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Non-uniform Specification and Large Deviations for Weak Gibbs Measures

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Abstract We establish bounds for the measure of deviation sets associated to continuous observables with respect to not necessarily invariant weak Gibbs measures. Under some mild assumptions, we obtain upper and lower bounds for the measure of deviation sets of some non-uniformly expanding maps, including quadratic maps and robust multidimensional non-uniformly expanding local diffeomorphisms. For that purpose, a measure theoretical weak form of specification is introduced and proved to hold for the robust classes of multidimensional non-uniformly expanding local diffeomorphisms and Viana maps.

Keywords Non-uniform specification · Large deviations · Hyperbolic times · Weak Gibbs measure

1 Introduction

The theory of Large Deviations concerns the study of the rates of convergence at which time averages of a given sequence of random variables converge to the limit distribution. An application of these ideas into the realm of Dynamical Systems is useful to estimate the velocity at which typical points of ergodic invariant measures converge to the corresponding space averages. More generally, given a continuous transformation f on a compact metric space M and a reference measure ν, one would like to provide sharp estimates for the ν-measure of the deviation sets

$$\left\{x \in M : \frac{1}{n}\sum_{j=0}^{n-1} g\big(f^j(x)\big) > c\right\}$$

for all continuous functions $g : M \to \mathbb{R}$ and real numbers c. To this purpose, a priori estimates on the measure of the dynamical balls

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$$B(x,n,\varepsilon)=\big\{y\in M:d\big(f^j(y),f^j(x)\big)\le\varepsilon,\forall 0\le j\le n\big\}$$

for $x\in M$, $\varepsilon>0$ and $n\ge 1$ are useful and somewhat necessary since points that belong to the same dynamical ball have nearby Birkhoff averages with respect to continuous functions.

Some large deviations ideas and techniques are particularly useful to the study of the thermodynamical formalism of transformations with some hyperbolicity. Recall that the variational principle for the pressure asserts that for every continuous potential ϕ

$$P_{\text{top}}(f,\phi)=\sup\left\{h_\eta(f)+\int\phi\,d\eta\right\},$$

where the supremum is taken over all invariant probability measures η. A measure μ that attains the supremum in the variational principle is called an equilibrium state for f with respect to the potential ϕ. A large deviations theory was developed for uniformly hyperbolic systems restricted to a basic piece of the non-wandering set and Hölder continuous potentials in both discrete and time-continuous settings. Indeed, such hyperbolic transformations admit a unique equilibrium state with respect to any Hölder continuous potential (see [11, 34, 35]), Young, Kifer and Newhouse [21, 22, 42] established, in the mid nineties, large deviation principles for this important open class of dynamical systems: the rate of decay is given explicitly in terms of the distance of all invariant measures η with "bad" space averages to the equilibrium state μ. Focusing on the discrete time case, the sharp lower and upper bounds obtained in [42] for the measure of deviation sets yield as a consequence that for any ergodic equilibrium state μ and every continuous observable g, the measure of the set of points whose time average $\frac{1}{n}\sum_{j=0}^{n-1}g(f^j(x))$ is far from the space average $\int g\,d\mu$ decreases exponentially fast. Two key ingredients to obtain the large deviations principle are that equilibrium states are Gibbs measures and that, when restricted to a basic piece of the non-wandering set, every uniformly hyperbolic dynamical system is semi-conjugated to a subshift of finite type that satisfies a very "strong mixing" condition known as specification. This notion, introduced by Bowen [10], means roughly that any finite sequence of pieces of orbit can be well approximated by periodic ones.

Our purpose here is to give a contribution for the ergodic theory of non-uniformly expanding maps in two directions. Namely, we introduce a measure-theoretical non-uniform specification property and obtain upper and lower large deviations bounds with respect to weak Gibbs measures as we now detail.

On the one hand, the specification property constitutes an important tool in dynamical systems which is useful e.g. to obtain uniqueness of equilibrium states for expansive transformations, to study large deviations or to study the multifractal formalism for associated to Birkhoff averages. However, and despite the fact that the later property holds for topologically mixing interval maps and dynamical systems with arbitrary small finite Markov partition, conceptually one cannot expect this to hold with great generality in the absence of uniform hyperbolicity. For that reason we introduce a measure theoretical non-uniform specification property and prove that it holds for a large class of robust non-uniformly expanding maps as in [39] and the multidimensional non-uniformly hyperbolic attractors with critical region considered in [38]. Such class of transformations that may not satisfy the strong specification property seem to constitute first multidimensional examples presenting a weak form of specification in a non-uniformly hyperbolic context. One should mention that other mild forms of specification were introduced by Saussol, Troubetzkoy, Vaienti [36] to study the relation between recurrence and dimension in dynamical systems, by Pfister, Sullivan [31] and Thompson [37] to the study of multifractal formalism for Birkhoff averages associated to beta-shifts, and also by Yamamoto [41] to study large deviations for automorphisms on compact metric abelian groups.

On the other hand, since the nineties many efforts have been made in the attempt to extend the theory of large deviations to the scope of non-uniformly hyperbolic dynamics and some important results in that direction have been obtained recently. Araújo and Pacífico [5] established large deviation upper bounds for the deviation sets of physical measures for non-uniformly expanding maps (in the sense of [1]). More recently, Melbourne and Nicol [25, 26] studied systems that admit some inducing Markov structure, and proved that the measure of points with atypical time averages for a Hölder continuous potential has the same decay rate as the inducing time itself. In particular, less than exponential rate of convergence to equilibrium is studied. Independently, in the case of exponential tail, Rey-Bellet and Young [33] obtained similar and sharper results. The construction of (countable) expanding Markov maps in [30] provides many examples where the previous results apply. Large deviations principles were also obtained by Yuri (see [44]) in the context of shifts with countably many symbols and by Comman and Rivera (see [17]) for non-uniformly expanding rational maps. Notice that in [5] the authors establish large deviation upper bounds with respect to Lebesgue measure, while in [17, 25, 26, 33, 44] the decay rate for the measure of deviation sets is studied with respect to the invariant probability measure. More recently, Chung [16] obtained some large deviation principles for the Lebesgue measure on Markov maps satisfying some technical conditions of similar flavor to our non-uniform specification property.

Hence, to the best of our knowledge, the theory of large deviations with respect to not necessarily invariant reference measures arising from thermodynamical formalism is far from complete.

Inspired by the pioneering work of Young [42] our purpose in this direction is to obtain large deviations estimates for non-uniformly expanding maps that exhibit the non-uniform specification property with respect to a not necessarily invariant weak Gibbs measure. Weak Gibbs measures are such that the measure of dynamical balls $B(x, n, \varepsilon)$ is given according to the nth Birkhoff sums of some potential on the orbit of x up to some multiplicative constant which has at most subexponential growth in n. See (2.3) below for the precise definition. Moreover, such measures arise naturally in the thermodynamical formalism of many non-uniformly expanding maps, where equilibrium states arise as invariant measures absolutely continuous with respect to some reference weak Gibbs measure as illustrated in Sect. 6. Roughly, one proves that the set of points whose time averages remain far from the space average with respect to the equilibrium measure decrease exponentially fast, with a decay rate which is related to the existence of invariant expanding measures with frequent hyperbolicity (we refer the reader to Theorem 2.2 for the precise statement). Since equilibrium states associated to uniformly expanding dynamics and Hölder continuous potentials satisfy the strong Gibbs property our results partially extend the ones of [42] to the non-uniformly expanding setting. In particular, the thermodynamical approach used here fails in the same extent to provide sharp large deviation bounds if there is non-uniqueness of equilibrium states. See for instance [21] for an example in the uniformly hyperbolic setting.

One should also mention that the large deviation results presented here are indeed complementary and extend the ones of obtained by Araújo and Pacífico [5] in the non-uniformly expanding setting. First, the reference measure is not necessarily Lebesgue and our assumptions in our Theorem 2.2 rely on the Gibbs property for the reference measure while the assumptions of [5] rely on the non-uniform hyperbolicity and slow recurrence condition to the critical region. Finally, we obtain large deviation lower bound estimates which were not available even in the case of the Lebesgue measure.

This paper is organized as follows. Our main results are stated along Sect. 2. In Sect. 3 we recall some necessary definitions and prove some preliminary lemmas. The proofs of our main results are given in Sects. 4 and 5. Finally, in Sect. 6 we present some examples and further questions.

2 Statement of Results

2.1 Abstract Theorem

Let $f : M \to M$ be a continuous transformation on a compact metric space M and let ν be some (not necessarily invariant) probability measure. In this section we state an abstract result on the deviation of Birkhoff averages given by continuous observables.

Given an observable $\phi : M \to \mathbb{R}$, we denote by $S_n\phi(x) = \sum_{j=0}^{n-1} \phi \circ f^j$ the nth Birkhoff sum of ϕ. Given a full ν-measure set $\Lambda \subset M$, denote by $\mathcal{F}(\Lambda)$ the set of continuous functions $\psi \in C(M, \mathbb{R})$ so that, there exists $\delta_0 > 0$ and for every $x \in \Lambda$ and $0 < \varepsilon < \delta_0$ there exists a sequence of positive constants $(K_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \frac{1}{n} \log K_n(x, \varepsilon) = 0$ and

$$K_n(x,\varepsilon)^{-1} e^{-S_n\psi(x)} \leq \nu\big(B(y,n,\varepsilon)\big) \leq K_n(x,\varepsilon) e^{-S_n\psi(x)} \tag{2.1}$$

for every $n \geq 1$ and every $y \in M$ satisfying $B(y, n, \varepsilon) \subset B(x, n, \delta_0)$. This is a generalization of the usual notion of Gibbs measure corresponding which can be obtained e.g. in the case that Λ is compact and $x \mapsto K_n(x, \varepsilon)$ is continuous and independent of n. Here we do not assume the compactness of Λ nor any regularity of the functions K_n. We define also $\delta(\varepsilon, \beta)$ as the exponential decay rate corresponding to the ν-measure of the points whose constants K_n grow at most β-exponentially, that is, if

$$\Delta_n(\beta) = \big\{x \in \Lambda : K_n(x,\varepsilon) < e^{\beta n}\big\}, \tag{2.2}$$

then set $\delta(\varepsilon, \beta) = \limsup_{n\to\infty} \frac{1}{n} \log \nu(\Delta_n^c(\beta))$. For notational simplicity, when no confusion is possible we shall omit the dependence on β in the definition of the set Δ_n. In a context of non-uniform hyperbolicity the quantity $\delta(\varepsilon, \beta)$ appears as the exponential decay of the instants of hyperbolicity, does not depend on ε and it is negative for interesting class of examples that appear in Sect. 6. Finally, the *relative entropy* of an f-invariant probability measure η is defined as $h_\nu(f, \eta) = \eta\text{-esssup}\, h_\nu(f, \cdot)$, where

$$h_\nu(f,x) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \nu\big(B(x,n,\varepsilon)\big), \quad \text{for all } x \in M.$$

We will also need the following:

Definition 2.1 We say that a map f satisfies the *specification property* if for any $\varepsilon > 0$ there exists an integer $N = N(\varepsilon) \geq 1$ such that the following holds: for every $k \geq 1$, any points $x_1, \ldots, x_k$, and any sequence of positive integers $n_1, \ldots, n_k$ and $p_1, \ldots, p_k$ with $p_i \geq N(\varepsilon)$ there exists a point x in M such that

$$d\big(f^j(x), f^j(x_1)\big) \leq \varepsilon, \quad \forall 0 \leq j \leq n_1$$

and

$$d\big(f^{j+n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(x), f^j(x_i)\big) \leq \varepsilon$$

for every $2 \leq i \leq k$ and $0 \leq j \leq n_i$.

Note that this notion of specification is purely topological and is slightly weaker than the one introduced by Bowen [10], that requires that any finite sequence of pieces of orbit is well approximated by periodic orbits. In fact, this condition is known to imply that the system is topologically mixing [10]. It might seem that specification is quite rare among most dynamical systems. However, Blokh [9] proved in a surprising way that the notions of specification and topologically mixing coincide for every one-dimensional *continuous* mapping. This is no longer true if the one-dimensional map fails to be continuous (see e.g. [14]). We refer the reader to [40] for more details on the specification property. Our first result is as follows.

