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RELIABILITY PAPER

Generalized q-Weibull model and the bathtub curve

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Abstract

Purpose – The purpose of this paper is to analyze mathematical aspects of the q-Weibull model and explore the influence of the parameter q.

Design/methodology/approach – The paper uses analytical developments with graph illustrations and an application to a practical example.

Findings – The q-Weibull distribution function is able to reproduce the bathtub shape curve for the failure rate function with a single set of parameters. Moments of the distribution are also presented.

Practical implications – The generalized q-Weibull distribution unifies various possible descriptions for the failure rate function: monotonically decreasing, monotonically increasing, unimodal and U-shaped (bathtub) curves. It recovers the usual Weibull distribution as a particular case. It represents a unification of models usually found in reliability analysis. Q-Weibull model has its inspiration in nonextensive statistics, used to describe complex systems with long-range interactions and/or long-term memory. This theoretical background may help the understanding of the underlying mechanisms for failure events in engineering problems.

Originality/value – Q-Weibull model has already been introduced in the literature, but it was not realized that it is able to reproduce a bathtub curve using a unique set of parameters. The paper brings a mapping of the parameters, showing the range of the parameters that should be used for each type of curve.

Keywords Failure rate, Generalized Weibull, Reliability, Bathtub curve, Reliability management, Distribution curves, Distribution functions

Paper type Research paper

1. Introduction

Reliability analysis widely uses Weibull (1951) distribution, that is a simple and powerful empirical model. Many branches of knowledge have applied this distribution. These are some recent examples: service operations (Hensley and Utley, 2011), the problem of the strength of a manufactured item against stress (Ali and Kannan, 2011) and large-scale information systems supporting infrastructures deterioration process formulated by a Weibull hazard model (Kobayashi and Kaito, 2011). Weibull probability density function (pdf) at time t, where $t < T$ and T is time to failure, is given by:

$$f(t) = \frac{\beta}{\eta - t_0}\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta - 1} \exp\left[-\left(\frac{t - t_0}{\eta - t_0}\right)^{\beta}\right] \quad (1)$$

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with $\beta > 0$, $\eta > t_0$, $t \geq t_0$, and $\int_0^\infty f(x)dx = 1$. Equation (1) may be viewed as a generalization of the exponential distribution, which is recovered if parameter β is taken as unity.

Various generalizations of Weibull model have been proposed: linear or nonlinear transformation of time, use of multiple distributions, time dependence of parameters, discrete, multivariate, stochastic models, etc. (Murthy *et al.*, 2004) for a comprehensive approach). Xie *et al.* (2000) compares the approximated exponential distribution using the average failure rate with the Weibull reliability. Almost all proposals of generalization of Weibull model share a common feature: they rely on the exponential framework (single exponential, exponentials of a variety of functions and so forth).

In the following we briefly point out some theoretical remarks about the emergence of exponential and non exponential distributions in statistical mechanics, which serve as motivation for our approach to the problem. Exponentials are usually found in non-interacting or weakly interacting systems. Systems that exhibit long-range (spatial) interactions, long-term (temporal) memory, effects of competition/cooperation, among others, usually can be classified as complex (Bak, 1997) and power-laws dominate their statistical distributions, in contrast to simple systems, that are the realm of exponential laws. Failure of a component may have many (recent or not) multiple and interacting causes, some of them acting on a cooperative and others on a conflictive basis, so it is not surprising that complex behavior may appear. If this happens, power-law-like expressions are expected to substitute exponentials in the statistical description.

Statistical mechanics of simple systems has a well established theoretical framework, and probability distributions with exponentials (e.g. Boltzmann weight, Maxwellian distribution among many others) are derived from Boltzmann-Gibbs-Shannon (BGS) entropy. On the other hand, theoretical basis of the statistical description of complex systems is object of intense current research.

