Rapid expansion and pseudo spectral implementation for reverse time migration in VTI media



2012 J. Geophys. Eng. 9 291

(http://iopscience.iop.org/1742-2140/9/3/291)

J. Geophys. Eng. **9** (2012) 291–301 doi:10.1088/1742-2132/9/3/291

# Rapid expansion and pseudo spectral implementation for reverse time migration in VTI media

**Reynam C Pestana**[1], **Bjørn Ursin**[2] **and Paul L Stoffa**[3]

[1] Center for Research in Geophysics and Geology (CPGG), Federal University of Bahia (UFBA), Rua Barão de Geremoabo, Salvador, Bahia, Brazil
[2] Department of Petroleum Engineering and Applied Geophysics, Norwegian University of Science and Technology (NTNU), SP Andersens vei 15A, N0-7491 Trondheim, Norway
[3] Institute for Geophysics, University of Texas at Austin, Austin, TX, USA

E-mail: reynam@cpgg.ufba.br, bjornu@ipt.ntnu.no and pauls@ig.utexas.edu

Received 6 October 2011
Accepted for publication 14 March 2012
Published 24 April 2012
Online at stacks.iop.org/JGE/9/291

**Abstract**
In isotropic media, we use the scalar acoustic wave equation to perform reverse time migration (RTM) of the recorded pressure wavefield data. In anisotropic media, P- and SV-waves are coupled, and the elastic wave equation should be used for RTM. For computational efficiency, a pseudo-acoustic wave equation is often used. This may be solved using a coupled system of second-order partial differential equations. We solve these using a pseudo spectral method and the rapid expansion method (REM) for the explicit time marching. This method generates a degenerate SV-wave in addition to the P-wave arrivals of interest. To avoid this problem, the elastic wave equation for vertical transversely isotropic (VTI) media can be split into separate wave equations for P- and SV-waves. These separate wave equations are stable, and they can be effectively used to model and migrate seismic data in VTI media where $|\epsilon - \delta|$ is small. The artifact for the SV-wave has also been removed. The independent pseudo-differential wave equations can be solved one for each mode using the pseudo spectral method for the spatial derivatives and the REM for the explicit time advance of the wavefield. We show numerically stable and high-resolution modeling and RTM results for the pure P-wave mode in VTI media.

**Keywords:** seismic waves, anisotropy, dispersion relation, vertical transversely isotropic, reverse time migration, rapid expansion method, pseudo spectral method

(Some figures may appear in colour only in the online journal)

## Introduction

Seismic anisotropy is observed in many exploration areas. Conventional isotropic migration methods are insufficient in these areas. Thus, where required by the analysis of the data, migration methods may be isotropic or vertical transversely isotropic (VTI) or tilted transversely isotropic (TTI).

Reverse time migration (RTM) is becoming the standard tool for imaging areas where large velocity contrast and/or steep dips pose imaging challenges, e.g. around and below salt bodies. RTM propagates the source wavefield forward in time and the receiver wavefield backward in time to image the subsurface reflectors. By using the two-way acoustic wave equation, it naturally takes into account both down-going and up-going waves and thus enables imaging of the turning waves and prismatic waves that are able to enhance the image of steep salt flanks and other steeply dipping events associated with complex structures. In recent years, RTM has gained popularity as computer power has increased, enabling its routine application to prestack seismic data.

Ignoring anisotropy may result in serious imaging problems for dipping reflectors beneath or inside anisotropic structures. To model anisotropic media, we should use the elastic wave equations where the P-wave and S-wave modes are intrinsically coupled. However, for imaging anisotropic media, particularly for marine data, the separation of P- and S-waves is often preferred. Thus, both the implementation of separation methods and the fully elastic solutions with appropriate corresponding imaging conditions remain a subject of ongoing research (Yan and Sava 2009, Lu 2010, Yan and Xie 2010).

For P-wave imaging purposes, the preferred approach has been to derive pseudo-acoustic wave equations that can be used in an efficient way for modeling and migration of seismic data acquired over anisotropic media. Alkhalifah (1998, 2000) derived an approximate dispersion relation for VTI media, simply by setting the shear wave velocity equal to zero along the vertical axis. But the implementation of the resulting equations in the space-time domain leads to complicated fourth-order partial differential equations. Other authors derive various second-order coupled systems (Zhou *et al* 2006, Du *et al* 2008) based on Alkhalifah's dispersion relations. Now the original fourth-order differential equation is represented as a coupled system of equations of second order in time that are much easier to implement. Using Hooke's law and the equations of wave motion, Duveneck *et al* (2008) derived coupled first-order and second-order VTI equations, but again they set the vertical shear wave velocity as zero.

Many researchers have implemented computationally efficient algorithms for these two-way wave equations for modeling and RTM in anisotropic media with the pseudo-acoustic approximation (Zhou *et al* 2006, Du *et al* 2008, Fowler *et al* 2010a, Song *et al* 2011) using second-order finite differences in time and higher order finite differences to compute the spatial derivatives.

Although this approach to defining a dispersion relation for a scalar wavefield has kinematics close to those of the P-arrivals in the real elastic vector wavefield, it can generate unwanted wave events. These events were initially categorized as numerical artifacts (Alkhalifah 2000). But later, Grechka *et al* (2004) identified them as the SV-component, because simply setting $v_s = 0$ does not result in the vanishing of the shear wave phase velocity everywhere in an acoustic VTI medium (Liu *et al* 2009). Methods have been proposed to suppress this artifact. For example, when the source point is located in an isotropic medium above the anisotropic medium, the artifact disappears.

To avoid this undesired SV-wave energy, different approaches have recently been proposed to model the pure P-wave mode (Etgen and Brandsberg-Dahl 2009, Liu *et al* 2009). Recently, Liu *et al* (2009) factored the dispersion relation presented by Alkhalifah (2000) and obtained two pseudo-partial differential equations. The P-wave equation is well-posed for any value of the anisotropic parameters, but the SV-wave becomes well-posed only when $\epsilon > \delta$ is satisfied.

The idea of obtaining a separate P-wave equation is not new. Fowler (2003) presents a systematic framework for deriving a variety of VTI approximations and he developed a sequence of well-defined approximations to the exact P-wave and SV-wave phase velocities. For example, the P-wave approximation present in Liu *et al* (2009) is equivalent to approximation P2 in table 2 of Fowler (2003). It is equivalent to the Muir–Dellinger approximation (Dellinger and Muir 1985, Dellinger *et al* 1993) which was later derived again by Stopin (2001). More recently, a P-wave dispersion relation for VTI media was presented by Etgen and Brandsberg-Dahl (2009) where they implemented the time stepping using the pseudo-analytical method. In fact, this approximation is the same one labeled P4 in table 2 of Fowler (2003) and it also has a long history and is originally credited to Harlan (1995). So the mode-separated dispersion relations for P- and SV-waves which we review below are not new and were developed and discussed in the references cited above.

For homogeneous media, all the above P or SV dispersion relations can be evaluated in the wavenumber domain and then used in an exponential or cosine time extrapolation operator. But, for heterogeneous VTI media, exact or approximated phase velocity functions cannot be easily implemented because they do not separate in the space and wavenumber domain. Recently, Etgen and Brandsberg-Dahl (2009), using an approximate dispersion relation for VTI media, solved the VTI time-wavenumber domain wave equation using the pseudo-analytical method as a time extrapolation procedure. Later, the pure P-wave equation, as derived using the Harlan (1995) approximation, which is indeed separable, was extended from VTI to TTI media using a matrix rotation, and then implemented again using the pseudo-analytical method (Crawley *et al* 2010) for the time advance of the wavefield.