Theorem 2.1 *Assume that* $h_{\text{top}}(f) < \infty$, *let* ν *be a probability measure and let* $\Lambda \subset M$ *be such that* $\nu(\Lambda) = 1$. *Given* $g \in C(M, \mathbb{R})$ *and* $c \in \mathbb{R}$, *if* $\psi \in \mathcal{F}(\Lambda)$ *then for every small* $\varepsilon, \beta > 0$ *it holds*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\left[x \in M : \frac{1}{n} S_n g(x) \geq c\right] \leq \max\left\{\delta(\varepsilon, \beta), \sup\left\{h_\eta(f) - \int \psi \, d\eta\right\} + \beta\right\}$$

where the supremum is over all invariant probability measures η *such that* $\int g\, d\eta \geq c$. *Moreover, it holds that*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c\right) \geq \sup\left\{h_\eta(f) - h_\nu(f, \eta)\right\},$$

where the supremum is taken over all ergodic *measures* η *satisfying* $\int g\, d\eta > c$. *Furthermore if* $\psi \in \mathcal{F}(\Lambda)$ *and* f *satisfies the specification property then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c\right) \geq \sup\left\{h_\eta(f) - \int \psi \, d\eta\right\},$$

where the supremum is taken over all invariant probability measures η *such that* $\eta(\Lambda) = 1$ *and* $\int g\, d\eta > c$.

This theorem generalizes Theorem 1 in [42], where $\Lambda = M$ was assumed to be compact and some uniform control on the measure of partition elements was required.

2.2 Deviation Bounds for Non-uniformly Expanding Maps

2.2.1 Context

Let M be a compact Riemannian manifold and let $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism outside of a compact critical or singular region $\mathcal{C}$. Assume:

(H) *f behaves like a power of the distance to the critical or singular set $\mathcal{C}$*: there exist $B > 1$ and $\beta \in (0, 1)$ such that for every $x, y \in M \setminus \mathcal{C}$ with $\text{dist}(x, y) < \text{dist}(x, \mathcal{C})/2$ and every $v \in T_x M$:
 (a) $\frac{1}{B} \text{dist}(x, \mathcal{C})^\beta \leq \frac{\|Df(x)v\|}{\|v\|} \leq B \,\text{dist}(x, \mathcal{C})^{-\beta}$;
 (b) $|\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\|\, | \leq B \frac{\text{dist}(x,y)}{\text{dist}(x,\mathcal{C})^\beta}$;
 (c) $|\log |\det Df(x)^{-1}| - \log |\det Df(y)^{-1}|\, | \leq B \frac{\text{dist}(x,y)}{\text{dist}(x,\mathcal{C})^\beta}$.

This condition was proposed in [1] as a multidimensional counterpart of the non-flat critical points in one-dimensional dynamics. We also assume the following condition on f:

(C) There exists $L > 0$ and $\gamma \in (0, 1)$ such that for any small $\varepsilon > 0$ every connected component in the preimage of a set of diameter ε is contained in a ball of radius $L\varepsilon^\gamma$.

This condition is clearly satisfied if f is a local diffeomorphism and, since f behaves like a power of the distance to $\mathcal{C}$, it is most likely to hold e.g. if $\mathcal{C}$ has empty interior. Such condition is satisfied by the class transformations with singularities (quadratic and Viana maps) considered in Sect. 6. Let $\phi : M\backslash\mathcal{C} \to \mathbb{R}$ be a Hölder continuous potential and assume:

(P1) There exists a probability measure ν that is positive on open sets, it is non-singular with respect to f with Hölder continuous Jacobian $J_\nu f = \lambda e^{-\phi}$, for some $\lambda > 0$. We will refer to ν as a *conformal measure* associated to ϕ;

(P2) (f, ν) has *non-uniform expansion*: there exists $\sigma > 1$ such that for ν-a.e. x

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \big\| Df\big(f^j(x)^{-1}\big)\big\| \le -2\log\sigma < 0$$

and

$$(\forall \varepsilon > 0)(\exists \delta > 0) \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \mathrm{dist}_\delta\big(f^j(x), \mathcal{C}\big) < \varepsilon,$$

where for any given $\delta > 0$, we let $\mathrm{dist}_\delta(x, \mathcal{C})$ be the δ-truncated distance from a point x to $\mathcal{C}$ defined as $\mathrm{dist}(x, \mathcal{C})$ if $\mathrm{dist}(x, \mathcal{C}) < \delta$ and equal to 1 otherwise.

These assumptions are quite natural in a context of non-uniform hyperbolicity and are verified by a large class of maps and potentials. For instance, if f is a non-uniformly expanding map (in the sense of [1]) and $\phi = -\log|\det Df|$ then the Lebesgue measure is a conformal measure that satisfies (P1) and (P2). Usually conformal measures appear as eigenmeasures associated to the dual $\mathcal{L}_\phi^*$ of the Ruelle-Perron-Frobenius operator

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y),$$

acting on the space of probability measures $\mathcal{M}$. Moreover, hypothesis (P1) and (P2) together with the fact that the potential ϕ is Hölder continuous yield that ν is a *weak Gibbs measure*: there are $P \in \mathbb{R}$ and $\delta > 0$ so that for any $0 < \varepsilon \le \delta$ and almost every x there is a sequence of positive numbers $(K_n)_{n\ge1}$ (depending also on ϕ) satisfying $\lim_{n\to\infty} \frac{1}{n} \log K_n(x, \varepsilon) = 0$ and

$$K_n(x, \varepsilon)^{-1} \le \frac{\nu(B(x, n, \varepsilon))}{e^{-Pn+S_n\phi(y)}} \le K_n(x, \varepsilon) \tag{2.3}$$

for every $y \in B(x, n, \varepsilon)$ (see e.g. [39]). Compare to Lemma 3.3 and Corollary 3.2 below. We say that n is a (σ, δ)-*hyperbolic time* for $x \in M$ (or hyperbolic time for short) if there is a small positive constant $b > 0$ such that

$$\prod_{j=n-k}^{n-1} \big\| Df\big(f^j(x)^{-1}\big)\big\| \le \sigma^{-k} \quad \text{and} \quad \mathrm{dist}_\delta\big(f^{n-k}(x), \mathcal{C}\big) > \sigma^{-bk}$$

for every $1 \le k \le n$. The non-uniform expansion condition (P2) guarantees the existence of infinitely many hyperbolic times ν-almost everywhere. We refer the reader to Sect. 3.2 for more details. Let H denote the set of points with infinitely many hyperbolic times, $n_1(\cdot)$ be the first hyperbolic time map and $\Gamma_n = \{x \in M : n_1(x) > n\}$. We say that a probability measure η is *expanding* if $\eta(H) = 1$. In particular, any invariant expanding measure has only positive Lyapunov exponents. We also assume:

(P3) There is a unique equilibrium state μ for f with respect to ϕ, it is absolutely continuous with respect to ν, there exists a positive constant $K > 0$ such that the density satisfies $d\mu/d\nu \geq K^{-1}$, and $n_1 \in L^1(\mu)$.

The last assumption above essentially means that the decay of the first hyperbolic time map is at least polynomial of order $n^{-(1+\varepsilon)}$, for some $\varepsilon > 0$. We refer the reader to the works [1, 13, 27, 39, 43, 44], just to quote some classes of maps and potentials that satisfy our assumptions.

2.2.2 Non-uniform Specification Property

In contrast to the topological concept of specification we introduce a **measure theoretical** notion.

Definition 2.2 We say that (f, μ) satisfy the **non-uniform specification property** if there exists $\delta > 0$ such that for μ-almost every x and every $0 < \varepsilon < \delta$ there exists an integer $p(x, n, \varepsilon) \geq 1$ satisfying

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} p(x, n, \varepsilon) = 0$$

and so that the following holds: given points $x_1, \ldots, x_k$ in a full μ-measure set and positive integers $n_1, \ldots, n_k$, if $p_i \geq p(x_i, n_i, \varepsilon)$ then there exists z that ε-shadows the orbits of each x_i during n_i iterates with a time lag of $p(x_i, n_i, \varepsilon)$ in between $f^{n_i}(x_i)$ and x_{i+1}, that is,

$$z \in B(x_1, n_1, \varepsilon) \quad \text{and} \quad f^{n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(z) \in B(x_i, n_i, \varepsilon)$$

for every $2 \leq i \leq k$.

These notions means that almost every finite pieces of orbits are approximated by a real orbit such that the time lag between two consecutive pieces of orbits is small proportion of the size of the piece of orbit being shadowed. Clearly, if the strong specification property holds then (f, η) satisfies the non-uniform specification property for every f-invariant probability measure η. Let us also mention that a notion of non-uniform specification property similar to the one introduced in [36] (using Pesin theory) would also be enough to obtain the lower bound estimates in Theorem 2.2 below. We shall not use or prove this fact here.

In opposition to the specification property we expect this weak form of specification to hold in a broad non-uniformly hyperbolic setting. We refer the reader to Sect. 6 for some examples in which the later condition holds but may fail to satisfy the specification property.

2.2.3 Deviation Bounds for Non-uniformly Expanding Maps

The following result extends the large deviation results proven in [42] for uniformly hyperbolic maps.

Theorem 2.2 *Let M be a compact manifold and $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism outside a critical or singular region $\mathcal{C}$ that satisfies (H) and (C). Let $\phi : M \setminus \mathcal{C} \to \mathbb{R}$ be a Hölder continuous potential and let ν and μ be probability measures given by (P1)–(P3). If $g \in C(M, \mathbb{R})$ and $c \in \mathbb{R}$ then*

$$\limsup_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M:\frac{1}{n}S_n g(x)\geq c\right)$$
$$\leq\max\left\{\sup\left\{-P+h_\eta(f)+\int\phi\,d\eta\right\},\limsup_{n\to\infty}\frac{1}{n}\log\mu(\Gamma_n)\right\}$$

where the supremum is taken over all invariant probability measures η such that $\int g\,d\eta\geq c$. If, in addition, f satisfies the specification property or (f,μ) satisfies the non-uniform specification property then

$$\liminf_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M:\frac{1}{n}S_n g(x)>c\right)\geq\sup\left\{-P+h_\eta(f)+\int\phi\,d\eta\right\},$$

where the supremum is taken over all invariant probability measures η such that $\eta(H)=1$, $\int g\,d\eta>c$ and $n_1\in L^1(\eta)$.

In consequence one can estimate the decay of the deviation set as follows:

Corollary 2.1 *Under the previous assumptions,*

$$\limsup_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M:\left|\frac{1}{n}S_n g(x)-\int g\,d\mu\right|\geq c\right)$$
$$\leq\max\left\{\sup\left\{-P+h_\eta(f)+\int\phi\,d\eta\right\},\limsup_{n\to\infty}\frac{1}{n}\log\mu(\Gamma_n)\right\}$$

where the supremum is taken over all invariant probability measures η such that $|\int g\,d\eta-\int g\,d\mu|\geq c$, and

$$\liminf_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M:\left|\frac{1}{n}S_n g(x)-\int g\,d\mu\right|>c\right)$$
$$\geq\sup\left\{-P+h_\eta(f)+\int\phi\,d\eta\right\}$$

where the supremum is taken over all invariant probability measures η such that $\eta(H)=1$, $|\int g\,d\eta-\int g\,d\mu|>c$ and $n_1\in L^1(\eta)$.

Some comments are in order. First notice that the upper bound estimate takes into account the loss of uniform expansion in terms of the decay of the first hyperbolic time map. In particular, if the first hyperbolic time map fails to have exponential decay then the right hand side in the previous upper bound is zero, since the other term is also non-positive. In [26] less than exponential deviations are proven for systems that admit a Young tower with inducing time has polynomial decay. More recently, in [4] a relation between the rate of decay of correlations and the large deviations with respect to the invariant measure has been established. This reenforces the idea that a condition on the tail of the first hyperbolic time map should not be easily removed in general. See Example 6.2 for a more detailed discussion. Moreover, we expect Theorem 2.1 to hold in the more general setting of zooming measures introduced in [30] since our ingredients are bounded distortion and growth to large scale. Finally, these results should also extend to the non-uniformly hyperbolic setting, e.g. the class of partially hyperbolic diffeomorphisms with contracting direction and center-unstable mostly expanding direction introduced in [1]. Since the construction of general equilibrium states for such class of maps is still not available, the lack of motivating examples lead us to state the results only in the non-uniformly expanding setting.

