex coordinate transformation discovered by Newman and Janis [3] to determine the metric of a rotating global monopole from its nonrotating counterpart that corresponds to the solution of Barriola and Vilenkin [2].

II. ROTATING GLOBAL MONOPOLE SPACETIME

The static solution of a global monopole in a O(3) broken symmetry model has been investigated by Barriola and Vilenkin [2]. They have shown that the gravitational field is described by a static and spherically symmetric metric with an additional solid angle deficit. The spacetime metric describing the region outside the core of the global monopole is given by Ref. [2]:

$$ds^2=\left(1-8\pi G\eta^2-\frac{2GM}{R}\right)d\tau^2-\left(1-8\pi G\eta^2-\frac{2GM}{R}\right)^{-1}\times dR^2-R^2(d\theta^2+\sin\theta^2 d\varphi^2), \tag{1}$$

where (τ,R,θ,φ) are spherical coordinates, η is the symmetry breaking scale, and M is the mass.

In order to write the metric given by Eq. (1) in a more appropriate form, let us introduce the following coordinate transformations:

$$t=b\tau,\quad r=b^{-1}R, \tag{2}$$

supplemented by the redefinition $\bar{M}\equiv b^{-3}M$, where $b^2=1-8\pi G\eta^2$. Doing this, the line element given by Eq. (1) turns into

$$ds^2=\left(1-\frac{2G\bar{M}}{r}\right)dt^2-\left(1-\frac{2G\bar{M}}{r}\right)^{-1}\times dr^2-b^2r^2(d\theta^2+\sin\theta^2 d\varphi^2). \tag{3}$$

This metric can be rewritten as

$$ds^2=\left(1-\frac{2G\bar{M}}{r}\right)du^2+2du\,dr-b^2r^2(d\theta^2+\sin\theta^2 d\varphi^2), \tag{4}$$

where the new variable u is defined by

$$u=t-r-2G\bar{M}\ln\left(\frac{r}{2G\bar{M}}-1\right). \tag{5}$$

It is worth noting that in the transformation given by Eq. (5) we are considering the region outside the monopole core, in which case the mass $\bar{M}$ can be considered as a constant.

*Electronic address: rmuniz@fisica.ufpb.br
†Electronic address: valdir@fisica.ufpb.br

From Eq. (4) we can read the contravariant components of the metric, namely,

$$g^{00}=0, \quad g^{11}=-\left(1-\frac{2G\bar{M}}{r}\right), \quad g^{01}=1,$$
$$g^{22}=-\frac{1}{b^2r^2}, \quad g^{33}=-\frac{1}{b^2r^2\sin^2\theta}, \tag{6}$$

which can be written in the alternate form

$$g^{\mu\nu}=l^\mu n^\nu+l^\nu n^\mu-m^\mu\bar{m}^\nu-m^\nu\bar{m}^\mu, \tag{7}$$

where the null tetrad vectors are

$$l^\mu=\delta_1^\mu, \quad n^\mu=\delta_0^\mu-\frac{1}{2}\left(1-\frac{2G\bar{M}}{r}\right)\delta_1^\mu,$$
$$m^\mu=\frac{1}{\sqrt{2}br}\left(\delta_2^\mu+\frac{i}{\sin\theta}\delta_3^\mu\right), \tag{8}$$
$$\bar{m}^\mu=\frac{1}{\sqrt{2}br}\left(\delta_2^\mu-\frac{i}{\sin\theta}\delta_3^\mu\right),$$

with $\bar{m}^\mu$ being the complex conjugate of m^μ.

Now, the radial coordinate r is allowed to take complex values and the tetrad can be rewriten as

$$l^\mu=\delta_1^\mu, \quad n^\mu=\delta_0^\mu-\frac{1}{2}\left[1-G\bar{M}\left(\frac{1}{r}+\frac{1}{\bar{r}}\right)\right]\delta_1^\mu,$$
$$m^\mu=\frac{1}{\sqrt{2}b\bar{r}}\left(\delta_2^\mu+\frac{i}{\sin\theta}\delta_3^\mu\right), \tag{9}$$
$$\bar{m}^\mu=\frac{1}{\sqrt{2}br}\left(\delta_2^\mu-\frac{i}{\sin\theta}\delta_3^\mu\right),$$

where $\bar{r}$ is the complex conjugate of r. Following Newman and Janis [3] let us perform the complex coordinate transformation

$$u'=u-ia\cos\theta, \quad r'=r+ia\cos\theta,$$
$$\theta'=\theta \quad \text{and} \quad \varphi'=\varphi \tag{10}$$

on the tetrad vetors l^μ, n^μ, and m^μ. If one now allows u' and r' to be real, we obtain