To solve the pure P-wave equation, for heterogeneous media, we need to use a mixed-domain method. Various mixed-domain methods for isotropic P-wave RTM have recently been presented by Soubaras and Zhang (2008), Zhang and Zhang (2009), Pestana and Stoffa (2010), Fowler *et al* (2010b), only to cite a few. All these methods can be extended to VTI and TTI media. In this paper, we use the rapid expansion method (REM) that is based on a Chebyshev series expansion for the time extrapolation. By coupling the REM with a pseduo spectral method for the spatial derivatives, we can accurately solve mixed-domain space-wave number equations. The implementation of the REM is straightforward for modeling problems (Kosloff *et al* 1989), but for RTM it has to be implemented in a recursive procedure as presented by Pestana and Stoffa (2010). Here we follow this approach to solve both the coupled system (Du *et al* 2008) and the separated pseudo-differential P-wave equation. We begin by reviewing the developments of the coupled and separated P-wave equations.

## Pseudo-acoustic wave equations

The scalar acoustic wave equation is commonly used in isotropic media to describe the propagation of P-waves through structures. However, anisotropic media are only correctly described by the elastic wave equation with intrinsically coupled P- and S-waves. Rather than solving the anisotropic elastic wave equation, several researchers have derived two-way pseudo-acoustic wave equations in anisotropic media (Alkhalifah 2000, Zhou *et al* 2006, Du *et al* 2008, Fowler *et al* 2010a, 2010b).

Acoustic anisotropy was introduced by setting the vertical shear wave velocity equal to zero ($v_{so} = 0$). The dispersion relation for waves in 3D acoustic VTI media (Alkhalifah 2000) is then given by

$$\omega^4 - \left[v_h^2 k_r^2 + v_{po}^2 k_z^2\right]\omega^2 - v_{po}^2\left(v_n^2 - v_h^2\right)k_r^2 k_z^2 = 0, \qquad (1)$$

where $v_{po}$ is the vertical P-wave velocity (along the symmetry axis), $v_h$ is the horizontal P-wave velocity and $v_n$ is the P-wave NMO-velocity. They are given by

$$\begin{aligned} v_h^2 &= v_{po}^2(1+2\epsilon) \\ v_n^2 &= v_{po}^2(1+2\delta), \end{aligned} \tag{2}$$

with $\epsilon$ and $\delta$ being the Thomsen (1986) parameters. The square of the vertical wavenumber is $k_z^2$, the square of the horizontal wavenumber is $k_r^2 = k_x^2 + k_y^2$ and $\omega$ is the angular frequency.

Introducing the following auxiliary function,

$$q(\omega, k_x, k_y, k_z) = \frac{\omega^2 + (v_n^2 - v_h^2)k_r^2}{\omega^2}\, p(\omega, k_x, k_y, k_z), \tag{3}$$

equation (1) can be written (Du *et al* 2008) as

$$\omega^2 p(\omega, k_x, k_y, k_z) = v_h^2 k_r^2 p(\omega, k_x, k_y, k_z) + v_{po}^2 k_z^2 q(\omega, k_x, k_y, k_z). \tag{4}$$

Applying an inverse Fourier transform to both sides of the previous two equations, using the correspondent relations $\mathrm{i}\omega \leftrightarrow \partial/\partial t$, $-\mathrm{i}k_x \leftrightarrow \partial/\partial x$, $-\mathrm{i}k_y \leftrightarrow \partial/\partial y$, $-\mathrm{i}k_z \leftrightarrow \partial/\partial z$, we obtain the following pseudo-acoustic VTI system of equations,

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} &= v_h^2\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + v_{po}^2\frac{\partial^2 q}{\partial z^2} \\ \frac{\partial^2 q}{\partial t^2} &= v_n^2\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2}\right) + v_{po}^2\frac{\partial^2 q}{\partial z^2}, \end{aligned} \tag{5}$$

or using the following matrix formulation (2D case), we have

$$\frac{\partial^2}{\partial t^2}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} v_h^2\frac{\partial^2}{\partial x^2} & v_{po}^2\frac{\partial^2}{\partial z^2} \\ v_n^2\frac{\partial^2}{\partial x^2} & v_{po}^2\frac{\partial^2}{\partial z^2} \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \tag{6}$$

which can also be rewritten as

$$\frac{\partial^2 U}{\partial t^2} = -A^2 U, \tag{7}$$

where $U = (p, q)$ and

$$-A^2 = \begin{pmatrix} v_h^2\frac{\partial^2}{\partial x^2} & v_{po}^2\frac{\partial^2}{\partial z^2} \\ v_n^2\frac{\partial^2}{\partial x^2} & v_{po}^2\frac{\partial^2}{\partial z^2} \end{pmatrix}. \tag{8}$$

Introducing this auxiliary function results in a coupled system of two wave equations which is more convenient and computationally efficient to solve. In both equations (1) and (6), there appears a spurious SV-wave (Grechka *et al* 2004) which introduces noise when modeling or migrating P-waves.

### *The REM—two coupled wave equations*

Considering now the system of equations derived by Du *et al* (2008) (equation (6)), it has a formal solution for the updated wavefield $U(t+\Delta t)$ that is given by

$$U(t+\Delta t) = -U(t-\Delta t) + 2\cos(A\Delta t)\,U(t). \tag{9}$$

Now we need to compute the cosine of a matrix acting on a vector. However, there is no closed form expression for the cosine of the matrix operator. We could use the Taylor expansion, but it requires many terms for large values of the argument of the cosine (Pestana and Stoffa 2010).

The numerical procedure for solving equation (9) in second temporal order can be expressed as

$$U(t+\Delta t) = 2U(t) - U(t-\Delta t) - \Delta t^2 A^2 U(t) \tag{10}$$

or

$$U(t+\Delta t) = 2U(t) - U(t-\Delta t) - \Delta t^2 A^2 U(t) + \frac{\Delta t^4 A^4}{12} U(t), \tag{11}$$

which is the standard fourth-order finite difference scheme (Etgen 1986).

A more efficient orthogonal polynomial series expansion for the cosine was presented by Tal-Ezer *et al* (1987) and applied for seismic modeling by Kosloff *et al* (1989). The method is called REM and was proposed by Pestana and Stoffa (2010) for RTM problems for the isotropic case.

For the REM, which is based on the Chebyshev polynomial expansion (Tal-Ezer *et al* 1987), the cosine function can be expressed as

$$\cos(A\Delta t) = \sum_{k=0}^{\infty} C_{2k} J_{2k}(R\Delta t)\, Q_{2k}\left(\frac{\mathrm{i}A}{R}\right), \tag{12}$$

where $C_{2k} = 1$ for $k = 0$ and $C_{2k} = 2$ for $k > 0$. $R$ is chosen as the largest eigenvalue of $A^2$. $J_{2k}$ is the Bessel function of the first kind order and $Q_{2k}$ are the modified Chebyschev polynomials that satisfy the following recurrence relations:

$$\begin{aligned} Q_0\left(\frac{\mathrm{i}A}{R}\right) &= I \quad \text{(the identity operator)} \\ Q_2\left(\frac{\mathrm{i}A}{R}\right) &= I - \frac{A^2}{R^2} \\ Q_{2k+2}\left(\frac{\mathrm{i}A}{R}\right) &= \left(-\frac{4A^2}{R^2} + 2I\right) Q_{2k} - Q_{2k-2}. \end{aligned} \tag{13}$$

For 3D wave propagation and considering the constant velocity case, the value of $R$ is given by $R = \pi v \sqrt{1/\Delta x^2 + 1/\Delta y^2 + 1/\Delta z^2}$. In general, for VTI media, $v$ should be replaced by $\max\{v_h, v_n\}$, the highest velocity in the grid. The sum in expression (12) is known to converge exponentially for $k > R\Delta t$; therefore, the summation can be safely truncated using a value of $k$ slightly greater than $R\Delta t$.