3 Preliminary Results

3.1 Metric Entropy

First we recall some definitions. Let $\varepsilon > 0$ and $n \geq 1$ be arbitrary. A set $E \subset M$ is (n, ε)-*separated* if $d_n(x, y) > \varepsilon$ for every $x, y \in E$ with $x \neq y$, where $d_n : M \times M \to \mathbb{R}_0^+$ is the metric given by

$$d_n(x, y) = \max_{0 \leq j \leq n-1} d\big(f^j(x), f^j(y)\big).$$

If, in addition, E has maximal cardinality we say that it is a *maximal* (n, ε)-*separated* set. Note that for any maximal (n, ε)-separated set E, the dynamical balls $B(x, n, \varepsilon)$ centered at points in E are pairwise disjoint and that the union $\bigcup_{x \in E} B(x, n, 2\varepsilon)$ covers M. We recall some properties of topological and metric entropy.

Proposition 3.1 [10] *Let $f : M \to M$ be a continuous map in a compact metric space M. If $h_{\text{top}}(f)$ denotes the topological entropy of f then*

$$h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon),$$

where $N(n, \varepsilon)$ the minimum number of (n, ε) dynamical balls necessary to cover M.

A metric counterpart of this result is as follows. Let η be an invariant probability measure and $\delta > 0$ arbitrary. Given $\varepsilon > 0$ let $N(n, \varepsilon, \delta)$ be the minimum number of (n, ε)-dynamical balls necessary to cover a set of measure larger than $1 - \delta$.

Proposition 3.2 [20, Theorem I.I] *Let $f : M \to M$ be a homeomorphism in a compact metric space M and η an f-invariant probability measure. Hence*

$$h_\eta(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon, \delta),$$

for every $\delta > 0$.

3.2 Hyperbolic Times

In this subsection we recall some properties of hyperbolic times.

Definition 3.1 We say that (f, η) is *non-uniformly expanding* if there exists $N \geq 1$ and $\sigma > 1$ such that almost every x satisfies

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \big\| Df^N\big(f^{jN}(x)^{-1}\big)\big\| \leq -2 \log \sigma < 0 \tag{3.1}$$

and the *slow recurrence condition*: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for μ-almost every point $x \in M$ it holds that

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} - \log \text{dist}_\delta\big(f^j(x), \mathcal{C}\big) < \varepsilon. \tag{3.2}$$

Let B, β be given by condition (H2) and take $0 < b < \{\frac{1}{2}, \frac{1}{2\beta}\}$. A sufficiency criterium for the existence of hyperbolic times is given as application of Pliss' lemma.

Lemma 3.1 [1, Lemma 5.4] *There exists constants $\theta > 0$ and $\delta > 0$ (depending only on f and c) such that if $x \in M \setminus \cup_n f^n(\mathcal{C})$ satisfies (3.1) and (3.2) then the following holds: for every large $N \geq 1$ there exist a sequence of integers $1 \leq n_1(x) < n_2(x) < \cdots < n_l(x) \leq N$, with $l \geq \theta n$ so that*

$$\prod_{j=n-k}^{n-1} \left\| Df\big(f^j(x)^{-1}\big)\right\| \leq \sigma^{-k} \quad \textit{and} \quad \mathrm{dist}_\delta\big(f^{n-k}(x), \mathcal{C}\big) > \sigma^{bk}. \tag{3.3}$$

One of the main features of hyperbolic times is stated below.

Lemma 3.2 [1, Lemma 2.7] *Given $c > 0$ and $\delta > 0$ there exists a constant $\delta_1 = \delta_1(c, \delta, f) > 0$ such that if n is a hyperbolic time for a point x then f^n maps diffeomorphically the dynamical ball $V_n(x) = B(x, n, \delta_1)$ onto the ball $B(f^n(x), \delta_1)$ around $f^n(x)$ and radius δ_1 and*

$$d\big(f^{n-j}(y), f^{n-j}(z)\big) \leq \sigma^{-\frac{j}{2}} d\big(f^n(y), f^n(z)\big)$$

for every $1 \leq j \leq n$ and every $y, z \in V_n(x)$.

Using that $J_\nu f = \lambda e^{-\phi}$ is Hölder continuous and the backward distances contraction at hyperbolic times we obtain a bounded distortion property.

Corollary 3.1 *There exists $K_0 > 0$ such that for every $y, z \in V_n(x)$*

$$K_0^{-1} \leq \frac{J_\nu f^n(y)}{J_\nu f^n(z)} \leq K_0.$$

3.3 Control of the Measure of Dynamical Balls

Now we prove a useful lemma on the measure of dynamical balls for weak Gibbs measures. In what follows δ_1 stands for the diameter of the hyperbolic ball as in Lemma 3.2.

Lemma 3.3 *For every $0 < \varepsilon < \delta_1$ there exists a positive constant $K(\varepsilon) > 0$ such that if n is a hyperbolic time for x and $B(y, n, \varepsilon) \subset B(x, n, \delta)$ then*

$$K(\varepsilon)^{-1} \leq \frac{\nu(B(y, n, \varepsilon))}{e^{-Pn + S_n\phi(y)}} \leq K(\varepsilon),$$

where $P = \log \lambda$.

Proof One has $f^n(B(y, n, \varepsilon)) = B(f^n(y), \varepsilon)$ by backward distance contraction at hyperbolic times. Hence, Corollary 3.1 asserts that

$$1 \geq \nu\big(B\big(f^n(y), \varepsilon\big)\big) = \int_{B(y,n,\varepsilon)} e^{-S_n\psi}\, d\nu \geq K_0^{-1} e^{Pn - S_n\phi(y)} \nu\big(B(y, n, \varepsilon)\big).$$

Using that ν is positive on open sets and the compactness of M it follows that the measure of every ball of radius ε is bounded away from zero. Thus the other inequality is obtained analogously. □

The following very interesting consequence is that dynamical balls have comparable measure that only depends on the center of the ball.

Corollary 3.2 *Assume that* $x \in H$*. For every* $0 < \varepsilon < \delta_1$ *and* $n \geq 1$ *there exists a positive constant* $K_n(x, \varepsilon) > 0$ *such that if* $B(y, n, \varepsilon) \subset B(x, n, \delta)$ *then*

$$K_n(x, \varepsilon)^{-1} \leq \frac{\nu(B(y, n, \varepsilon))}{e^{-Pn+S_n\phi(y)}} \leq K_n(x, \varepsilon).$$

Proof Given an arbitrary n write $n_i(x) \leq n < n_{i+1}(x)$, where n_i and n_{i+1} are consecutive hyperbolic times for x. Using that $B(y, n, \varepsilon) \subset B(y, n_i(x), \varepsilon)$ it is clear that

$$\begin{aligned}\nu\big(B(y, n, \varepsilon)\big) &\leq K(\varepsilon) e^{(\sup|\phi|+|P|)(n-n_i(x))} e^{-Pn+S_n\phi(y)} \\ &\leq K_n(x, \varepsilon) e^{-Pn+S_n\phi(y)},\end{aligned}$$

with $K_n(x, \varepsilon) = K(\varepsilon) \exp[(\sup|\phi| + |P|)(n - n_i(x))]$ (depends only on the center x). This finishes the proof of the corollary. □

Now we prove that the constants K_n have subexponential growth with respect to every invariant expanding measure such that the first hyperbolic time map is integrable. More precisely,

Lemma 3.4 *Let* η *be an* f*-invariant and expanding probability measure so that* $n_1 \in L^1(\eta)$ *and let* $K_n(x, \varepsilon)$ *be given as above. Then,*

$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log K_n(x, \varepsilon) = 0 \tag{3.4}$$

for η*-almost every* x*. In consequence,* $\psi = \phi - P$ *belongs to* $\mathcal{F}(H)$*.*

Proof This proof resembles the one of Proposition 3.8 in [27]. Let η be an f-invariant, expanding probability measure so that $n_1 \in L^1(\eta)$ and take $\beta > 0$ arbitrary. Given $x \in H$, $n \geq 1$ and $0 < \varepsilon < \delta_1$ recall that $K_n(x, \varepsilon) \leq K(\varepsilon) \exp[(|P| + \sup|\phi|) n_1(f^{n_i(x)}(x))]$, where $n_i(x) \leq n \leq n_{i+1}(x)$ are consecutive hyperbolic times for x. Set $C_\beta = \beta n/(|P| + \sup|\phi|) - \log K(\varepsilon)$. If $K_n(x, \varepsilon) > e^{\beta n}$ this implies that $n_1(f^k(x)) > C_\beta n \geq C_\beta k$, where $k = n_i(x)$. This shows that

$$\begin{aligned}\big\{x \in H : K_n(x, \varepsilon) > e^{\beta n} \text{ i.o.}\big\} &\subset \big\{x \in H : n_1\big(f^n(x), \varepsilon\big) > e^{\beta n} \text{ i.o.}\big\} \\ &\subset \bigcup_{n \geq 1} \big\{x \in H : n_1\big(f^n(x)\big) > C_\beta n\big\}.\end{aligned}$$

Furthermore, using the invariance of η and the integrability assumption

$$\sum_{n=1}^{+\infty} \eta\big(x \in H : n_1\big(f^n(x)\big) > C_\beta n\big) = \sum_{n=1}^{+\infty} \eta\big(x \in H : n_1(x) > C_\beta n\big) \leq \int n_1 \, d\eta < \infty.$$

Using Borel-Cantelli lemma this proves that $K_n(x, \varepsilon) \leq e^{\beta n}$ for all but finitely many values of n for η-almost every x. Since β was taken arbitrary, this completes the proof of the first claim above.

Using that $n_1 \in L^1(\mu)$ and $d\mu/d\nu$ is bounded from below by a constant (recall assumption (P3)) it follows that (3.4) holds ν-almost everywhere. Together with Corollary 3.2 this shows that $\psi = \phi - P$ belongs to $\mathcal{F}(H)$ and that ν is a weak Gibbs measure. This finishes the proof of the lemma. □

Remark 3.1 It follows from (3.4) and the definition of the constants $K_n(x,\varepsilon)$ that if $n_1 \in L^1(\eta)$ then given $\beta > 0$, for η-almost every x there exists $n_x \geq 1$ such that $n - n_i(x) \leq \beta n$ for every $n \geq n_x$. In fact we prove even more: given $\beta > 0$ then for η-almost every x there exists $n_x \geq 1$ such that $n_{i+1}(x) - n_i(x) \leq \beta n$ for every $n \geq n_x$.

4 Abstract Deviation Bounds

In this section we prove Theorem 2.1. Upper and lower bounds for the measure of the deviation sets are given separately.

4.1 Upper Bound

Let $g \in C(M,\mathbb{R})$, $c \in \mathbb{R}$ and $\psi \in \mathcal{F}(\Lambda)$ be fixed. We want to prove that for every small $\varepsilon, \beta > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\left[x \in M : \frac{1}{n} S_n g(x) \geq c\right] \leq \max\left\{\delta(\varepsilon,\beta), \sup\left\{h_\eta(f) - \int \psi \, d\eta\right\} + \beta\right\}$$

where the supremum is taken over all invariant probability measures η such that $\int g\, d\eta \geq c$. We use the following result from Calculus (see e.g. [40, Lemma 9.9]).

Lemma 4.1 *Given $n \geq 1$, real numbers $(a_i)_{i=1..n}$ and $0 \leq p_i \leq 1$ such that $\sum_{i=1}^n p_i = 1$ then*

$$\sum_{i=1}^{n} p_i(a_i - \log p_i) \leq \log\left(\sum_{i=1}^{n} e^{a_i}\right),$$

and the equality holds if and only if $p_i = \frac{e^{a_i}}{\sum_j e^{a_j}}$.