The definition of the nonextensive entropy (Tsallis, 1988), which is a generalization of BGS entropy (by means of a parameter q, also known as entropic index), has introduced the possibility to extend statistical mechanics to complex systems in a coherent and natural way. The developments surpassed the bounds of physics and have lead to applications in different areas, including topics in applied mathematics (Tsallis, 2009). We focus on the q-exponential function, which naturally appears in nonextensive formalism, defined as:

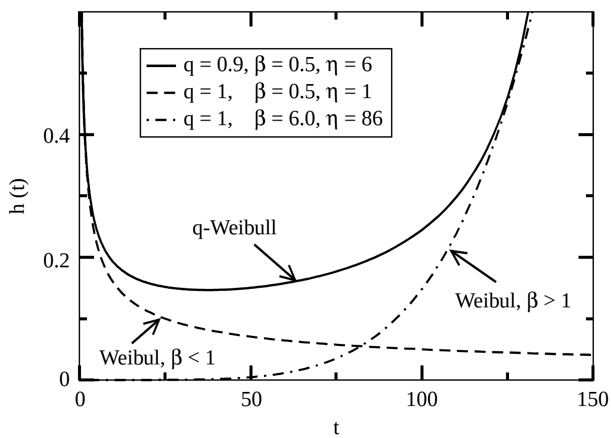
$$\exp_q(x) = \begin{cases} (1+(1-q)x)^{(1/(1-q))}, & \text{if } (1+(1-q)x) \geq 0 \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

with $x, q \in \mathrm{R}$. The q-exponential is reduced to the usual exponential function in the limiting case $q \rightarrow 1$ ($\exp_1 x = \exp x$), and thus equation (2) is a generalization of the later. The definition of the q-exponential brings a cut-off condition that prevents negative or even complex values. This is an important feature whenever the function is to be associated with probabilities. For certain values of the parameters the q-exponential presents a cross-over between an exponential behavior and a power-law regime ($\exp_q(-ax)$ with $a > 0$ and $q > 1$ is asymptotically a power-law for large x, leading to fat-tailed distributions).

The q-exponential has been applied to different contexts in pure and applied mathematics. For the present purposes we are particularly interested in the applications in probability distributions. The q-gaussian distribution (Tsallis *et al.*, 1996; Prato and Tsallis, 1999) generalizes the gaussian (recovered for $q = 1$), and also the Cauchy-Lorentz distribution (recovered for $q = 2$), among others. The central limit theorem has been generalized into its "q-version" in Tsallis (2005) and Umarov *et al.* (2008).

If we look to Weibull distribution on the light of nonextensive statistics, a natural step forward is its generalization with q-exponentials, and this was done in Picoli *et al.* (2003), with applications in frequency distributions for different systems. To the best of our knowledge, the first use of q-Weibull distribution in reliability analysis was presented in Costa *et al.* (2006). It was applied to describe time-to-breakdown during the dielectric breakdown regime of ultra-thin oxides in electronic devices. q-Weibull pdf was also used to model data of the New York Stock Exchange and the Helsinki Stock Exchange (Vuorenmaa, 2006).

The aim of the present paper is to recall q-Weibull model and to analyze some features and details that are important to reliability analysis and were not covered earlier. It is a continuation of a previous paper (Sartori *et al.*, 2009), in which we have done a preliminary study of the applicability of q-Weibull distribution. The present paper shows that q-Weibull distribution is able to reproduce various types of failure rate behaviors: monotonically decreasing, monotonically increasing, unimodal and U-shaped (bathtub curve). The possibility to use q-Weibull to describe the bathtub curve was not realized by previous papers, nor the bathtub curve was well described by other models using a single set of parameters for its three characteristic regions. Before introducing the model (what is done in the next section), we show Figure 1 that compares Weibull distribution with the q-Weibull distribution. Two curves of the Weibull distribution are displayed, a decreasing function (with shape parameter $\beta < 1$), and an increasing function (with shape parameter $\beta > 1$). The q-Weibull model approximates both curves, for small and large values of time, and properly interpolates in-between, generating the curve with the bathtub shape.



Notes: The displayed q-Weibull distribution is a generalization of ordinary Weibull and is able to represent the bathtub curve; the values of the parameters were chosen just to give a good visual representation

Figure 1.
Comparison of two instances of the Weibull distribution, the decreasing curve with shape parameter $\beta = 0.5$, and the increasing curve with shape parameter $\beta = 6$

Section 2 introduces the model and some of its features are shown in Section 3. Section 4 brings an example and our conclusions and final remarks are developed in Section 5.

2. *q*-Weibull failure rate model

The q-Weibull model is obtained from the classical Weibull model (equation (1)) by the substitution of the exponential function by a q-exponential (see details in Costa *et al.* (2006)):

$$f_q(t) = (2-q)\frac{\beta}{\eta - t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1}\exp_q\left[-\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]. \quad (3)$$

The factor $(2-q)$ and the constraint $q < 2$ are necessary due to normalization requirements. The ordinary Weibull pdf is recovered in the limit $q \to 1$, and coherently equation (1) shall now be denoted as $f_1(t)$. η is the scale parameter and t_0 is the location parameter of q-Weibull model as well as in Weibull model. However, both the parameters β and q control the shape of the q-Weibull distribution, while in the Weibull model, only the parameter β affects its shape.