In conventional RTM or seismic modeling, the spatial derivatives are usually calculated by finite-difference schemes. For the coupled VTI equations (5) or (6), finite-difference methods are readily employed. The spatial operators can be of low order (e.g. fourth) or higher order (e.g. eighth or more). But the time integration would usually be of low order (e.g. second). To maintain stability, small time steps are required to satisfy the stability condition. But, to avoid numerical dispersion, even smaller time steps are needed (Kosloff *et al* 1989). This leads to increases in the computation time over that required to satisfy the stability condition alone (Pestana and Stoffa 2010).

Several seismic modeling algorithms have been proposed with the goal of obtaining more accurate results with less numerical dispersion due to the spatial and time discretizations

(Etgen and Brandsberg-Dahl 2009, Soubaras and Zhang 2008, Fowler *et al* 2010b). For example, Soubaras and Zhang (2008) proposed a polynomial expansion for the time integration where the coefficients are estimated by some optimization procedure.

In our numerical implementations, we use a pseudo spectral method for the spatial derivatives to avoid all numerical dispersion problems (Carcione *et al* 2002). We also use the REM as described above for both stable and accurate time extrapolations (Carcione *et al* 2002, Kosloff *et al* 1989). Since the REM uses a series of orthogonal Chebyshev polynomials, this is more efficient than using Taylor series expansions when large time steps are used (Pestana and Stoffa 2010).

## Separate P- and SV-wave equations

To avoid the undesired SV-wave energy, different approaches have been proposed to model the pure P-wave mode (Dellinger and Muir 1985, Dellinger *et al* 1993, Harlan 1995, Stopin 2001, Fowler 2003, Etgen and Brandsberg-Dahl 2009, Liu *et al* 2009).

Recently, Liu *et al* (2009) factored the dispersion relation (1) and obtained two separate P- and SV-wave dispersion relations:

$$\omega^2 = \frac{1}{2}\left[v_h^2 k_r^2 + v_{po}^2 k_z^2\right] \pm \frac{1}{2}\left[v_h^2 k_r^2 + v_{po}^2 k_z^2\right] \times \left[1 + \frac{4v_{po}^2\left(v_n^2 - v_h^2\right)k_r^2 k_z^2}{\left[v_h^2 k_r^2 + v_{po}^2 k_z^2\right]^2}\right]^{1/2}. \quad (14)$$

Expanding the square root to first order ($\sqrt{1+X} = 1 + \frac{1}{2}X$), we obtain

$$\omega^2 \approx v_{po}^2 k_z^2 + v_h^2 k_r^2 + \frac{\left(v_n^2 - v_h^2\right)k_r^2 k_z^2}{k_z^2 + F k_r^2} \qquad \text{(for P-wave)} \quad (15)$$

and

$$\omega^2 \approx -\frac{\left(v_n^2 - v_h^2\right)k_r^2 k_z^2}{k_z^2 + F k_r^2} \qquad \text{(for SV-wave)}, \quad (16)$$

where $F = \frac{v_h^2}{v_{po}^2} = 1 + 2\epsilon$

For the SV-wave equation to be stable, it is required that $v_h^2 - v_n^2 \geqslant 0$ or $\epsilon \geqslant \delta$. This requirement does not, however, represent realistic SV-wave propagation.

The P-wave equation (15) can also be derived using approximation P2 in table 2 of Fowler (2003) which is equivalent to the Muir–Dellinger approximation (Dellinger and Muir 1985, Dellinger *et al* 1993).

We start with the exact dispersion relation for VTI media as derived by Tsvankin (1996):

$$\frac{v^2(\theta)}{v_{po}^2} = 1 + \epsilon \sin^2\theta - \frac{f}{2} \pm \frac{f}{2}\left[1 + \frac{2\epsilon\sin^2\theta}{f}\right] \times \left[1 - \frac{2(\epsilon-\delta)\sin^2 2\theta}{f\left(1 + \frac{2\epsilon\sin^2\theta}{f}\right)^2}\right]^{1/2}, \quad (17)$$

where $\theta$ is the phase angle measured from the symmetry axis. The plus sign corresponds to the P-wave and the minus sign corresponds to the SV-wave.

Here

$$f = 1 - \left(\frac{v_{so}}{v_{po}}\right)^2. \quad (18)$$

One again expands the square root to first order ($\sqrt{1-X} = 1 - \frac{1}{2}X$) obtaining the approximations

$$\frac{v^2(\theta)}{v_{po}^2} \approx 1 + 2\epsilon\sin^2\theta - \frac{(\epsilon-\delta)\sin^2 2\theta}{2\left(1 + \frac{2\epsilon\sin^2\theta}{f}\right)} \qquad \text{P-wave} \quad (19)$$

and

$$\frac{v^2(\theta)}{v_{po}^2} \approx \frac{v_{so}^2}{v_{po}^2} + \frac{(\epsilon-\delta)\sin^2 2\theta}{2\left(1 + \frac{2\epsilon\sin^2\theta}{f}\right)} \qquad \text{SV-wave}. \quad (20)$$

We have $\sin(\theta) = \frac{v(\theta)k_r}{\omega}$ and $\cos(\theta) = \frac{v(\theta)k_z}{\omega}$ so that

$$v^2(\theta) = \frac{\omega^2}{k_r^2 + k_z^2}. \quad (21)$$

The results are the dispersion relations

$$\omega^2 = v_{po}^2 k_z^2 + v_h^2 k_r^2 - \frac{2v_{po}^2(\epsilon-\delta)k_r^2 k_z^2}{k_z^2 + F k_r^2} \quad (22)$$

for P-waves and

$$\omega^2 = v_{so}^2\left(k_r^2 + k_z^2\right) + \frac{2v_{po}^2(\epsilon-\delta)k_r^2 k_z^2}{k_z^2 + F k_r^2} \quad (23)$$

for SV-waves. Here

$$F = 1 + \frac{2\epsilon}{f} = \frac{v_h^2 - v_{so}^2}{v_{po}^2 - v_{so}^2}. \quad (24)$$

The equations (22) and (23) are good approximations for the P- and SV-wave dispersion relations if

$$\left|\frac{2(\epsilon-\delta)\sin^2 2\theta}{f\left(1 + \frac{2\epsilon\sin^2\theta}{f}\right)^2}\right| \ll 1. \quad (25)$$

They are exact for an elliptic medium where $\epsilon = \delta$ (see equation (17)).

We note that the separated equations are equivalent to equations P8 and VS8 in table 2 of Fowler (2003). In Fowler (2003), a common derivation and comparison of these TI approximations are given; see also Dellinger and Muir (1985), Dellinger *et al* (1993) and Stopin (2001). We note that both equations are better suited for the computationally more expensive pseudo spectral method rather than finite-difference methods of solution.