Let B_n denote the set of points $x \in M$ so that $S_n g(x) \geq cn$. Recall that Λ is a ν-full measure set and, for every $x \in \Lambda$ and every small $\varepsilon > 0$ it holds that

$$\nu\big(B(x,n,\varepsilon)\big) \leq K_n(x,\varepsilon) e^{-S_n\psi(x)},$$

with $\limsup_n \frac{1}{n} \log K_n(x,\varepsilon) = 0$. Let $\beta > 0$ and $0 < \varepsilon < \delta_0$ be arbitrary small and $n \geq 1$ be fixed. Then $B_n \subset \Delta_n^c \cup (B_n \cap \Delta_n)$, where Δ_n is as in (2.2). Moreover, if $E_n \subset B_n \cap \Delta_n$ is a maximal (n,ε)-separated set, $B_n \cap \Delta_n$ is contained in the union of the dynamical balls $B(x,n,2\varepsilon)$ centered at points of E_n and, consequently,

$$\nu(B_n) < \nu\big(\Delta_n^c\big) + e^{\beta n} \sum_{x\in E_n} e^{-S_n\psi(x)} \tag{4.1}$$

for every n. Now, consider the probability measures σ_n and η_n given by

$$\sigma_n = \frac{1}{Z_n} \sum_{x\in E_n} e^{-S_n\psi(x)} \delta_x \quad \text{and} \quad \eta_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \sigma_n,$$

where $Z_n = \sum_{x\in E_n} e^{-S_n\psi(x)}$, and let η be a weak* accumulation point of the sequence $(\eta_n)_n$. It is not hard to check that η is an f-invariant probability measure. Assume $\mathcal{P}$ is a partition of M with diameter smaller than ε and $\eta(\partial\mathcal{P}) = 0$. Each element of $\mathcal{P}^{(n)}$ contains at most one point of E_n. By the previous lemma

$$H_{\sigma_n}\big(\mathcal{P}^{(n)}\big) - \int S_n\psi\, d\sigma_n = \log\left(\sum_{x\in E_n} e^{-S_n\psi(x)}\right)$$

which, as in the usual proof of the variational principle (see [40, Pages 219–221]), guarantees that

$$\limsup_{n\to\infty} \frac{1}{n}\log Z_n \le h_\eta(f) - \int \psi\, d\eta. \tag{4.2}$$

Observe also that $\int \psi\, d\eta \ge c$ by weak* convergence since E_n is contained in B_n and

$$\int g\, d\eta_n = \frac{1}{n}\sum_{j=0}^{n-1} \frac{1}{Z_n}\sum_{x\in E_n} e^{S_n\phi(x)}.g\circ f^j(x) \ge c.$$

Finally, it follows from (4.1) and (4.2) that for every $\beta > 0$

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\nu(B_n) &\le \max\left\{\delta(\varepsilon,\beta), h_\eta(f) - \int\psi\, d\eta + \beta\right\}\\ &\le \max\left\{\delta(\varepsilon,\beta), \sup\left\{h_\xi(f) - \int \psi\, d\xi\right\} + \beta\right\},\end{aligned}$$

where the supremum is over all invariant probability measures. This completes the proof of the first statement in Theorem 2.1.

4.2 Lower Bound Using Ergodic Measures

Let $g : M \to \mathbb{R}$ be a continuous map, take $c \in \mathbb{R}$ and a $\beta > 0$ small. If η is an ergodic probability measure such that $\int g\, d\eta > c$ we claim that

$$\liminf_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M : \frac{1}{n}S_n g(x) > c\right) \ge h_\eta(f) - h_\nu(f,\eta) - 2\beta.$$

Denote by B_n the set of points $x \in M$ such that $S_n g(x) > cn$ and fix $\delta_2 = \frac{1}{2}(\int g d\eta - c)$. Notice that $h_\eta(f) \le h_{\text{top}}(f) < \infty$ and that we may assume $h_\nu(f,\eta) < \infty$ (because otherwise there is nothing to prove). Hence η-almost every point x satisfies

$$h_\nu(f,x) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty} -\frac{1}{n}\log\nu\big(B(x,n,\varepsilon)\big) \le h_\nu(f,\eta) < \infty. \tag{4.3}$$

Since η is ergodic then $\frac{1}{n}S_n g(x) \to \int g\, d\eta$ for η-almost every x. Choose $\xi > 0$ by uniform continuity so that $|g(x) - g(y)| < \delta_2$ whenever $d(x,y) < \xi$. Observe that if $n_0 = n_0(\beta) \ge 1$ is large and $\delta \in (0,\xi)$ is small enough then the set D of points $x \in M$ satisfying

$$\frac{1}{n}S_n g(x) > c + \delta_2 \quad \text{and} \quad \nu\big(B(x,n,\varepsilon)\big) \ge e^{-[h_\nu(f,\eta)+\beta]n} \tag{4.4}$$

has η-measure at least $\frac{1}{2}$, and that the minimal number $N(n, 2\varepsilon, \frac{1}{2})$ of $(n, 2\varepsilon)$-dynamical balls necessary to cover a set of η-measure at least $\frac{1}{2}$ satisfies

$$N\left(n, 2\varepsilon, \frac{1}{2}\right) \ge e^{[h_\eta(f)-\beta]n} \tag{4.5}$$

for every $n \ge n_0$ and every $0 < \varepsilon \le \delta$. Indeed, the existence of such n_0 and δ is a consequence of Proposition 3.2, the definition of relative entropy and ergodicity. Moreover, it follows from our choice of ξ and δ_2 that

$$B_n \supset \bigcup_{x\in D} B(x,n,\varepsilon) \supset D$$

for all $n \geq n_0$ and $0 < \varepsilon < \xi$. So, if $\varepsilon > 0$ is small and $E_n \subset D$ is a maximal (n,ε)-separated set, using that the dynamical balls $B(x,n,\varepsilon)$ centered at points in E_n are pairwise disjoint contained in B_n and the union $\cup_{x\in E_n} B(x,n,2\varepsilon)$ covers D, relations (4.4) and (4.5) yield that

$$\nu(B_n) \geq \nu\Big(\bigcup_{x\in E_n} B(x,n,\varepsilon)\Big) \geq \sum_{x\in E_n} \nu\big(B(x,n,\varepsilon)\big) \geq e^{[h_\eta(f)-h_\nu(f,\eta)-2\beta]n}$$

whenever $n \geq n_0$, which proves our claim. The second assertion in Theorem 2.1 follows from the arbitrariness of β.

Remark 4.1 Since $B_n \supset B_n \cap \Delta_n$, where Δ_n is as in (2.2) then $\nu(B_n) \geq \nu(B_n \cap \Delta_n)$. However, we have no estimate whatsoever for the measure of the intersection in terms of $\nu(\Delta_n)$. Hence the previous result shows only that the measure of the points with predetermined Birkhoff averages decreases at most exponentially fast.

4.3 Lower Bound over All Invariant Measures

The proof of the last statement in Theorem 2.1 is divided in two steps. First we prove the lower bound when the supremum is restricted over ergodic measures. Afterwards we deduce the general bound using that every invariant measure can be approximated by a finite collection of ergodic measures and the specification to "glue" together finite pieces of orbits. We begin by the following lemma.

Lemma 4.2 *If $g \in C(M,\mathbb{R})$, $c \in \mathbb{R}$ and $\psi \in \mathcal{F}(\Lambda)$ then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\Big(x \in M : \frac{1}{n} S_n g(x) > c\Big) \geq h_\eta(f) - \int \psi \, d\eta$$

for every ergodic probability measure η such that $\eta(\Lambda) = 1$ and $\int g\, d\eta > c$.

Proof Fix $g \in C(M,\mathbb{R})$, $c \in \mathbb{R}$ and $\psi \in \mathcal{F}(\Lambda)$, and denote by B_n the set of points $x \in M$ such that $S_n g(x) > cn$. Let $\beta > 0$ be a small constant and $\delta_2 = \frac{1}{2}(\int g\, d\eta - c)$. Let $\xi > 0$ be given by uniform continuity such that $|g(x) - g(y)| < \delta_2$ for any points $x, y \in M$ at distance smaller than ξ. As before, if n_0 is large enough and $0 < \varepsilon < \xi$ is small then the set D of points $x \in \Lambda$ satisfying

$$\frac{1}{n} S_n g(x) > c + \delta_2, \qquad K_n(x,\varepsilon)^{-1} \geq e^{-\beta n} \quad \text{and} \quad \frac{1}{n} S_n \psi(x) < \int \psi \, d\eta + \beta \tag{4.6}$$

for every $n \geq n_0$ has η-measure at least $\frac{1}{2}$ and $N(n, 2\varepsilon, \frac{1}{2}) \geq e^{(h_\eta(f)-\beta)n}$ for every $n \geq n_0$. Then, using that $B_n \supset \bigcup_{x\in D} B(x,n,\varepsilon) \supset D$ it follows that

$$\nu(B_n) \geq \sum_{x\in E_n} \nu\big(B(x,n,\varepsilon)\big) \geq e^{[h_\eta(f) - \int \psi\, d\eta - 3\beta]n}$$

for every maximal (n,ε)-separated set $E_n \subset D$. This proves that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\Big(x \in M : \frac{1}{n} S_n g(x) > c\Big) \geq h_\eta(f) - \int \psi\, d\eta - 3\beta.$$

Since β was taken arbitrary the statement in the lemma follows directly. □

The following result asserts that any invariant probability measure can be approximated by a finite convex combination of ergodic measures supported in Λ.

Lemma 4.3 *Let $\eta = \int \eta_x d\eta(x)$ be the ergodic decomposition an f-invariant probability measure η such that $\eta(\Lambda) = 1$. Given $\beta > 0$ and a finite set $(\psi_j)_{1\le j\le r} \subset C(M)$ of continuous functions, there are positive real numbers $(a_i)_{1\le i\le k}$ satisfying $a_i \le 1$ and $\sum a_i = 1$, and finitely many points $x_1, \dots, x_k$ such that the ergodic measures $\eta_i = \eta_{x_i}$ from the ergodic decomposition satisfy*

(i) $\eta_i(\Lambda) = 1$;
(ii) $h_{\hat\eta}(f) \ge h_\eta(f) - \beta$; *and*
(iii) $|\int \psi_j\, d\hat\eta - \int \psi_j\, d\eta| < \beta$ *for every* $1 \le j \le r$;

where $\hat\eta = \sum_{i=1}^k a_i \eta_i$.

Proof Fix the f-invariant probability measure η such that $\eta(\Lambda) = 1$. By ergodic decomposition theorem and convexity of the entropy, we can write $\eta = \int \eta_x\, d\eta(x)$ and $h_\eta(f) = \int h_{\eta_x}(f)\, d\eta(x)$, where each η_x denotes an ergodic component of η. Clearly $\eta_x(\Lambda) = 1$ for η-almost every x. Let $\mathcal{P}$ be a small finite partition of the space $\mathcal{M}(\Lambda)$ of invariant probability measures supported in Λ such that

$$\left| \int \psi_j\, d\xi_1 - \int \psi_j\, d\xi_2 \right| < \beta \tag{4.7}$$

for every $1 \le j \le e$ and every pair of probability measures ξ_1, ξ_2 in the same partition element. Set $k = \#\mathcal{P}$ and $a_i = \eta(P_i)$ for every element P_i in $\mathcal{P}$. For every $1 \le i \le k$ pick an ergodic measure $\eta_i = \eta_{x_i} \in P_i$ satisfying $h_{\eta_x}(f) \le h_{\eta_i}(f) + \beta$ for η-almost every $\eta_x \in P_i$. Part (i) in the lemma is immediate. On the other hand, (ii) follows because

$$h_\eta(f) = \int h_{\eta_x}(f)\, d\eta(x) \le \sum_{i=1}^k a_i h_{\eta_i}(f) + \beta = h_{\hat\eta}(f) + \beta.$$

Finally, (4.7) implies that

$$\left| \int \psi_j\, d\eta - \int \psi_j\, d\hat\eta \right| = \left| \int \left(\int \psi_j\, d\eta_x \right) d\eta(x) - \sum_{i=1}^k a_i \int \psi_j\, d\eta_i \right| \le \sum_{i=1}^k a_i \beta = \beta$$

for every j. This proves (iii) and finishes the proof of the lemma. □

Now we will finish the proof of Theorem 2.1.