The q-Weibull distribution is also a generalization of Burr XII distribution function (Burr, 1942):

$$f(t) = ck\frac{t^{c-1}}{s^c}\left[1+\left(\frac{t}{s}\right)^c\right]^{-k-1} \quad (k>0, c>0, s>0), \quad (4)$$

if the parameters of q-Weibull are taken as $\beta = c$, $\eta = s/(k+1)^{1/c}$ and $q = (k+2)/(k+1) > 1$. It is worth a mention that q-Weibull is a generalization of Burr XII, and not the opposite, as claimed by Nadarajah and Kotz (2006), once equation (4) demands $q > 1$, while equation (3) is also defined for $q \leq 1$. Burr XII distribution can assume different shapes, which allow it to be a good candidate to fit various lifetimes data.

The q-Weibull reliability function is consistently given by $R_q(t) = \int_t^\infty f_q(t')dt'$, i.e.:

$$\begin{aligned} R_q(t) &= \left[1-(1-q)\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]_+^{(2-q)/(1-q)} \\ &= \left[\exp_q\left[-\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]\right]^{2-q}, \end{aligned} \quad (5)$$

where we use the symbol $[A]_+$ (first line of equation (5)), that means that $[A]_+ = A$ if $A \geq 0$ and $[A]_+ = 0$ if $A < 0$. This is already implicit in equation (2): we use it here and also in some equations in the following just to remind the reader of the cut-off condition of the q-exponential. To deduce equation (5) we have used the following property of the q-exponential function:

$$\int \exp_q(ax)dx = \frac{1}{(2-q)a}[\exp_q(ax)]^{2-q}. \quad (6)$$

Note that $(\exp_q x)^a \neq \exp_q(ax)$ for $q \neq 1$, but:

$$(\exp_q x)^a = \exp_{1-(1-q)/a}(ax), \quad (7)$$

So that equation (5) may be alternatively written as $R_q(t) = \exp_{q'}[-(2-q)((t-t_0)/(\eta-t_0))^{\beta}]$ with $q' = 1/(2-q)$. The interested reader may find more properties of q-exponentials at Yamano (2002).

The cumulative distribution function $F_q(t)$ is the complement to the reliability function, $F_q(t) = 1 - R_q(t)$, and the instantaneous failure rate, defined as $h_q(t) \equiv f_q(t)/R_q(t)$ is generalized to:

$$\begin{aligned} h_q(t) &= \frac{(2-q)\beta}{\eta-t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1} \times \left[1-(1-q)\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]_+^{-1} \\ &= \frac{(2-q)\beta}{\eta-t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1} \times \left[\exp_q\left[-\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta}\right]\right]^{q-1}, \end{aligned} \tag{8}$$

which is consistently reduced to the usual Weibull version as $q \rightarrow 1$:

$$h_1(t) = \frac{\beta}{\eta - t_0}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-1}. \tag{9}$$

This is precisely the origin of the difference of behaviors between usual ($q = 1$) and q-Weibull models: the integral of an ordinary exponential is an exponential (except from a multiplicative constant), and they cancel out in the expression for the failure rate with $q = 1$. That does not happen with $h_q(t)$, due to the property given by equation (6).

Equation (8) is able to represent four different types of failure rate function, according to the values of the parameters, besides the constant type (with $q = 1$ and $\beta = 1$). $h_q(t)$ is monotonically decreasing for $1 < q < 2$ and $0 < \beta < 1$, monotonically increasing for $q < 1$ and $\beta > 1$, unimodal for $1 < q < 2$ and $\beta > 1$ and U-shaped (bathtub curve) for $q < 1$ and $0 < \beta < 1$. The non-monotonic hazard function cited by Vuorenmaa (2006) corresponds to the unimodal type and the bathtub shape was not covered by that paper. Figure 2 shows the four possibilities (detailed analysis of the q parameter is performed in Section 3), and Figure 3 shows the corresponding four unreliability curves.