For implementation, we multiply both sides of equations (22) and (23) by the wavefield $P(k_r, k_z, \omega)$ followed by an inverse Fourier transform and then using the relation $i\omega \leftrightarrow \partial/\partial t$, we obtain the following P and SV wave equations in the time-wavenumber domain for VTI media:

$$\frac{\partial^2 P}{\partial t^2} = -\left\{v_{po}^2 k_z^2 + v_h^2 k_r^2 - \frac{2v_{po}^2(\epsilon-\delta)k_r^2 k_z^2}{k_z^2 + F k_r^2}\right\} P \quad (26)$$

for P-waves and

$$\frac{\partial^2 P_{SV}}{\partial t^2} = -\left\{v_{so}^2\left(k_r^2 + k_z^2\right) + \frac{2v_{po}^2(\epsilon-\delta)k_r^2 k_z^2}{k_z^2 + F k_r^2}\right\} P_{SV} \quad (27)$$

for SV-waves.

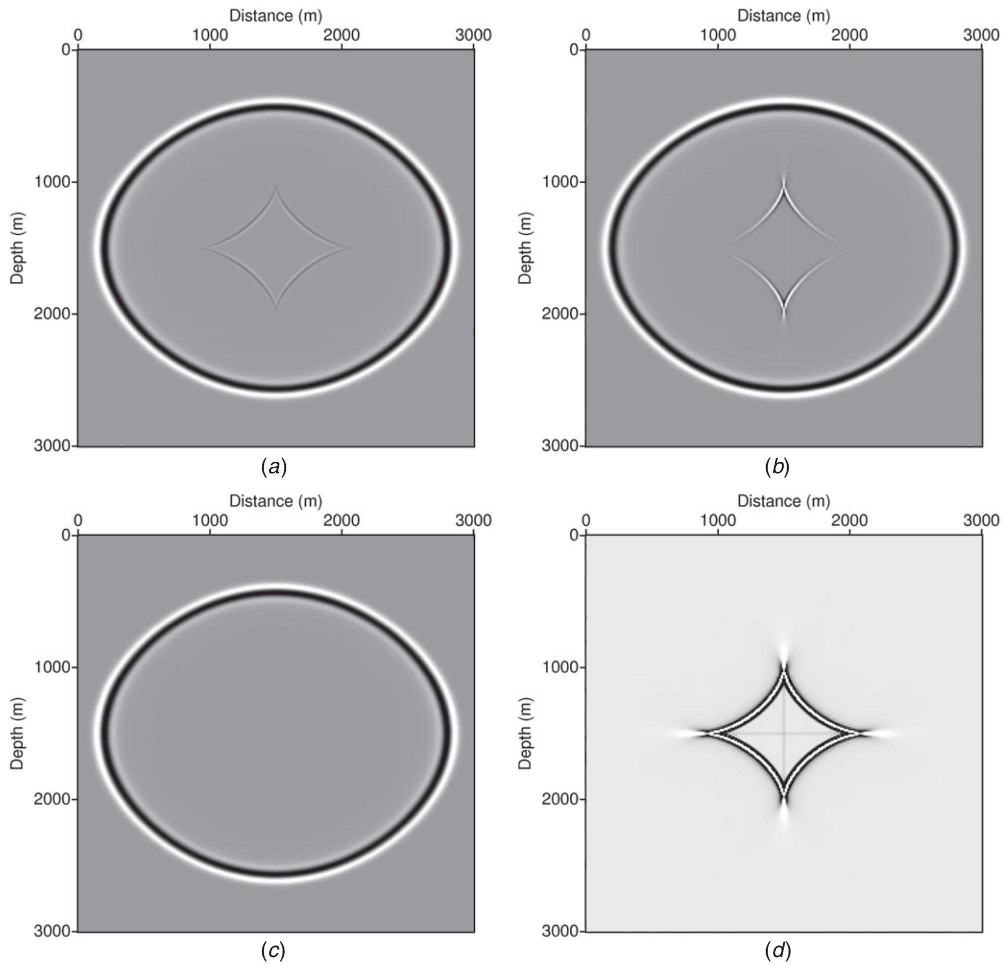



**Figure 1.** Impulse response: (*a*) p-wavefield and (*b*) q-wavefield, of the Du *et al* (2008) equations solved by the REM. They clearly show the spurious SV-wave artifacts. (*c*) P and S wavefields (*d*) from the decoupled P- and SV-wave equations proposed by Liu *et al* (2009) also solved by the REM.

In the appendix, using a high-order approximation for the square root, we derive from (17) separate wave equations for P- and SV-waves. However, due to computation issues, the $F_1$ and $F_2$ terms in the denominators of these equations have to be constant, as we explain later. Moreover, equations (A.9) and (A.10) reduce to equations (22) and (23), respectively.

When we set $v_{so} = 0$ (or $f = 1$), equations (22) and (23) reduce to (19) and (20). If instead we set $\epsilon = 0$ in this expression, then $F = 1$, and equation (22) reduces to the dispersion relation used by Etgen and Brandsberg-Dahl (2009). This one is labeled P4 in Fowler (2003) and is more properly credited to Harlan (1995). It is also equivalent to the 'weak-anisotropy-squared' approximation used by Stopin (2001).

### *The REM—separate equations for P- and SV-waves*

Based on the work of Zhang and Zhang (2009), the two-way wave equation can be transformed to a first-order equation in time given by

$$\left(\frac{\partial}{\partial t} + \mathrm{i}\Phi\right) P(x, y, z, t) = 0, \tag{28}$$

where $P$ is the complex pressure wavefield and $\Phi$ is a pseudo-differential operator in the space domain. In isotropic media, the operator is defined by $\Phi = v\sqrt{-\nabla^2}$ or by its symbol $\varphi = v(x, y, z)\sqrt{k_x^2 + k_y^2 + k_z^2}$ where $v$ is the velocity in the space domain.

To produce anisotropic wave propagation, without adding spurious waves, considering $F$ equal to a constant, we can use expression (22), and in this case we have

$$\varphi = \sqrt{v_{po}^2 k_z^2 + v_h^2 k_r^2 - \frac{\left(v_h^2 - v_n^2\right) k_r^2 k_z^2}{k_z^2 + F k_r^2}}. \tag{29}$$

The solution of equation (29) is given by

$$P(t + \Delta t) = \mathrm{e}^{-\mathrm{i}\Phi\Delta t} P(t). \tag{30}$$

Adding $P(t - \Delta t) = \mathrm{e}^{\mathrm{i}\Phi\Delta t} P(t)$ to equation (30), we obtain

$$p(t + \Delta t) + p(t - \Delta t) = 2\cos(\Phi\Delta t) p(t). \tag{31}$$

Now we can revert to $p$ since the imaginary part is decoupled (cosine is real). Since cosine is an even function, its expansion contains only powers of $\Phi^2$.