Proof of Theorem 2.1 (continuation) Take $g \in C(M, \mathbb{R})$, $c \in \mathbb{R}$, $\psi \in \mathcal{F}(\Lambda)$ and let η be an invariant probability measure such that $\eta(\Lambda) = 1$ and $\int g\, d\eta > c$. Denote by B_n the set of points $x \in M$ such that $S_n g(x) > cn$. Take $\beta > 0$ arbitrary small, $\delta_2 = \frac{1}{5}(\int g\, d\eta - c)$ and the measure $\hat\eta = \sum_{i=1}^k a_i \eta_i$ given by Lemma 4.3 that satisfies

$$h_{\hat\eta}(f) \ge h_\eta(f) - \beta, \qquad \int g\, d\hat\eta \ge \int g\, d\eta - \beta \quad \text{and} \quad \int \psi\, d\hat\eta \le \int \psi\, d\eta + \beta.$$

Since β is small we can assume $\int g\, d\hat\eta > c + 4\delta_2$. Now we claim that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu \left(x \in M : \frac{1}{n} S_n g(x) > c \right) \ge h_{\hat\eta}(f) - \int \psi\, d\hat\eta - 4\beta. \tag{$\star\star$}$$

As before, we may choose n_0 sufficiently large and δ small enough so that, for every $1 \le i \le k$, the set D_i of points $x \in \Lambda$ such that

$$\frac{1}{n} S_n g(x) > \int g \, d\eta_i - \beta, \qquad \frac{1}{n} S_n \psi(x) < \int \psi \, d\eta_i + \beta \quad \text{and} \quad K_n(x,\varepsilon)^{-1} \ge e^{-\beta n}$$

for every $n \ge n_0$ and $0 < \varepsilon \le \delta$ has η_i-measure at least $\frac{1}{2}$. Hence, given large n, small $\varepsilon > 0$ and $1 \le i \le k$ we proceed as in the proof of Lemma 4.2 to obtain a finite set $E_n^i \subset D_i$ so that

(1) E_n^i is a maximal $([a_i n], \varepsilon)$-separated set in D_i;
(2) $\#E_n^i \ge e^{(h_{\eta_i}(f)-\beta)[a_i n]}$; and
(3) for every $x \in E_n^i$ it holds

$$\frac{1}{[a_i n]} S_{[a_i n]} g(x) > \int g \, d\eta_i - \beta \quad \text{and} \quad \frac{1}{[a_i n]} S_{[a_i n]} \psi(x) < \int \psi \, d\eta_i + \beta.$$

By the specification property, for every sequence $(x_1, x_2, \dots, x_k)$ with $x_i \in E_n^i$ there exists $x \in M$ that ε-shadows each x_i during $[a_i n]$ iterates with a time lag of $N(\varepsilon)$ iterates in between. Consequently, if n is large and $\tilde{n} = \sum_i [a_i n] + kN(\varepsilon)$ then $S_{\tilde{n}} g(x) > (c + 2\delta_2)\tilde{n}$. Since the dynamical ball $B(x, \tilde{n}, \varepsilon/8)$ is contained in $B_{\tilde{n}} \cap B(x_1, \tilde{n}, \delta_0)$ for every large n, it follows from (2.1) that

$$\nu\big(B(x,\tilde{n},\varepsilon)\big) \ge K_{\tilde{n}}(x_1,\varepsilon)^{-1} e^{-S_{\tilde{n}}\psi(x)} \ge e^{-\beta\tilde{n}} e^{-(\int \psi \, d\eta_i + 2\beta)\tilde{n}}.$$

On the other hand, there are at least $\#E_n^1 \times \cdots \times \#E_n^k$ such pairwise disjoint dynamical balls contained in $B_{\tilde{n}}$. It follows that

$$\nu(B_{\tilde{n}}) \ge \sum_x \nu\big(B(x,\tilde{n},\varepsilon)\big) \ge e^{[h_{\hat{\eta}}(f) - \int \psi d\hat{\eta} - 4\beta]\tilde{n}}$$

for every large n, which gives $(\star\star)$. Since β was chosen arbitrarily small then

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c \right) \ge h_\eta(f) - \int \psi \, d\eta - 6\beta,$$

which proves the third part in Theorem 2.1 and finishes its proof. □

5 Deviation Estimates for Non-uniformly Expanding Maps

In this section we use some of the ideas involved in the proof of Theorem 2.1 together with the key notion of non-uniform specification to prove the large deviation bounds in Theorem 2.2. Through the section, let M be a compact manifold and $f : M \to M$ be a $C^{1+\alpha}$ local diffeomorphism outside a critical/singular region $\mathcal{C}$ that satisfies (H). Let $\phi : M \setminus \mathcal{C} \to \mathbb{R}$ be a Hölder continuous potential such that (P1)–(P3) hold. Denote by ν the corresponding weak Gibbs measure and by μ the unique equilibrium state for f with respect to ϕ.

5.1 Upper Bound

In this subsection we obtain an upper bound for the measure of the deviation set of non-uniformly expanding maps.

Lemma 5.1 *If $g \in C(M,\mathbb{R})$ and $c \in \mathbb{R}$ then it holds that*

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) \geq c \right) \\ \leq \max\left\{ \sup\left\{ -P + h_\eta(f) + \int \phi \, d\eta \right\}, \limsup_{n\to\infty} \frac{1}{n} \log \mu(\Gamma_n) \right\},$$

where the supremum is taken over all f-invariant measures η such that $\int g \, d\eta \geq c$.

Proof Let $\beta > 0$ be given. First we observe that the computations in Lemma 3.4 show that $\psi = \phi - P \in \mathcal{F}(H)$ and that there exists $C_\beta > 0$ such that the set Δ_n as in (2.2) is given by

$$\Delta_n \supset \left\{ x \in H : n_1\left(f^{n_i(x)}(x) \right) \leq C_\beta n \right\}$$

where $n_i(x) \leq n \leq n_{i+1}(x)$ are consecutive hyperbolic times for x. In particular, using that $d\mu/d\nu \geq K^{-1}$, computations analogous to the ones in the proof of Lemma 3.4 also give that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \nu\left(\Delta_n^c\right) &\leq \limsup_{n\to\infty} \frac{1}{n} \log \nu \left[\bigcup_{1\leq k\leq n} \left\{ x \in H : n_i(x) = k, n_1\left(f^k(x)\right) > C_\beta n \right\} \right] \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log\left(K n \mu\left(x \in H : n_1(x) > C_\beta n \right)\right) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log\left(K C_\beta^{-1} n \mu\left(x \in H : n_1(x) > n \right)\right) \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \mu(\Gamma_n). \end{aligned}$$

Hence $\delta(\varepsilon, \beta)$ does not depend neither on ε or β. Thus, the lemma is an immediate consequence of the first part in Theorem 2.1. □

5.2 Lower Bound Estimates

To obtain lower bound estimates one technical difficulty to overcome is that no a priori estimates for the measure of dynamical balls hold for specified orbits even if the dynamical system satisfies the specification property. Here we make use of approximation Lemma 4.3 and specification properties to prove the following lower bound for the measure of deviation sets.

Proposition 5.1 *Assume that $g \in C(M,\mathbb{R})$ and $c \in \mathbb{R}$. If either f satisfies the specification property or (f,μ) satisfies the non-uniform specification property then*

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c \right) \geq -P + h_\eta(f) + \int \phi \, d\eta,$$

for every invariant and expanding *probability measure η satisfying $\int g \, d\eta > c$ and $n_1 \in L^1(\eta)$.*

Proof Note that $\psi = \phi - P \in \mathcal{F}(H)$ by Lemma 3.4. Set $g \in C(M,\mathbb{R})$ and $c \in \mathbb{R}$, and let B_n be the set of points $x \in H$ such that $S_n g(x) > cn$. Fix $\beta > 0$ arbitrary small and let η be an f-invariant and expanding probability measure such that $\int g \, d\eta > c$ and $n_1 \in L^1(\eta)$. Set also $\delta_2 = \frac{1}{5}(\int g \, d\eta - c)$. Observe that almost every ergodic component η_x of the invariant measure η satisfy $n_1 \in L^1(\eta_x)$. It follows from Lemma 4.3 that there are exists a probability

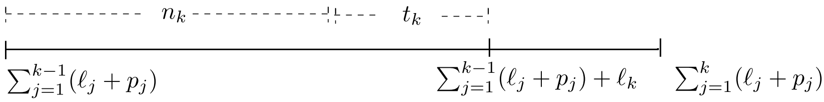

Fig. 1 Combinatorics of the specified orbit

vector $(a_1,\dots,a_k)$ and f-invariant ergodic probability measures $(\eta_i)_{1\le j\le k}$ such that $\hat\eta = \sum a_j\eta_j$ satisfies

$$h_{\hat\eta}(f)\ge h_\eta(f)-\beta, \qquad \int g\,d\hat\eta \ge \int g\,d\eta-\beta \quad\text{and}\quad \int \psi\,d\hat\eta \le \int \psi\,d\eta+\beta.$$

Moreover, it is not hard to check that we can assume $n_1\in L^1(\eta_j)$ for every $1\le j\le k$. So, the Ergodic Theorem and Remark 3.1 guarantee that one can pick $n_0\ge 1$ large and δ small enough such that, for every $1\le j\le k$, the set D_j of points $x\in H$ such that

$$n_{i+1}(x)-n_i(x)\le \beta n, \qquad \frac{1}{n}S_n g(x) > \int g\,d\eta_j-\beta \quad\text{and}\quad \frac{1}{n}S_n\psi(x) < \int \psi\,d\eta_j+\beta,$$

for every $n\ge n_0$, has η_j-measure larger than $\frac12$. Recall that $n_i(x)\le n<n_{i+1}(x)$ are consecutive hyperbolic times for x. If $0<\varepsilon\ll\delta_1$ (as in Lemma 3.2) is small then $|g(x)-g(y)|<\delta_2$ whenever $|x-y|<\varepsilon$. As in Sect. 4.2, for every large n and small $\varepsilon>0$ there exists a set $E_n^j\subset D_j$ such that

(1) E_n^j is a maximal $([a_jn],\varepsilon)$-separated set;
(2) $\#E_n^j\ge e^{(h_{\eta_j}(f)-\beta)[a_jn]}$;
(3) for every $x\in E_n^j$ it holds

$$\frac{1}{[a_jn]}S_{[a_jn]}g(x) > \int g\,d\eta_j-\beta \quad\text{and}\quad \frac{1}{[a_jn]}S_{[a_jn]}\psi(x) < \int \psi\,d\eta_j+\beta.$$

Condition (1) yield that the dynamical balls $B(x,[a_jn],\varepsilon)$ centered at points in E_n^j are pairwise disjoint. We divide the remaining of the proof in two cases:

First case f satisfies the specification property

Given any sequence $(z_1,z_2,\dots,z_k)$ with $z_j\in E_n^j$ there exists some point $z\in M$ that ε-shadows each z_j during $\ell_j:=[a_jn]$ iterates with a time lag of $p_j=N(\varepsilon)$ iterates as in Definition 2.1. Let $n_j:=n_i(z_j)$ denote the last hyperbolic time for z_j smaller than ℓ_j and write $\ell_j=n_j+t_j$ for some $t_j\ge 0$. See Fig. 1 above.