$$l'^\mu=\delta_1^\mu, \quad n'^\mu=\delta_0^\mu-\frac{1}{2}\left[1-2G\bar{M}\left(\frac{r'}{r'^2+a^2\cos^2\theta'}\right)\right]\delta_1^\mu,$$
$$m'^\mu=\frac{1}{\sqrt{2}b(r'+ia\cos\theta')}\left[ia\sin\theta'(\delta_0^\mu-\delta_1^\mu)+\delta_2^\mu+\frac{i}{\sin\theta'}\delta_3^\mu\right], \tag{11}$$
$$\bar{m}'^\mu=\frac{1}{\sqrt{2}b(r'-ia\cos\theta')}\left[-ia\sin\theta'(\delta_0^\mu-\delta_1^\mu)+\delta_2^\mu-\frac{i}{\sin\theta'}\delta_3^\mu\right],$$

where $\bar{m}'^\mu$ is, as stated, defined as the complex conjugate of m'^μ.

The metric

$$g'^{\mu\nu}=l'^\mu n'^\nu+l'^\nu n'^\mu-m'^\mu\bar{m}'^\nu-m'^\nu\bar{m}'^\mu \tag{12}$$

can be shown to correspond to a rotating global monopole with angular momentum per unit mass a. From Eq. (11) we can read off the contravariant and covariant components of the metric. The covariant components of the metric (12) in terms of coordinates (u',r',θ',φ') reads

$$g'_{\mu\nu}=\begin{bmatrix} 1-\frac{2G\bar{M}r'}{r'^2+a^2\cos^2\theta'} & 1 & 0 & \frac{2G\bar{M}ar'\sin^2\theta'}{r'^2+a^2\cos^2\theta'} \\ \cdot & 0 & 0 & -a^2\sin^2\theta' \\ \cdot & \cdot & -b^2(r'^2+a^2\cos^2\theta') & 0 \\ \cdot & \cdot & \cdot & g'_{\varphi'\varphi'} \end{bmatrix}, \tag{13}$$

where

$$g'_{\varphi'\varphi'}=-\sin^2\theta'\left[b^2r'^2+\frac{2G\bar{M}a^2r'\sin^2\theta'}{r'^2+a^2\cos^2\theta'}+a^2(b^2\cos^2\theta'+\sin^2\theta')\right]. \tag{14}$$

The metric given by Eq. (13) is not in the appropriate form to investigate if it corresponds to a rotating global monopole. Thus one might want to further transform it into one written in Boyer-Lindquist [7] coordinates:

$$t=u'-r'-G\bar{M}\ln(r'^2-2G\bar{M}r'+a^2)-\frac{2G^2\bar{M}^2\arctan\left(\frac{r'-G\bar{M}}{\sqrt{a^2-G^2\bar{M}^2}}\right)}{\sqrt{a^2-G^2\bar{M}^2}},$$

$$r=r',\quad \theta=\theta',$$

$$\varphi=\varphi'+\frac{a\arctan\left(\frac{r-G\bar{M}}{\sqrt{a^2-G^2\bar{M}^2}}\right)}{\sqrt{a^2-G^2\bar{M}^2}}. \tag{15}$$

Using these transformations, the metric given by Eq. (13) turns into

$$g_{\mu\nu}=\begin{bmatrix} 1-\frac{2G\bar{M}r}{r^2+a^2\cos^2\theta} & 0 & 0 & -\frac{2G\bar{M}ar\sin^2\theta}{r^2+a^2\cos^2\theta} \\ \cdot & g_{rr} & 0 & g_{r\varphi} \\ \cdot & \cdot & -b^2(r^2+a^2\cos^2\theta) & 0 \\ \cdot & \cdot & \cdot & g_{\varphi\varphi} \end{bmatrix}, \tag{16}$$

where

$$g_{rr}\equiv\frac{-r^2+a^2[(1-b^2)\sin^2\theta-\cos^2\theta]}{r^2-2G\bar{M}r+a^2}+(1-b^2)\frac{a^2\sin^2\theta[2G\bar{M}r-a^2(1-\sin^4\theta)]}{(r^2-2G\bar{M}r+a^2)^2},$$

$$g_{r\varphi}\equiv-\frac{(1-b^2)a[r^2\sin^2\theta-a^2\cos^2\theta(1+\cos^2\theta)]}{r^2-2G\bar{M}r+a^2},$$

$$g_{\varphi\varphi}\equiv-\sin^2\theta\{b^2r^4+[1-(1-2b^2)\cos^2\theta]a^2r^2+2G\bar{M}a^2r\sin^2\theta+a^4\cos^2\theta(b^2\cos^2\theta+\sin^2\theta)\}/(r^2+a^2\cos^2\theta).$$

Setting $b=1$ in the metric (16), we get the Kerr metric in Boyer-Lindquist coordinates, as it should be. On the other hand, if we set $a=0$ (with $b\neq 1$), we obtain the solution of Barriola and Vilenkin given in Eq. (1).