In order to have an efficient numerical scheme, we require that

$$\Phi^2 = \sum_j f_j(x)\, g_j(k) \tag{32}$$

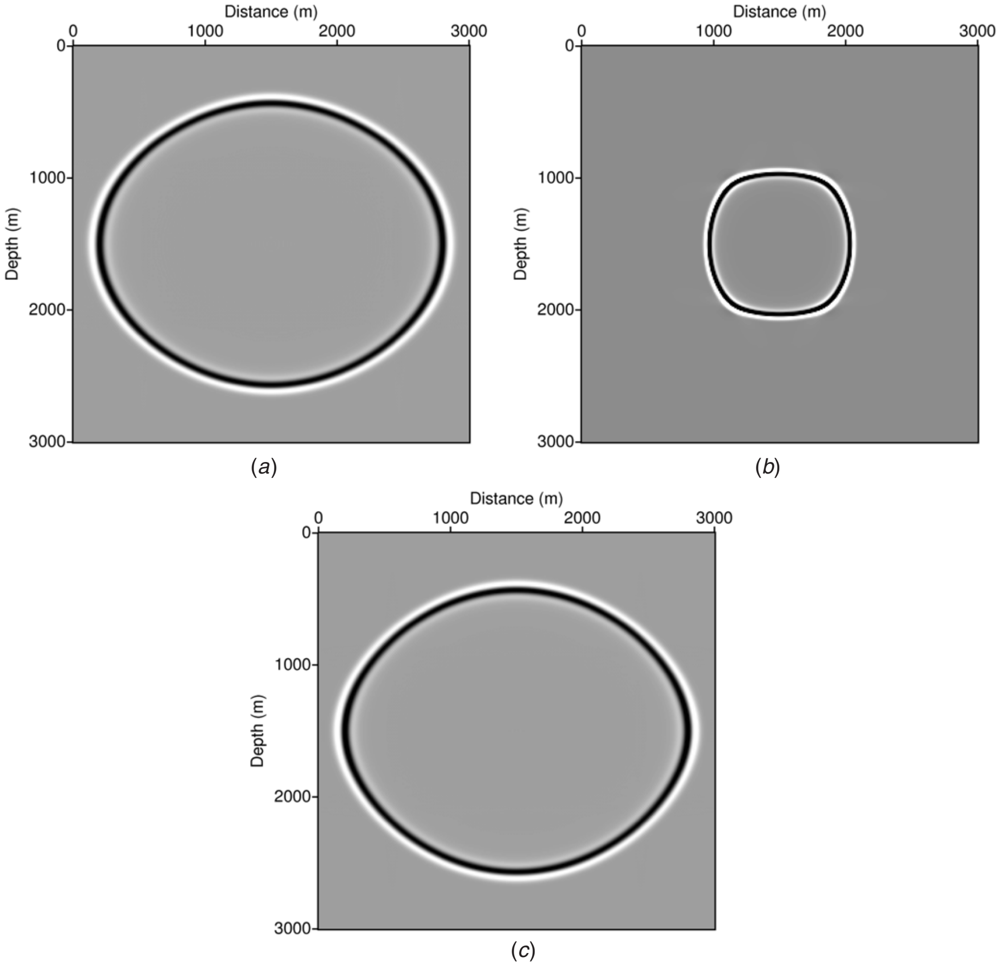



**Figure 2.** Impulse response: (*a*) pure P-wavefield and (*b*) SV-wavefield using the method presented here. The REM was used to solve equations (22) and (23). (*c*) P-wavefield solved by the REM of the dispersion relation given by equation (22) with $F = 1$).

so that, approximately,

$$\Phi^2\, p = \sum_j f_j(x)\, \mathfrak{F}^{-1}\{g_j(k)\, \mathfrak{F}(p)\}, \qquad (33)$$

where $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ denote forward and inverse spatial Fourier transforms, respectively. Such a separation is possible in equation (29) only if the factor $F$ is a constant, independent of $x$.

The cosine function can now be evaluated by the REM. We note that equations (26) and (27) are easily solved using a pseudo spectral method and their solution using a finite-difference implementation would be challenging due to the presence of the wave numbers in the denominators. To evaluate these would require further approximations. Using the REM combined with a pseudo spectral method for the spatial derivatives provides a highly accurate, numerically stable algorithm.

Additionally, using parallel computers for evaluation of the Fourier transforms makes our pseudo spectral implementation computationally feasible. For example, in 3D we simply have to replace $k_r^2$ by $(k_x^2 + k_y^2)$ in equations (26) and (27). The required forward 3D Fourier transform can be done using one bank of multiple nodes (Chu 2009). We then require two inverse 3D Fourier transforms, given sufficient nodes, both of which can also be done in parallel using two banks of nodes. Thus, the total FFT effort is the time required for one forward and inverse transform performed using multiple nodes.

The pseudo spectral REM implementation of the separated VTI equation has been extended to TTI by Zhan *et al* (2012). In this case, the number of fast Fourier transforms increases to 7 for a 2D problem and 21 for the 3D case. But these can also all be done in parallel.

## Numerical results

In our first modeling experiment, time snapshots ($t = 0.4$ s) of wave propagation in a homogeneous VTI medium ($v_{po} = 3000\ \mathrm{m\,s^{-1}}$, $\epsilon = 0.24$ and $\delta = 0.1$) are simulated with the source pulse in the center of the model. Here $\mathrm{d}x = \mathrm{d}z = 0.01$ km and we used the pseudo spectral method for the spatial derivatives in all cases.

Figures 1(*a*) and (*b*) correspond to the same time snapshots from the simulations using the REM to the system of equations (6) of Du *et al* (2008) for the p- and q-wavefields. In these figures, a diamond-shaped spurious SV-wavefront can be seen. Figure 1(*c*) shows the P-wave and figure 1(*d*) shows the SV component computed from the decoupled P- and SV-wave equations (15) and (16) proposed by Liu *et al* (2009) and also solved by the REM. The system of equations introduced by Du