Therefore, if we set $p_k=0$ and take $\tilde n=\sum_{j=1}^k(\ell_j+p_j)$ one can use that $\max\{k,t_k\}\le \beta\ell_k\ll\beta(\tilde n-t_k)$ to deduce that

$$\begin{aligned}
S_{\tilde n}g(z) &\ge \sum_{j=1}^k S_{\ell_j}g(z_j)-\delta_2\sum_{j=1}^k\ell_j-\sup|g|kN(\varepsilon)\\
&\ge \sum_{j=1}^k\Big(\int g\,d\eta_j-\delta_2-\beta\Big)\ell_j-\sup|g|kN(\varepsilon)\\
&> (c+3\delta_2)n-\sup|g|kN(\varepsilon)\\
&\ge (c+3\delta_2)\tilde n-2\sup|g|kN(\varepsilon)\\
&\ge (c+2\delta_2)\tilde n
\end{aligned}$$

provided that n is large enough. Hence $B(z,\tilde{n},\varepsilon)\subset B_{\tilde{n}}$ and there are at least $e^{(h_{\hat{\eta}}(f)-2\beta)\tilde{n}}$ such distinct dynamical balls. Since each of the points x_i were chosen in a full measure set then the weak Gibbs property yields an estimate for the measure of the corresponding dynamical balls. However, no a priori estimates on the measure of the specified orbit z is guaranteed. We claim that

$$\begin{aligned}\nu\big(B(z,\tilde{n},\varepsilon)\big)&\geq e^{-2k\sup|\psi|\beta\tilde{n}}\left(\prod_{j=1}^{k}e^{-S_{\ell_i}\psi(z_j)}\right)\\&\geq e^{-2k\sup|\psi|\beta\tilde{n}}e^{-2\beta\tilde{n}}e^{-\tilde{n}\int\psi\,d\eta}\end{aligned}\tag{5.1}$$

for every large n.

Proof of the claim Let L and γ be given by condition (C). For notational simplicity set $\tilde{z}_{k+1}=f^{\sum_{j=1}^{k}(\ell_j+p_j)}(z)\in B(z_{k+1},\ell_{k+1},\varepsilon)$. Since $B(z,\tilde{n},\varepsilon)$ contains $B(z,\tilde{n}+n_{k+1},\varepsilon)$, where ℓ_k+n_{k+1} denotes the first hyperbolic time for z_k larger than ℓ_k, first we show that

$$f^{\tilde{n}+n_{k+1}}\big(B(z,\tilde{n}+n_{k+1},\varepsilon)\big)=B\big(f^{\tilde{n}+n_{k+1}}(z),\varepsilon\big).\tag{5.2}$$

Since ℓ_k+n_{k+1} is a hyperbolic time for z_k and $\varepsilon\ll\delta_1$ then there exists backward distance contraction and $f^{\ell_k+n_{k+1}}(B(\tilde{z}_k,\ell_k+n_{k+1},2\varepsilon))=B(f^{\ell_k+n_{k+1}}(\tilde{z}_k),2\varepsilon)$. In fact, using $d(f^{n_k}(\tilde{z}_k),f^{n_k}(z_k))<\varepsilon$, that β is fixed arbitrary small and $\ell_k-n_k<\beta n$ (recall the definition of the set D_k) then

$$\operatorname{diam}\big(B(\tilde{z}_k,\ell_k+n_{k+1},\varepsilon)\big)\leq\operatorname{diam}\big(B(\tilde{z}_k,n_k,\varepsilon)\big)\leq\varepsilon\sigma^{-\frac{1}{2}n_k}\leq\varepsilon\sigma^{-\frac{1}{2}[a_k-\beta]n}\ll\varepsilon$$

provided that n is large enough. Again, if n is large, using property (C) on the diameter of preimages this yields that there exists $\tilde{L}>0$ so that

$$\begin{aligned}\operatorname{diam}\Big(f^{-p_{k-1}-t_{k-1}}\big(B(\tilde{z}_k,\ell_k+n_{k+1},\varepsilon)\big)\Big)&\leq\tilde{L}\big[\varepsilon\sigma^{-\frac{1}{2}[a_k-\beta]n}\big]^{\gamma^{p_{k-1}+t_{k-1}}}\\&\leq\tilde{L}\big[\varepsilon\sigma^{-\frac{1}{2}[a_k-\beta]n}\big]^{\gamma^{N(\varepsilon)+\beta n}}\ll\varepsilon\end{aligned}$$

and, consequently, $f^{-p_{k-1}-t_{k-1}}(B(\tilde{z}_k,\ell_k+n_{k+1},\varepsilon))\subset B(f^{n_{k-1}}(\tilde{z}_{k-1}),2\varepsilon)$. Recall that $\tilde{z}_k=f^{\ell_{k-1}+p_{k-1}}(\tilde{z}_{k-1})$, that n_{k-1} is a hyperbolic time for z_{k-1} and there exists backward distance contraction in the dynamical ball of radius $\delta_1\gg2\varepsilon$. Hence $f^{n_{k-1}}(B(\tilde{z}_{k-1},n_{k-1},2\varepsilon))=B(f^{n_{k-1}}(\tilde{z}_{k-1}),2\varepsilon)$ and

$$\begin{aligned}&B(\tilde{z}_{k-1},\ell_{k-1}+p_{k-1}+\ell_k+n_{k+1},\varepsilon)\\&\quad=B(\tilde{z}_{k-1},n_{k-1},\varepsilon)\cap f^{-\ell_{k-1}-p_{k-1}}\big(B(\tilde{z}_k,\ell_k+n_{k+1},\varepsilon)\big)\\&\quad=B(\tilde{z}_{k-1},n_{k-1},\varepsilon)\cap f^{-n_{k-1}}\Big[f^{-p_{k-1}-t_{k-1}}\big(B(\tilde{z}_k,\ell_k+n_{k+1},\varepsilon)\big)\Big]\\&\quad\subset B(\tilde{z}_{k-1},n_{k-1},\varepsilon)\cap f^{-n_{k-1}}\Big[B\big(f^{n_{k-1}}(\tilde{z}_{k-1}),2\varepsilon\big)\Big].\end{aligned}$$

Consequently, the dynamical ball $B(\tilde{z}_{k-1},\ell_{k-1}+p_{k-1}+\ell_k+n_{k+1},\varepsilon)$ is mapped diffeomorphically by $f^{\ell_{k-1}+p_{k-1}+\ell_k+n_{k+1}}$ onto the ball centered at $f^{\ell_k+n_{k+1}}(\tilde{z}_k)$ with radius ε. Using the same argument as above recursively we obtain (5.2) as desired.

It remains to compute the measure of $B(z,\tilde{n}+n_{k+1},\varepsilon)$. Using (5.2), the fact that ν is a conformal measure with Jacobian $J_\nu f=e^{-\psi}$ and the bounded distortion property at hyperbolic times (see Corollary 3.1) it follows that

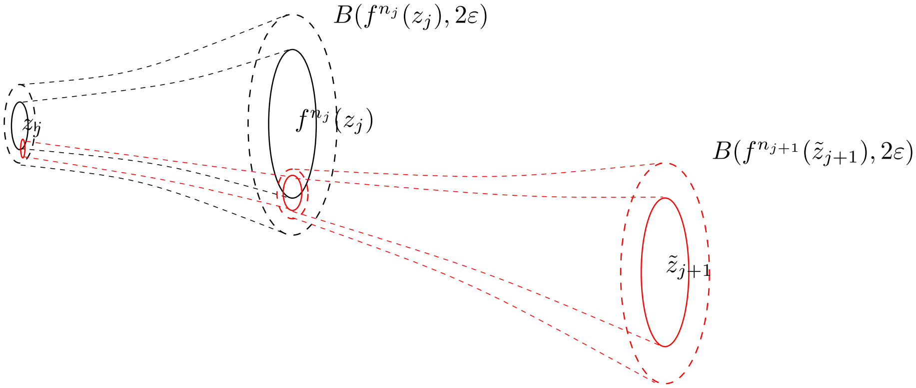

Fig. 2 (Color online) Concatenation of dynamical balls

$$\nu\big(B\big(f^{\tilde n+n_{k+1}}(z),\varepsilon\big)\big)=\int_{B(z,\tilde n+n_{k+1},\varepsilon)} e^{S_{\tilde n+n_{k+1}}\psi(y)}\,d\nu(y)$$
$$\leq K_0^k e^{\sup|\psi|[n_{k+1}+\sum_j(t_j+p_j)]}\left(\prod_{j=1}^k e^{S_{\ell_i}\psi(z_j)}\right)\nu\big(B(z,\tilde n+n_{k+1},\varepsilon)\big).$$

Since ν is an open measure then every ball of radius ε has measure at least $C_\varepsilon>0$. Finally, using that $n_k\leq\ell_k\leq(\ell_k+n_{k+1})$ are consecutive hyperbolic times for z_k then $n_{k+1}\leq(\ell_k+n_{k+1})-n_k\leq\beta\ell_k=\beta[a_kn]\ll\beta\tilde n$ and so

$$\nu\big(B(z,\tilde n,\varepsilon)\big)\geq\nu\big(B(z,\tilde n+n_{k+1},\varepsilon)\big)$$
$$\geq C_\varepsilon K_0^{-k}\left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)}\right)e^{-\sup|\psi|[kN(\varepsilon)+(k+1)\beta\tilde n]}$$
$$\geq e^{-2k\sup|\psi|\beta\tilde n}\left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)}\right)$$
$$\geq e^{-2k\sup|\psi|\beta\tilde n}\exp\left(\sum_{j=1}^k\left(-\int\psi\,d\eta_j-\beta\right)\ell_j\right)$$
$$\geq e^{-2k\sup|\psi|\beta\tilde n}e^{-2\beta\tilde n}e^{-\tilde n\int\psi\,d\eta}$$

for every large n, which proves our claim. □

We are now in a position to finish the proof of the first case of the proposition. Indeed, note that we obtain as a direct consequence of (5.1) that $\log\nu(B_{\tilde n})\geq(h_\eta(f)-\int\psi d\eta-5\beta-2\beta k\sup|\psi|)\tilde n$ for every large n. Since β was arbitrary this shows that

$$\liminf_{n\to\infty}\frac{1}{n}\log\nu\left(x\in M:\frac{1}{n}S_ng(x)>c\right)\geq h_\eta(f)-\int\psi\,d\eta.$$

Second case (f,μ) satisfies the non-uniform specification property

In the case that (f,μ) satisfies the non-uniform specification property the computations are similar to the previous ones with the difference that the time lags given by non-uniform specification may be unbounded. Take n_0 large and δ small so that

$$p(x,n,\varepsilon) \le \beta n$$

for every $x \in D_i$, $0 < \varepsilon \le \delta$, $1 \le i \le k$ and $n \ge n_0$. Through the remaining of the proof set also $p_j := \max_{x\in E_n^j} p(x,n,\varepsilon)$.

For every sequence $(z_1, z_2, \dots, z_k)$ with $z_j \in E_n^j$ there exists some point $z \in M$ that ε-shadows, in the non-uniform metric, each z_i during $\ell_j := [a_j n]$ iterates with a time lag of p_j iterates. Moreover, if $\tilde n = \sum_{j=1}^k (\ell_j + p_j)$, the set of points z obtained as above are $(\tilde n, \varepsilon)$ separated and there are at least $e^{(h_{\hat\eta}(f)-2\beta)\tilde n}$ such points. Since $\beta > 0$ is small, observe that

$$S_{\tilde n} g(z) \ge \sum_{j=1}^k S_{\ell_j} g(z_j) - \delta_2 \sum_{j=1}^k \ell_j - \sup|g| \sum_{j=1}^k p_j, \tag{5.3}$$

which is bounded from below by

$$\sum_{j=1}^k \left(\int g\, d\eta_j - 2\beta - \delta_2 \right) \ell_j > \left(\int g\, d\eta - 2\delta_2 \right) n$$
$$> (c + 2\delta_2) n + (c+\delta_2) \frac{\beta n}{\sup|g|} > (c+\delta_2)\tilde n$$

for every large n. It follows from our choice of ε that $B(z,\tilde n,\varepsilon) \subset B_{\tilde n}$. Given z as above, the computations involved in the proof of (5.1) give that

$$\nu\big(B(z, \tilde n + n_{k+1}, \varepsilon)\big) \ge C(\varepsilon)^{-1} K_0^{-k} e^{-\sup|\psi|[n_{k+1}+\sum_j (t_j+p_j)]} \left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)} \right)$$
$$\ge e^{-3(k+1)\sup|\psi|\beta\tilde n} \left(\prod_{j=1}^k e^{-S_{\ell_i}\psi(z_j)} \right)$$

and, consequently, $\nu(B(z,\tilde n,\varepsilon)) \ge e^{-3(k+1)\sup|\psi|\beta\tilde n} e^{-2\beta\tilde n} e^{-\tilde n \int \psi\, d\eta}$ for every large integer n. Henceforth, $\log \nu(B_{\tilde n}) \ge (h_\eta(f) - \int \psi\, d\eta - 5\beta - 3(k+1)\beta \sup|\psi|)\tilde n$ for large n. Since both $0 < \varepsilon < \delta$ and $\beta > 0$ were chosen arbitrarily small and $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{p(x,n,\varepsilon)}{n} = 0$ for almost every x one obtains

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu\left(x \in M : \frac{1}{n} S_n g(x) > c \right) \ge h_\eta(f) - \int \psi\, d\eta.$$

The proof of the proposition is now complete. □

6 Some Applications

6.1 One-Dimensional Examples

Large deviations estimates for one-dimensional non-uniformly expanding maps were obtained only by Keller and Nowicki [23] for quadratic maps satisfying the Collet-Eckmann condition and by Araújo and Pacífico [5] to non-uniformly expanding quadratic maps. The first authors proved a large deviations principle for observables of bounded variation and the second authors obtained upper bounds for the measure of the deviation sets of any continuous observable. Using that every topologically mixing and continuous interval map satisfies specification (see [9]) we will now discuss applications of our results to some important classes of examples.