For $q < 1$, equation (8) presents a divergence that defines the maximum allowed time (lifetime deadline) at:

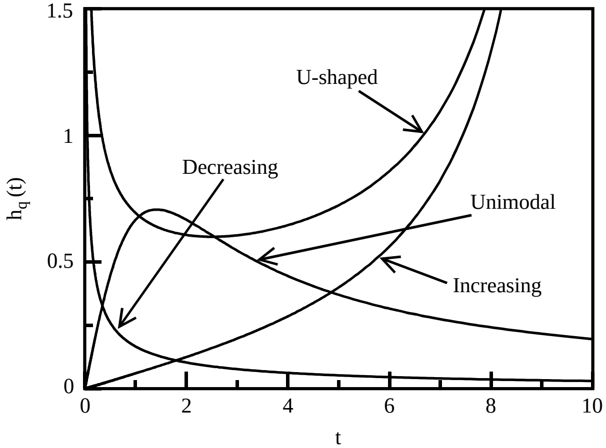
$$t_{max} = t_0 + (\eta - t_0)(1-q)^{-1/\beta}. \tag{10}$$

Finite t_{max} corresponds to a relaxation of the constraint usually imposed to a cumulative failure rate function $H_q(t) = \int_0^t h_q(t)dt$ (Pham and Lai, 2007): it is normally expected that $H_1 \rightarrow \infty$ at $t \rightarrow \infty$. According to q-Weibull model, $H_{q<1} \rightarrow \infty$ at $t \rightarrow t_{max} < \infty$. That is to say that ordinary Weibull is unlimited, while q-Weibull (with $q < 1$) is limited to t_{max}. Coherently, $\lim_{q\rightarrow(1^-)} t_{\max} \rightarrow \infty$, as q approaches the unity from the left.

The time derivative of the q-failure rate is:

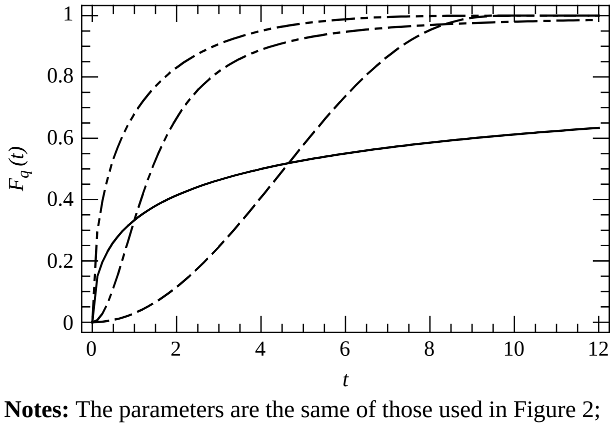
$$h'_q(t) = \frac{(2-q)\beta(\beta-1)}{(\eta-t_0)^2}\left(\frac{t-t_0}{\eta-t_0}\right)^{\beta-2} \frac{[1-((1-q)/(1-\beta))((t-t_0)/(\eta-t_0))^{\beta}]}{[1-(1-q)((t-t_0)/(\eta-t_0))^{\beta}]_+^2}. \tag{11}$$

For the unimodal case ($1 < q < 2$ and $\beta > 1$) and for the U-shaped case ($q < 1$ and $0 < \beta < 1$), the root of equation (11) is located at:



Notes: Values of the parameters were chosen to give a good visualization of the curves in the same figure; the four types are: (i) monotonically decreasing function: (q = 1.5, = 0.5, = 1); (ii) monotonically increasing function: (q = 0.5, β = 2, η = 7.071, evaluated from equation (10) with t_{max} = 10); (iii) unimodal function: (q = 1.5, β = 2, η = 1); (iv) U-shaped (bathtub curve): (q = 0.5, β = 0.5, η = 2.5, evaluated from equation (10) with t_{max} = 10

Figure 2. Types of failure rate curves described by q-Weibull



Notes: The parameters are the same of those used in Figure 2; the four types of failure rate associated are: (i) monotonically decreasing (solid line); (ii) monotonically increasing (dashed line); (iii) unimodal (dot-dashed line); (iv) U-shaped (bathtub curve) (dot-dot-dashed line)

Figure 3. Unreliability curves of the q-Weibull distribution

$$t^* = t_0 + (\eta - t_0)\left(\frac{1-\beta}{1-q}\right)^{1/\beta}, \tag{12}$$

which corresponds to the extreme value (maximum for unimodal case, minimum for bathtub case):

$$h_q(t^*) = \frac{2-q}{\eta - t_0}\left(\frac{1-\beta}{1-q}\right)^{(\beta-1)/\beta}. \tag{13}$$

Figure 4 shows the change of sign in time derivative of $h_q(t)$.