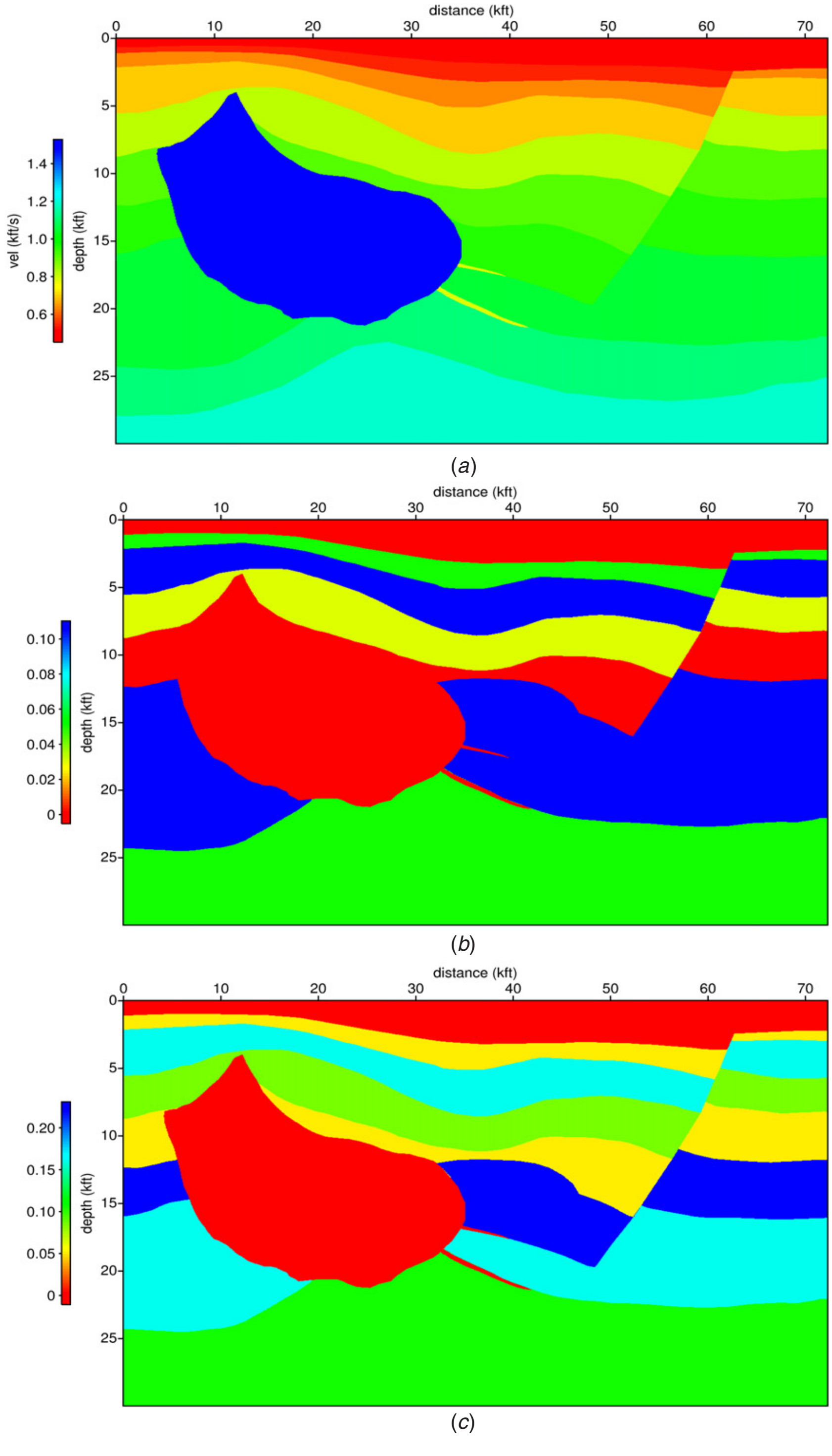
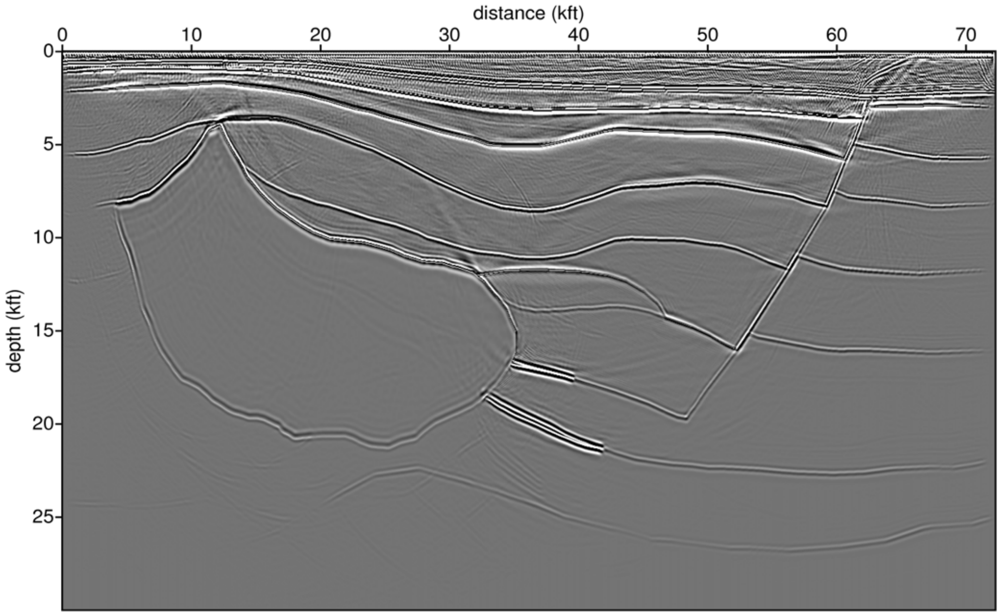


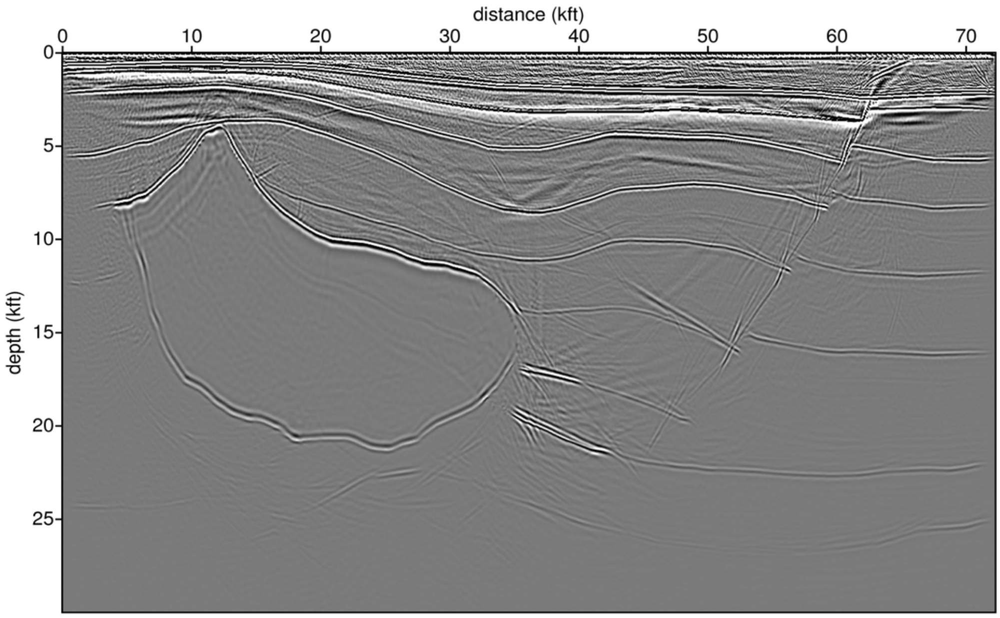

**Figure 3.** Salt model: (*a*) velocity field, (*b*) delta and (*c*) epsilon.

*et al* (2008) produces a strong unwanted spurious SV-wave, and it is possible that SV-wave artifacts will contaminate RTM images. Using the decoupled P- and SV-waves, it is clear that figure 1(*c*) has only a P-wave component, while figure 1(*d*) has only a SV-wave component. Therefore, a pure P-wave equation offers a better imaging alternative since it does not have the SV-wave artifacts.

To demonstrate the applicability of the separated P-wave equation using our numerical technique, we use the same anisotropic model parameters. We also use the same grid and implementation method. In figure 2(*a*), we have the same time snapshot ($t = 0.4$ s) from simulation using the REM for the pure P-wave (equation (26)) and in figure 2(*b*), we have the SV-wave component (equation (27)). Both were computed using $F = \frac{1+2\epsilon-\gamma^2}{1-\gamma^2}$ with $\gamma^2 = \frac{v_{so}^2}{v_{po}^2} = 1/4$. In figure 2(*c*), we also have the simulation of the pure P-wave, but with $F = 1$, which is the dispersion relation used by Etgen and Brandsberg-Dahl (2009). The pure-P-wave results presented in figures 2(*a*) and (*c*) are quite close. And the SV-wave using the separated SV-wave equation proposed here (equation (27)) is stable. In this




(*a*)




(*b*)

**Figure 4.** (*a*) Anisotropic RTM by the REM using the Du *et al* (2008) equations and the correct VTI model parameters. (*b*) Isotropic RTM also solved by the REM.

SV-wave equation, the vertical shear wave velocity is not zero which removes the SV-wavefront artifact and results in stable wave propagation (Tsvankin 2001).

Next an anisotropic salt model is used to test the performance of imaging quality with both the coupled and separated equations solved by the pseudo spectral method and the REM. The input 2D synthetic data were generated for this model using elastic finite-difference modeling. The vertical P-wave velocity is shown in figure 3(*a*). The $\epsilon$ and $\delta$ parameters are shown in figures 3(*b*) and (*c*), respectively.