Example 6.1 (Non-uniformly expanding quadratic maps) We consider the class of quadratic maps f_a on the real line given by

$$f_a(x) = 1 - ax^2.$$

In [8], Benedicks and Carleson proved the existence of a positive Lebesgue measure set of parameters $\Omega \in [0, 2]$ such that for every $a \in \Omega$ the quadratic map f_a has positive Lyapunov exponent and an unique absolutely continuous invariant probability measure μ_a supported on $[f^2(0), f(0)]$. In fact, these maps are topologically mixing on $[f^2(0), f(0)]$ and $d\mu_a/dLeb \in L^p$ for every $p < 2$. It follows from the previous discussion that each f_a satisfies the specification property. Moreover, the same argument used in [6] to deal with infinitely many critical points is enough to guarantee that $Leb(\Gamma_n)$ decays exponentially fast (cf. [5, Sect. 2.1]).

Now we notice that all invariant measures are expanding. Indeed, on the one hand [12, Proposition 3.1] establishes for S-unimodal maps and any invariant measure $\lambda(\mu) \geq \lambda_{\text{per}}$, where $\lambda(\mu) = \int \log|f'|\, d\mu$ is the integrated Lyapunov exponent of μ and λ_{per} is the infimum of Lyapunov exponents among periodic orbits. On the other hand, it follows from [24] that the Collet-Eckmann is equivalent to $\lambda_{\text{per}} > 0$.

Since there exist many invariant probability measures with integrable first hyperbolic time map we proceed to show that the measure of the deviation sets is exponential. Using that $d\mu_a/dLeb \in L^p$ for any $p \in (1, 2)$ and that $Leb(\Gamma_n)$ decreases exponentially fast then, if $q > 1$ satisfies $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality

$$\mu_a(\Gamma_n) = \int 1_{\Gamma_n}\, d\mu_a = \int 1_{\Gamma_n} \frac{d\mu_a}{d\nu}\, d\nu \leq \left\| \frac{d\mu_a}{d\nu} \right\|_p Leb(\Gamma_n)^q$$

also decreases exponentially fast and $n_1 \in L^1(\mu_a)$. Since μ_a is an equilibrium state for $\phi_a = -\log|f_a'|$ then it follows from Ruelle-Pesin's formulas that $P = 0$. Moreover, $\nu = Leb$ is an expanding conformal measure and so

$$\limsup_{n\to\infty} \frac{1}{n} \log Leb\left(x \in M : \left|\frac{1}{n} S_n g(x) - \int g\, d\mu_a\right| \geq c\right) \leq -\alpha$$

where

$$\alpha = \min\left\{- \lim_{n\to\infty} \frac{1}{n} \log \mu_a(\Gamma_n), \sup\left\{-h_\eta(f) + \int \log\left|f_a'\right| d\eta\right\}\right\} > 0,$$

and the supremum in the right hand term is over all invariant measures η such that $|\int g\, d\eta - \int g\, d\mu_a| \geq c$. Analogously,

$$\liminf_{n\to\infty} \frac{1}{n} \log Leb\left(x \in M : \left|\frac{1}{n} S_n g(x) - \int g\, d\mu_a\right| > c\right) \geq -\beta$$

where $0 < \beta = \sup\{-h_\eta(f) + \int \log|f_a'|\, d\eta\}$ and the supremum is taken over all invariant measures η such that $n_1 \in L^1(\eta)$ and $|\int g\, d\eta - \int g\, d\mu_a| > c$.

In the following examples we obtain some large deviation estimates for maps of the interval with intermittency behavior with respect to some equilibrium states. In particular we consider the case of the physical and the maximal entropy measure. Moreover, we discuss the presence of the condition on the decay of the first hyperbolic time map in the large deviations upper bound.

Example 6.2 (Intermittency phenomena) Given $\alpha \in (0,1)$, let $f : [0,1] \to [0,1]$ be the $C^{1+\alpha}$ transformation of the interval given by

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \le x \le \frac{1}{2} \\ 2x-1 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$$

known as the Maneville-Pommeau map. This transformation has 0 as an indifferent fixed point (that is $Df(0)=1$) and expansion everywhere else. The map presents an intermittency phenomenon. We provide bounds for the measure of deviation sets in the context of the SRB measure and the case of the maximal entropy measure.

(a) SRB measure

It is known that f has a finite absolutely continuous invariant probability measure μ with polynomial decay of correlations of order $\mathcal{O}(n^{\frac{1}{\alpha}-1})$. In fact that is also the decay of the tail of the first hyperbolic time with respect to $m = Leb$.

By Ruelle-Pesin's formula, μ is an equilibrium state for f with respect to the potential $\phi = -\log|Df|$ with pressure $P := P(\phi) = 0$. In fact, μ and δ_0 are the unique ergodic equilibrium states for ϕ, and so any other equilibrium state is of the form $t\mu + (1-t)\delta_0$ for some $t \in (0,1)$. In addition, it is proved in [19] that $d\mu/dm \approx x^{-\alpha}$. However, $n_1 \notin L^1(\mu)$. Roughly, partitioning the unit interval according to the sequence $(\frac{1}{n})_n$ it follows that

$$\begin{aligned}\int n_1\, d\mu &\ge \sum_{n\ge1} n_1\Big(\frac{1}{n+1}\Big)\mu\Big(\Big[\frac{1}{n+1},\frac{1}{n}\Big]\Big) \approx \sum_{n\ge1} n_1\Big(\frac{1}{n+1}\Big) m\Big(\Big[\frac{1}{n+1},\frac{1}{n}\Big]\Big)\Big(\frac{1}{n}\Big)^{-\alpha} \\ &= \sum_{n\ge1} n^\alpha n_1\Big(\frac{1}{n+1}\Big) m\Big(\Big[\frac{1}{n+1},\frac{1}{n}\Big]\Big),\end{aligned}$$

which is infinite because $n_1 \ge 1$. In consequence the Lebesgue measure of deviation sets decrease polynomially and $\limsup \frac{1}{n}\log\mu(\Gamma_n) = 0$. Since f admits a finite and generating Markov partition then it satisfies the specification property. Therefore, it follows from Theorem 2.1 that for every *continuous* observable g

$$\liminf_{n\to\infty} \frac{1}{n}\log Leb\Big[x \in M : \Big|\frac{1}{n}S_n g(x) - \int g\, d\mu\Big| > c\Big] \ge \sup_\eta\Big\{h_\eta(f) - \int \log|Df|\, d\eta\Big\} \tag{6.1}$$

where η denotes an invariant measure so that $\eta(H)=1$, $|\int g\, d\eta - \int g\, d\mu| > c$ and $n_1 \in L^1(\eta)$, and

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\log Leb\Big(x\in M : \Big|\frac{1}{n}S_n g(x) - \int g\, d\mu_\phi\Big| \ge c\Big) \\ &\quad \le \max\Big\{\sup\Big\{-P + h_\eta(f) + \int \phi\, d\eta\Big\}, \limsup_{n\to\infty}\frac{1}{n}\log\mu_\phi(\Gamma_n)\Big\}\end{aligned} \tag{6.2}$$

where the supremum is taken over all invariant probability measures η such that $|\int g\, d\eta - \int g\, d\mu_\phi| \ge c$. In the case that $g(0) = \int g\, d\mu$ then any invariant probability measure η considered in the right hand side of (6.1) and (6.2) is far from the convex hull generated by the equilibrium states *Leb* and δ_0. Hence it holds that the supremum over all invariant probability measures η such that $|\int g\, d\eta - \int g\, d\mu_\phi| \ge c$ satisfies

$$\sup\Big\{-P + h_\eta(f) + \int \phi\, d\eta\Big\} < 0 = \limsup_{n\to\infty}\frac{1}{n}\log\mu_\phi(\Gamma_n)$$

and that the measure of deviation sets decrease at most exponentially fast. Some results in [4] relate decay of correlations, the decay of the tail of inducing maps and the rate of decay of the deviations with respect to the invariant probability measure. The tail of the first hyperbolic time in (6.2) gives an indication that this relation can be expected to hold also in the case of deviations with respect to the Lebesgue measure. Let us also point out that the results obtained by Chung [16] yield a large deviations principle, where the rate function is not strictly concave due to the non-uniqueness of equilibrium states. In particular, for an open and dense set of observables (namely those satisfying $g(0) \neq \int g\, d\mu$) it follows that deviations are sub-exponential. In fact, we note that polynomial upper and lower bounds for *Hölder continuous* observables have been established in [25, 26, 32].

(b) Equilibrium states for potentials with small variation

It was obtained in [39] that f admits a unique equilibrium state μ_ϕ with respect to any Hölder continuous potential ϕ such that $\sup\phi - \inf\phi < \log 2$. Moreover, μ_ϕ is absolutely continuous with respect to a weak Gibbs conformal measure ν_ϕ, is expanding and $\mu_\phi(\Gamma_n)$ decays exponentially fast. Hence it follows from Corollary 2.1 that for every continuous observable g it holds that

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_\phi\left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\, d\mu_\phi \right| \geq c \right) \leq \max\left\{ \sup\left\{ -P + h_\eta(f) + \int \phi\, d\eta \right\}, \limsup_{n\to\infty} \frac{1}{n} \log \mu_\phi(\Gamma_n) \right\}$$

where the supremum is taken over all invariant probability measures η such that $|\int g\, d\eta - \int g\, d\mu_\phi| \geq c$, and also that

$$\liminf_{n\to\infty} \frac{1}{n} \log \nu_\phi\left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\, d\mu_\phi \right| > c \right) \geq \sup\left\{ -P + h_\eta(f) + \int \phi\, d\eta \right\}$$

where the supremum is taken over all invariant probability measures η such that $\eta(H) = 1$, $|\int g\, d\eta - \int g\, d\mu_\phi| > c$ and $n_1 \in L^1(\eta)$. Since the equilibrium state is unique the right hand side of both expressions above is strictly negative, which yields that the measure of deviation sets decrease exponentially fast. We also remark that if η is an f-invariant probability measure with $\eta \neq \delta_0$ then it follows from Birkhoff's ergodic theorem that its Lyapunov exponent is $\int \log |f'|\, d\eta > 0$. Therefore for an open and dense class of continuous observables g (namely those that satisfy $g(0) \neq \int g\, d\mu_\phi$) if c is small enough then

$$\limsup_{n\to\infty} \frac{1}{n} \log \nu_\phi\left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\, d\mu_\phi \right| \geq c \right) \leq \sup\left\{ -P + h_\eta(f) + \int \phi\, d\eta \right\}$$

where the supremum is taken over all invariant expanding probability measures η such that $|\int g\, d\eta - \int g\, d\mu_\phi| \geq c$. This indicates that it may be possible to establish a large deviations principle for this non-uniformly expanding dynamics using expanding measures. We refer the reader to Sect. 7 for further discussion.

6.2 Higher Dimensional Examples

The next class of examples are multidimensional local diffeomorphisms obtained by local bifurcation of expanding maps and were introduced in [1]. Although the original expanding maps satisfy the specification property we point out that the same should not hold for the perturbations.