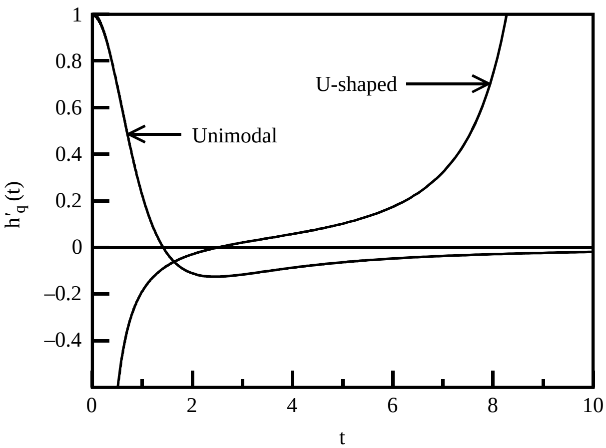
The time derivative of the usual ($q = 1$) Weibull failure rate is a monotonic power-law:

$$h_1'(t) = \frac{\beta(\beta-1)}{(\eta - t_0)^2}\left(\frac{t-t_0}{\eta - t_0}\right)^{\beta-2}, \tag{14}$$

Hence it is unable to represent the whole bathtub curve. $h_1'(\mathrm{t}) < 0$ for $0 < \beta < 1$, and this situation can just describe the warm in phase. Wear out phase needs $h_1'(\mathrm{t}) > 0$, and this happens in usual Weibull for $\beta > 1$. Description of intermediary random failure phase happens by imposing $\beta = 1$. q-Weibull failure rate reproduces the whole curve by a continuous function with the same set of parameters.

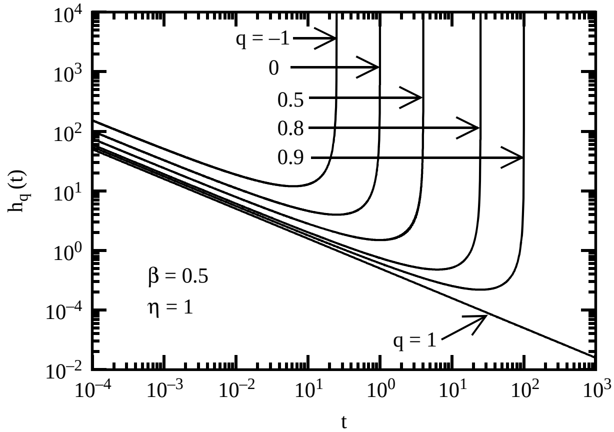
3. Influence of the parameter *q*

In order to exhibit the effect of the parameter $q < 1$ on the q-Weibull model, let us consider the instance $\beta = 0.5$. First we keep parameter η constant (let us assume $\eta = 1$ for simplicity). The usual ($q = 1$) Weibull does not present a limiting lifetime (i.e. $t_{max} = \infty$). As q departs from unity (from below), lifetime deadline gets smaller values, as Figure 5 shows. Second, let us keep t_{max} constant (we choose the instance $\beta = 0.5$ and $t_{max} = 100$),



Notes: Parameters are the same of those in Figure 2 (monotonic cases are not shown); the change of sign in $h'_q(\mathrm{t})$ is responsible for the proper description of the whole bathtub curve

Figure 4.
Time derivative $h_q'(\mathrm{t})$, given by equation (11), with $q = 0.5$, $\beta = 0.5$ and $\eta = 2.5$ (corresponding to the U-shaped curve) and $q = 1.5$, $\beta = 2$ and $\eta = 1$ (corresponding to the unimodal curve)



Notes: All curves are calculated with β = 0.5 and η = 1; limiting lifetime comes from $t = \infty$, for q = 1 to closer and finite values, as the parameter q departs from unity from below

Figure 5.
q-Weibull failure rate curve as a function of time for different values of $q < 1$ in log-log scale

so η is obtained according to equation (10). Figure 6 shows curves for different values of q. As q approaches unity (from below), intermediate random failure phase decreases and minimum of failure rate (equation (13)) increases. Particularly $\lim_{q\to(1^-)} h_q(t^*) \to \infty$. Minimum value of h_q is found at $\lim_{\beta\to 1}\lim_{q\to -\infty} h_q(t^*) = 1/t_{max}$.