Figure 4(*a*) shows the anisotropic RTM result of the P-wavefield, using the REM for the Du *et al* (2008) system of equations (6). This result was obtained using the correct model parameters but with $v_{so} = 0$. For comparison, we show the isotropic RTM results (figure 4(*b*)) which were imaged using the vertical P-wave velocity. In figure 4(*a*), the anisotropic migration improves the image of the steeply dipping reflectors including the faulted bed and salt body. It also correctly images the reflector in the center of the section which is caused by variations only in the anisotropic parameters. This event nearly disappears in the isotropic image and generates some noise artifacts.

In figure 5(*a*), we show the result of prestack RTM using our equation (26) with $F$ computed with $\frac{v_{so}^2}{v_{po}^2} = 1/4$ and $\epsilon$ equal to the maximum value in the model. Figure 5(*b*) shows the prestack RTM image obtained using $F = 1$ as proposed by Etgen and Brandsberg-Dahl (2009). We see that these two images are very similar to the results obtained using the

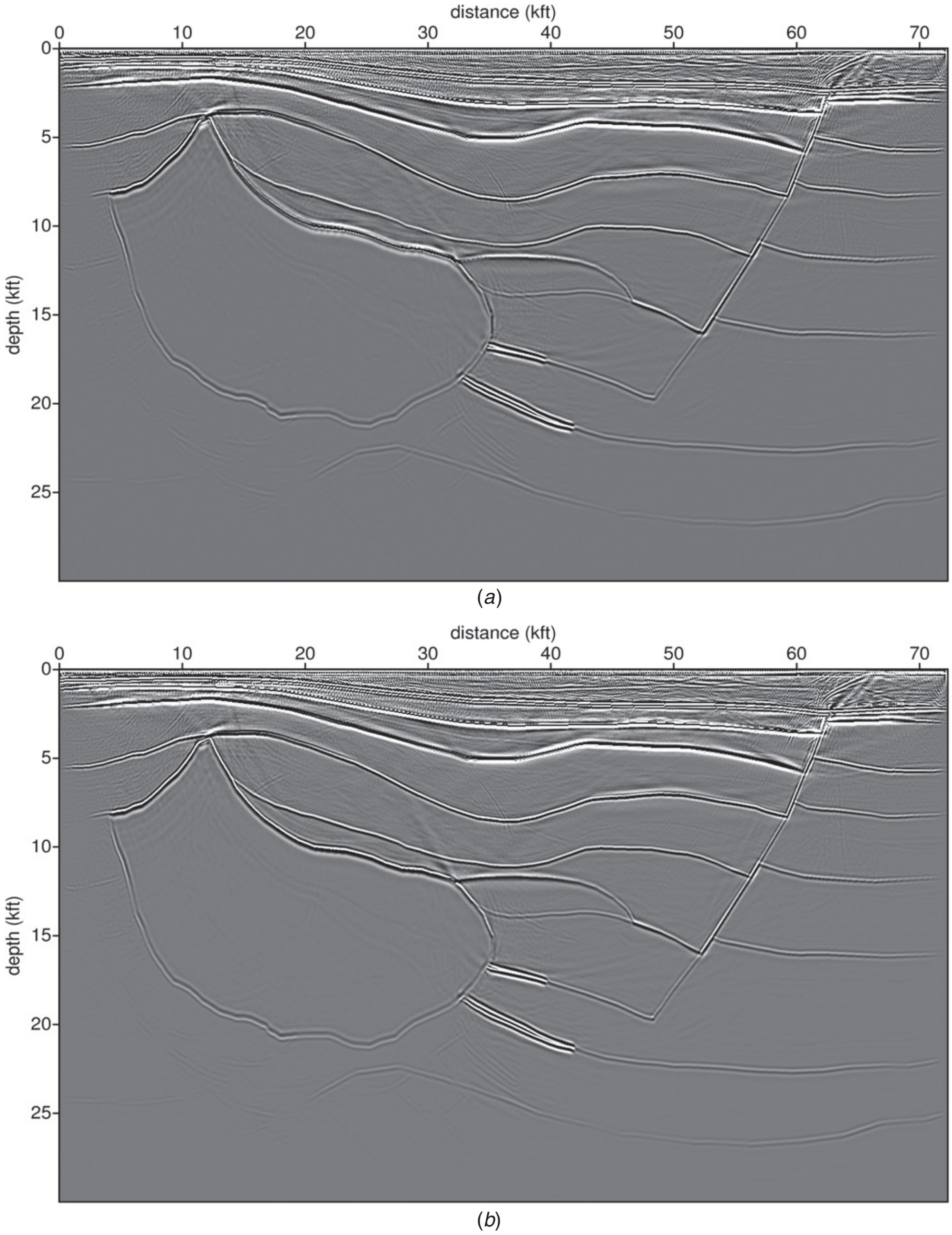



**Figure 5.** Anisotropic RTM by the REM with our pure-P-wave equation (equation (22)) (*a*) and with the solution proposed by Etgen and Brandsberg-Dahl (2009) (equation (22) with $F = 1.0$) (*b*).

equations from Du *et al* (2008). But the separated P-wave equation and the one proposed by Etgen and Brandsberg-Dahl (2009) are computationally more efficient.

## Conclusions

The pseudo-acoustic approximation for wave propagation in VTI media, where $v_{so} = 0$, results in a reasonable approximation for P-waves, but generates a degenerate SV-wave. This results in SV-wave noise for P-wave propagation. Separating the exact dispersion relation for VTI media results in stable approximate equations for both P- and SV-waves. These can be implemented efficiently using a pseudo spectral method for the space derivatives and the REM for the explicit time marching. This combination yields a practical and computationally efficient method of solution which is numerically stable and has minimum numerical dispersion.

## Acknowledgments

RCP and PLS would like to acknowledge support received for this research from King Abdullah University of Science and Technology (KAUST). BU has received support from VISTA and the Norwegian Research Council through the ROSE project. The authors would like to thank Paul Fowler for a detailed review of an early version of the manuscript. Finally, the authors also thank Amerada Hess for making the synthetic data set available.

## Appendix

*Improved dispersion relations*

We may write equation (17) as

$$\frac{v^2(\theta)}{v_{po}^2} = \frac{1}{2}\left[A + 2\,\frac{v_{so}^2}{v_{po}^2} \pm A\left(1 - \frac{B}{A^2}\right)^{1/2}\right], \qquad \text{(A.1)}$$

with

$$A = f + 2\epsilon \sin^2\theta$$
$$B = 2f(\epsilon - \delta)\sin^2 2\theta. \qquad \text{(A.2)}$$

We first use the approximation $(1-X)^{1/2} = 1 - \frac{1}{2}X$ to obtain equations (22) and (23).