Now, consider the approximation in which we neglect terms of order $\geqslant a^2/r^2$. Therefore, metric (16) becomes

$$g_{\mu\nu}\simeq\begin{bmatrix} 1-\frac{2G\bar{M}}{r} & 0 & 0 & -\frac{2aG\bar{M}\sin^2\theta}{r} \\ \cdot & -\left(1-\frac{2G\bar{M}}{r}\right)^{-1} & 0 & -\frac{(1-b^2)a\sin^2\theta}{1-\frac{2G\bar{M}}{r}} \\ \cdot & \cdot & -b^2r^2 & 0 \\ \cdot & \cdot & \cdot & -b^2r^2\sin^2\theta \end{bmatrix}. \tag{17}$$

Note that if we set $b=1$ we get the Lense-Thirring metric and if we simultaneously set $a=0$ and neglect terms containing $\bar{M}$, we get the point global monopole [2].

In the approximation in which we are neglecting terms of order a^2/r^2 and up, the Einstein tensor is given by

$$G^{\mu}_{\nu} \simeq \begin{bmatrix} \frac{1-b^2}{b^2r^2} & 0 & 0 & -\frac{2aG\bar{M}(1-b^2)}{b^4r^5} \\ \cdot & \frac{1-b^2}{b^2r^2} & 0 & -\frac{a(1-b^2)^2}{b^4r^4\left(1-\frac{2G\bar{M}}{r}\right)} \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}. \tag{18}$$

It is worth commenting that the off-diagonal elements go to zero more quickly than the diagonal ones. Therefore, for r sufficiently large, we get

$$G^{\mu}_{\nu} \approx \begin{bmatrix} \frac{1-b^2}{b^2r^2} & 0 & 0 & 0 \\ \cdot & \frac{1-b^2}{b^2r^2} & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}. \tag{19}$$

Doing the inverse coordinate transformation corresponding to Eq. (2) and substituting $b^2=1-8\pi G\eta^2$, we get

$$G^{\mu}_{\nu} \approx \begin{bmatrix} \frac{8\pi G\eta^2}{R^2} & 0 & 0 & 0 \\ \cdot & \frac{8\pi G\eta^2}{R^2} & 0 & 0 \\ \cdot & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & 0 \end{bmatrix}, \tag{20}$$

which is the Einstein tensor corresponding to the global monopole solution of Barriola and Vilenkin far from the monopole core (point global monopole).

Therefore, for suficiently large value of r, the energy-momentum tensor T^{μ}_{ν} associated with metric (16) is the same as the global monopole of Barriola and Vilenkin; that is, $T^t_t \simeq T^r_r \simeq \eta^2/R^2$, and the other components are, approximately, zero. In other words, for sufficiently large values of r, the metric given by Eq. (16) obeys Einstein's equations with energy-momentum tensor, which is the same of the global monopole solution.

III. CONCLUDING REMARKS

The metric obtained in this paper and given by Eq. (16) corresponds to the nonrotating global monopole solution of Barriola and Vilenkin [2] in the limit when the angular momentum vanishes. In the rotating case we get the Kerr solution setting $b=1$, and in the nonrotating case we get the Schwarzschild solution for this same value of parameter b. We also get from our results that the Einstein tensor corresponding to the static solution is reobtained in the appropriate limit when r is sufficiently large.

The method of complex coordinate transformation to obtain a rotating solution from its static counterpart works for a class of solutions that contains the Schwarzschild solution as a special case. Taking into account the fact that we can obtain the Schwarzschild solution from the global monopole by setting $b=1$, we may conclude that this method works also in this case. Therefore, the metric we have found bears the same relation to the corresponding static solution as the Kerr metric bears to the Schwarzschild metric. As this method works to construct the Kerr solution from the Schwarzschild one, we can conclude that it works to contruct the solution given by metric (16)—that is, the one that corresponds to the rotating counterpart of the static solution of Barriola and Vilenkin [2], which we are calling the rotating global monopole solution.

ACKNOWLEDGMENTS

This work was partially supported by CNPq.

[1] T. W. B. Kibble, J. Phys. A **9**, 1387 (1976); A. Vilenkin, Phys. Rep. **121**, 263 (1985).
[2] M. Barriola and A. Vilenkin, Phys. Rev. Lett. **63**, 341 (1989).
[3] E. T. Newman and A. I. Janis, J. Math. Phys. **6**, 915 (1965).
[4] E. T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, J. Math. Phys. **6**, 918 (1965).
[5] Hongsu Kim, Phys. Rev. D **59**, 064002 (1999).
[6] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992); M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, Phys. Rev. D **48**, 1506 (1993).
[7] R. H. Boyer and R. W. Lindquist, J. Math. Phys. **8**, 265 (1967); B. Carter, Phys. Rev. **174**, 1559 (1968).