Example 6.3 Let f_0 be an expanding map in $\mathbb{T}^n$ and take a periodic point p for f_0. Let f be a C^1-local diffeomorphism obtained from f_0 by a bifurcation in a small neighborhood U of p in such a way that:

(1) every point $x \in \mathbb{T}^n$ has some preimage outside U;
(2) $\|Df(x)^{-1}\| \le \sigma^{-1}$ for every $x \in \mathbb{T}^n \setminus U$, and $\|Df(x)^{-1}\| \le L$ for every $x \in \mathbb{T}^n$ where $\sigma > 1$ is large enough or $L > 0$ is sufficiently close to 1;
(3) f is topologically exact: for every open set U there is $N \ge 1$ for which $f^N(U) = \mathbb{T}^n$.

It follows from [39] that f has a unique (ergodic) equilibrium state μ for the Hölder continuous potential $\phi = -\log|\det Df|$, it is absolutely continuous with respect to the conformal measure $\nu = Leb$ with density bounded away from zero and infinity, and it is expanding. We note also that the equilibrium state μ also satisfies the non-uniform specification property.

Lemma 6.1 *(f, μ) satisfies the non-uniform specification property.*

Proof First we note that since M is compact and f is topologically exact then for every $\varepsilon > 0$ there exists $N_\varepsilon \ge 1$ such that $f^{N_\varepsilon}(B) = \mathbb{T}^n$ for every ball B of radius ε. Indeed, for every x let $N(x,\varepsilon) \ge 1$ be the minimum integer such that $f^{N(x,\varepsilon)}(B(x,\varepsilon/3)) = \mathbb{T}^n$. By compactness the open cover $(B(x,\varepsilon/3))_{x\in\mathbb{T}^n}$ admits a finite covering $(B(x_i,\varepsilon/3))_{i=1..n}$. Hence, if $N_\varepsilon = \max\{N(x_i,\varepsilon) : i = 1..n\}$ then any ball B of radius ε contains a ball $B(x_j,\varepsilon/3)$, for some j, and so $f^{N_\varepsilon}(B) = \mathbb{T}^n$.

It follows from [39] that the equilibrium state μ is absolutely continuous with respect to a conformal measure ν with density bounded away from zero and infinity and $n_1 \in L^1(\mu)$. Moreover, the sequence $n_k(\cdot)$ of hyperbolic times is non-lacunar, that is $\frac{n_{k+1}-n_k}{n_k} \to 0$ at almost every x. Therefore, if $0 < \varepsilon < \delta$, n is large and $n_k(x) < n < n_{k+1}(x)$ are consecutive hyperbolic times then clearly $B(x, n_{k+1}, \varepsilon) \subset B(x, n, \varepsilon)$ and

$$f^{n_{k+1}+N_\varepsilon}\big(B(x, n_{k+1}, \varepsilon)\big) = f^{N_\varepsilon}\big(B\big(f^{n_{k+1}}(x), \varepsilon\big)\big) = \mathbb{T}^n.$$

Thus for any given $y \in \mathbb{T}^n$ and proximity $\zeta > 0$ there exists $z \in B(x,n,\varepsilon)$ so that $f^{N_\varepsilon+n_{k+1}(x)-n}(f^n(z)) = f^{N_\varepsilon+n_{k+1}(x)}(z) \in B(y,\zeta)$. Take $p(x,n,\varepsilon) = N_\varepsilon + n_{k+1}(x) - n$. Then for any $x_1,\dots,x_m$ in a full μ-measure set, any positive integers $k_1,\dots,k_m$ and $p_i \ge p(x_i,n_i,\varepsilon)$ there exists $z \in \mathbb{T}^n$ such that $z \in B(x_1,n_1,\varepsilon)$ and $f^{n_1+p_1+\cdots+n_{i-1}+p_{i-1}}(z) \in B(x_i,n_i,\varepsilon)$ for every $2 \le i \le k$. To obtain the non-uniform specification property just note that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{p(x,n,\varepsilon)}{n} \le \lim_{\varepsilon\to 0}\limsup_{k\to\infty}\frac{N_\varepsilon + n_{k+1}(x) - n_k(x)}{n_k(x)} = 0. \qquad \square$$

Using the non-uniform specification property we obtain from Theorem 2.2 that

$$\limsup_{n\to\infty}\frac{1}{n}\log Leb\left(x \in M : \left|\frac{1}{n}S_n g(x) - \int g\,d\mu\right| \ge c\right)$$
$$\le \max\left\{\sup\left\{-P + h_\eta(f) + \int \phi\,d\eta\right\}, \limsup_{n\to\infty}\frac{1}{n}\log\mu(\Gamma_n)\right\},$$

where the supremum taken over all invariant probability measures η satisfying $|\int g\,d\eta - \int g\,d\mu| > c$, and also

$$\liminf_{n\to\infty} \frac{1}{n} \log Leb\left(x \in M : \left| \frac{1}{n} S_n g(x) - \int g\,d\mu \right| > c \right) \geq \sup\left\{ -P + h_\eta(f) + \int \phi\,d\eta \right\},$$

where the supremum taken over expanding f-invariant probability measures η such that $n_1 \in L^1(\eta)$ and $|\int g\,d\eta - \int g\,d\mu| > c$. Note that both rates are exponential.

We also prove that a robust class of multidimensional non-uniformly expanding maps with singularities also satisfy this weak form of specification.

Example 6.4 (Viana maps) In [38], the author introduced a robust class of multidimensional non-uniformly hyperbolic maps with singularities commonly known as Viana maps. More precisely, these are obtained as C^3 small perturbations of the skew product ϕ_α of the cylinder $S^1 \times I$ given by

$$\phi_\alpha(\theta, x) = \big(d\theta (\mathrm{mod}\ 1), 1 - ax^2 + \alpha \cos(2\pi\theta)\big),$$

where $d \geq 16$ is an integer, a is a Misiurewicz parameter for the quadratic family, and α is small. These maps admit a unique SRB measure μ (it is absolutely continuous with respect to $m = Leb$, has only positive exponents and $d\mu/dm \in L^p(m)$ where $p = d/(d-1)$) and are strong topologically mixing on the attractor $\Lambda = \bigcap_{n\geq 0} \phi_\alpha^n(S^1 \times I)$: for every open set A there exists a positive integer $n = n(A)$ such that $\phi_\alpha^n(A) = \Lambda$. See [2, 7, 38] for more details.

We show that (f, μ) satisfies the non-uniform specification using that the sequence of hyperbolic times is non-lacunar, that is, $\frac{n_{k+1}-n_k}{n_k} \to 0$ at almost every x and that the image of hyperbolic balls grow to Λ after finitely many iterates.

First we observe that n_1 is integrable with respect to the SRB measure μ. In fact, since the tail of the first hyperbolic time map decays subexponentially fast with respect to m in particular one has $n_1 \in L^q(m)$ for every $q \geq 1$. Using once more Cauchy-Schwartz inequality and that $d\mu/dm \in L^p(m)$ it follows that $n_1 \in L^1(\mu)$ and, consequently, the sequence $n_j(\cdot)$ of hyperbolic times is non-lacunar (see e.g. [39, Corollary 3.8]).

The arguments used in the proof of Theorem C in [7] give that for every image of a rectangle at a hyperbolic time grow to Λ after a finite number ℓ of iterates. Since *Leb* almost every point has infinitely many hyperbolic times then f is topologically exact. Therefore, the argument that (f, μ) satisfies the non-uniform specification property goes along the same lines used in proof of Lemma 6.1.

For completeness, let us mention that during the refereeing process it has been announced in [4] that Viana maps have at least stretched exponential large deviations with respect to the invariant SRB measure μ.

7 Recent Developments and Some Future Perspectives

Both weak specification properties and the theory of large deviations have been target of recent intense study. In this section we discuss some recent results and establish a connection with some future perspectives.

Weak Specification Properties The important notion of (strong) specification introduced by Bowen in [10] allowed to deduce that uniformly hyperbolic maps are rich from the ergodic theory viewpoint. In fact, this property play a key role in the proof that these maps have a unique equilibrium state for every Hölder continuous potential, that the topological entropy coincides with the exponential growth rate of the set $Per_n(f)$ of periodic points of period n. Moreover, it has important connections with the study of Poincaré recurrence and large deviations.

Hence, it is important to understand which topological or measure-theoretical weaker forms of specification do non-uniformly hyperbolic dynamical systems satisfy, which ergodic properties are obtained as consequence and the relation between the topological and measure-theoretical notions. Firstly let us mention that Oliveira [28] proved for C^1-endomorphisms that every ergodic and invariant probability measure with only positive Lyapunov exponents satisfies the non-uniform specification property introduced in [36] and deduced interesting results concerning Poincaré recurrence. Very recently, Oliveira and Tian [29] announced that every ergodic hyperbolic measure preserved by a $C^{1+\alpha}$-diffeomorphism satisfy both the non-uniform specification property of [36] as the one introduced here. In consequence, since Dirac measures satisfy the non-uniform specification property trivially, all invariant probability measures for the Maneville-Pommeau transformation in Example 6.2 do satisfy the non-uniform specification property. In fact this is already a consequence from the fact that the Maneville-Pommeau transformation satisfies the (strong) specification property. So, having in mind the results obtained in [3, 15] it would be very interesting to answer the following question:

Question 1 *Let $f : M \to M$ be a C^1-local diffeomorphism on a compact Riemannian manifold M and assume that (robustly) every f-invariant ergodic probability measure μ satisfies the non-uniform specification property. Does the map f satisfy the specification property?*

Another interesting topic is to relate specification properties with the presence of discontinuities in the system. Indeed, Buzzi [14] proved that contrary to the characterization due to Blokh [9] for continuous interval maps there exists a large class of topologically mixing but discontinuous maps of the interval (including β-transformations) so that the set of parameters for which the strong specification property holds although dense has zero Lebesgue measure. In fact, it was communicated to us by Dan Thompson that there are beta-transformations that do not satisfy the non-uniform specification property for any full supported probability measure. So we pose the following question:

Question 2 *Does the set of parameters for which β-transformations satisfy the non-uniform specification property for any full supported invariant probability measure have positive Lebesgue measure?*

Clearly this set contains the set of parameters for which strong specification property holds, that has zero Lebesgue measure. Note that an affirmative answer to the previous questions would be a contribution for a better understanding of the non-uniform specification property would give a wider class of examples for which our results apply.

Large Deviations Many recent contributions to the theory of large deviations in non-uniformly hyperbolic dynamical systems have been given. In fact, as discussed in the introduction, the existence of Markov towers allowed Melbourne, Nicol [26] and Rey-Bellet, Young [33] to obtain large deviation principles with respect to the invariant probability measure for the original dynamical system.

More recently, Chung [16] has also obtained large deviation principle with respect to the (not necessarily invariant) Lebesgue measure for Markov tower maps induced by return time functions satisfying some technical conditions with similar flavor to our non-uniform specification property. Such results apply e.g. for a Markov tower obtained from the Maneville-Pommeau with a rate function expressed by the pressure function computed using only expanding measures. The results in Example 6.2 are a first step to prove that a large deviations principle hold for the original Maneville-Pommeau transformation. More generally,

Question 3 *Let μ be the unique equilibrium state for f with respect to a potential ϕ, absolutely continuous with respect to a* (*not necessarily invariant*) *weak Gibbs measure*, *and whose first hyperbolic time map has exponential tail. Does there exists a large deviation principle with respect to the weak Gibbs measure? If so*, *can the rate function be described using only thermodynamical quantities at invariant expanding measures and the tail of the first hyperbolic time map?*

We expect that our results can extend to the partially hyperbolic setting. However, despite the fact that equilibrium states in a broad non-uniformly hyperbolic context are expected to be absolutely continuous with respect probability measures that exhibit some weak Gibbs property on Pesin local unstable leaves, the general thermodynamical formalism even for partially hyperbolic dynamical systems is far from being completely understood. Finally, let us mention that in the case of SRB measures some large deviations upper bounds were obtained in [5]. Moreover, some large deviations lower bounds have also been announced recently by Hirayama and Sumi [18] for hyperbolic measures that satisfy a measure-theoretical transversality condition.

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