Next we use the approximation

$$(1-X)^{1/2} = 1 - \frac{1}{2}X - \frac{1}{8}X^2 + \cdots \approx 1 - \frac{\frac{1}{2}X}{1 - \frac{1}{4}X} \qquad \text{(A.3)}$$

to obtain

$$A\left(1 - \frac{B}{A^2}\right)^{1/2} \approx A - \frac{2BA}{4A^2 - B}. \qquad \text{(A.4)}$$

We compute

$$\frac{BA}{4A^2 - B} = \frac{(\epsilon-\delta)\sin^2 2\theta\ \left(1 + \frac{2\epsilon}{f}\sin^2\theta\right)}{2\left[1 + \frac{2(\epsilon+\delta)}{f}\sin^2\theta + \left(\frac{2(\epsilon-\delta)}{f} + 4\left(\frac{\epsilon}{f}\right)^2\right)\sin^4\theta\right]}. \qquad \text{(A.5)}$$

We neglect the second-order terms in $\epsilon$ and $\delta$ to obtain

$$\frac{BA}{4A^2 - B} \approx \frac{(\epsilon-\delta)\sin^2 2\theta}{2\left[1 + \frac{2(\epsilon+\delta)}{f}\sin^2\theta + \frac{2(\epsilon-\delta)}{f}\sin^4\theta\right]}. \qquad \text{(A.6)}$$

When equation (A.6) is used in equation (A.1), we obtain an improved approximation for P-waves:

$$\frac{v^2(\theta)}{v_{po}^2} = 1 + 2\epsilon\sin^2\theta - \frac{(\epsilon-\delta)\sin^2 2\theta}{2\left[1 + \frac{2(\epsilon+\delta)}{f}\sin^2\theta + \frac{2(\epsilon-\delta)}{f}\sin^4\theta\right]} \qquad \text{(A.7)}$$

and

$$\frac{v^2(\theta)}{v_{po}^2} = \frac{v_{so}^2}{v_{po}^2} + \frac{(\epsilon-\delta)\sin^2 2\theta}{2\left[1 + \frac{2(\epsilon+\delta)}{f}\sin^2\theta + \frac{2(\epsilon-\delta)}{f}\sin^4\theta\right]} \qquad \text{(A.8)}$$

for SV-waves.

The new dispersion relation for P-waves is

$$\omega^2 = v_{po}^2 k_z^2 + v_h^2 k_r^2 - \frac{v_{po}^2\, 2(\epsilon-\delta) k_r^2 k_z^2 \left(k_r^2 + k_z^2\right)}{k_z^4 + 2F_1 k_r^2 k_z^2 + F_2 k_r^4} \qquad \text{(A.9)}$$

and for SV-waves

$$\omega^2 = v_{so}^2\left(k_r^2 + k_z^2\right) + \frac{v_{po}^2\, 2(\epsilon-\delta) k_r^2 k_z^2 \left(k_r^2 + k_z^2\right)}{k_z^4 + 2F_1 k_r^2 k_z^2 + F_2 k_r^4}. \qquad \text{(A.10)}$$

Here,

$$F_1 = 1 + \frac{\epsilon + \delta}{f} \quad \text{and} \quad F_2 = 1 + \frac{4\epsilon}{f}. \qquad \text{(A.11)}$$

If se set $F_1 = F_2 = 1$ in these expressions, equations (A.9) and (A.10) reduce to equations (22) and (23) for $F = 1$.

## References


Alkhalifah T 1998 Acoustic approximations for processing in transversely isotropic media *Geophysics* **63** 623–31

Alkhalifah T 2000 An acoustic wave equation for anisotropic media *Geophysics* **65** 1239–50

Carcione J M, Herman G C and ten Kroode A 2002 Seismic modeling *Geophysics* **67** 1304–25

Chu C 2009 Seismic modeling and imaging with the Fourier method: Numerical analyses and parallel implementation strategies *PhD Thesis* The University of Texas, Austin, TX

Crawley S, Brandsberg-Dahl S, McClean J and Chemingui N 2010 TTI reverse time migration using the pseudo-analytic method *Leading Edge* **29** 1379–84

Dellinger J and Muir F 1985 Two domains of anisotropic *Stanford Explor. Proj. Rep.* **44** 59–62

Dellinger J, Muir F and Karrenbach M 1993 Anelliptic approximations for TI media *J. Seism. Explor.* **2** 23–40

Du X, Fletcher R and Fowler P J 2008 A new pseudo-acoustic wave equation for VTI media *70th EAGE Annual Conf. and Exhibition* Extended Abstract H033

Duveneck E, Milcik P M B P and Perkins C 2008 Acoustic VTI wave equations and their application for anisotropic reverse-time migration *78th Annual Inter. Meeting, SEG* Expanded Abstract pp 2186–90

Etgen J 1986 High-order finite-difference reverse time migration with the two-way nonreflecting wave equation *Stanford Explor. Proj. Rep.* **48** 133–46

Etgen J and Brandsberg-Dahl S 2009 The pseudo-anatical method: application of pseudo-Laplacians to acoustic and acoustic anisotropic wave propagation *79th Annual Int. Meeting, SEG* Expanded Abstract pp 2552–5

Fowler P J 2003 Practical VTI approximations: a systematic anatomy *J. Appl. Geophys.* **54** 347–68

Fowler P J, Du X and Fletcher R P 2010a Coupled equations for reverse time migration in transversely isotropic media *Geophysics* **71** S11–22

Fowler P J, Du X and Fletcher R P 2010b Recursive integral extrapolation methods for scalar waves *80th Annual Inter. Meeting, SEG* Expanded Abstract pp 3210–15

Grechka V, Zhang L and Rector J W 2004 Shear waves in acoustic anisotropic media *Geophysics* **69** 576–82

Harlan W 1995 Flexible seismic traveltime tomography applied to diving waves *Stanford Explor. Proj. Rep.* **89** 145–64

Kosloff D, Filho A Q, Tessmer E and Behle A 1989 Numerical solution of the acoustic and elastic wave equations by a new rapid expansion method *Geophys. Prospect.* **37** 383–94

Liu F, Morton S A, Jiang S, Ni L and Leveille J P 2009 Decoupled wave equation for P and SV waves in an acoustic VTI media *79th Annual Inter. Meeting, SEG* Expanded Abstract pp 2844–8

Lu R 2010 Elastic RTM: anisotropic wave-mode separation and converted-wave polarization correction *80th Annual Int. Meeting, SEG* Expanded Abstract pp 3171–5

Pestana R C and Stoffa P L 2010 Time evolution of wave equation using rapid expansion method *Geophysics* **75** T121–31

Song X, Fomel S, Ying L and Ding T 2011 Lowrank finite-difference for wave extrapolation *81st Annual Int. Meeting, SEG* Expanded Abstract pp 3372–6

Soubaras R and Zhang Y 2008 Two-step explicit marching method for reverse time migration *70th EAGE Conf. and Exhibition (Rome, Italy)* pp 2272–76

Stopin A 2001 Comparison of equations in TI medium *9th Int. Workshop on Seismic Anisotropic*

Tal-Ezer H, Kosloff D and Koren Z 1987 An accurate scheme for forward seismic modeling *Geophys. Prospect.* **35** 479–90

Thomsen L 1986 Weak elastic anisotropy *Geophysics* **51** 1954–66

Tsvankin I 1996 P-wave signature and notation for transversely isotropic media: an overview *Geophyscis* **61** 467–83

Tsvankin I 2005 *Seismic Signatures and Analysis of Reflection Data in Anisotropic Media* 2nd edn (Amsterdam: Elsevier)

Yan J and Sava P 2009 Elastic wave-mode separation for VTI media *Geophysics* **74** WB19–32

Yan R and Xie X B 2010 The new angle-domain imaging condition for elastic RTM *80th Annual Int. Meeting, SEG* Expanded Abstract pp 3181–6

Zhan G, Pestana R C and Stoffa P L 2012 Decoupled equations for reverse time migration in tilted transversely isotropic media *Geophysics* **77** T37–45

Zhang Y and Zhang G 2009 One-step extrapolation method for reverse time migration *Geophysics* **74** A29–33

Zhou H G Z G and Bloor R 2006 An anisotropic acoustic wave equation for modeling and migration in 2D TTI media *76th Annual Int. Meeting, SEG* Expanded Abstract pp 194–8