Influence of q on unimodal case ($1 < q < 2$ and $\beta > 1$) can be viewed in Figure 7. There is a displacement of the maximum failure rate as q approaches the value 2.

For $1 < q < 2$ and $0 < \beta < 1$, q-Weibull failure rate is a monotonically decreasing function and Figure 8 shows examples.

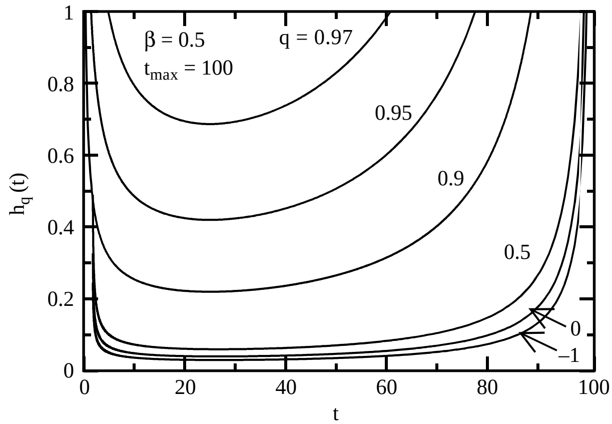
4. Examples

We illustrate the flexibility and the reach of the q-Weibull distribution, in comparison to the usual Weibull model, with three examples, extracted from components of oil wells.

We maximize the coefficient of determination R^2 of the estimated unreliability $\hat{F}_i = (i - 0.3)/(n + 0.4)$ (Bernard's approximation), for each sample i (n is the total number of samples), properly linearized as $y_i = \ln[-\ln_{q'}(1 - \hat{F}_i)]$ vs $x_i = \ln(t_i - t_0)$, with $q' = 1/(2 - q)$. Note that the usual procedure must be changed, as we are dealing with the q-Weibull model, and thus there is a q-logarithm within the expression of y_i. As an additional criterion to evaluate the goodness of the fittings, we also evaluate the mean squared error (MSE).

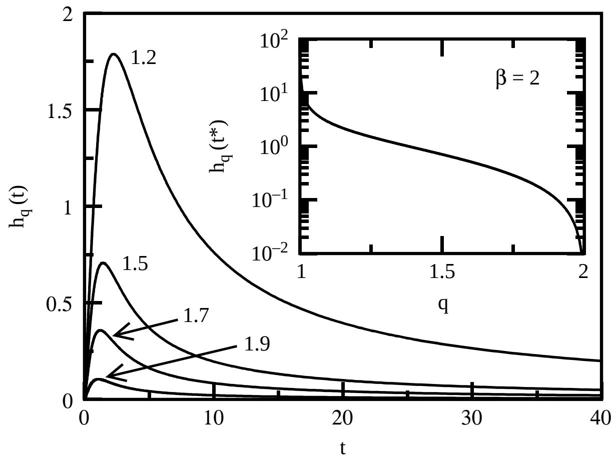
When the censoring of the data is simple type-I, type-II or multiply censored data $\hat{F}_i$ is corrected as done with Weibull distribution (Rinne, 2008).

Tables I-III present times to failure data (in days) of oil pumps, pumping rods, and production tubings, respectively. In Table II, 1,448 times to failure of pumping rods were grouped within 20 time intervals, and the relative frequency of occurrence was used to estimate the unreliability, for each interval. Table III shows 115 different values of time to failure (repetitions were excluded from 438 samples in order to reduce the size of



Notes: All curves are calculated with $\beta = 0.5$ and $t_{max} = 100$, so η is taken from equation (10): $\eta = 0.09, 0.25, 1, 25, 100, 400$ corresponds to $q = 0.97, 0.95, 0.9, 0.5, 0, -1$, respectively; as q approaches unity, intermediate random failure phase decreases, and minimum value of failure rate $h_q(t^*)$ increases ($h_q(t^*) \to \infty$ for $q \to 1$). As $q \to -\infty$, curves tend to a lower bound (this particular instance, $h_q(t^*) = 0.02$, from equation (9) with $\beta = 0.5$ and $t_{max} = 100$)

Figure 6.
q-Weibull failure rate curve as a function of time for different values of $q < 1$

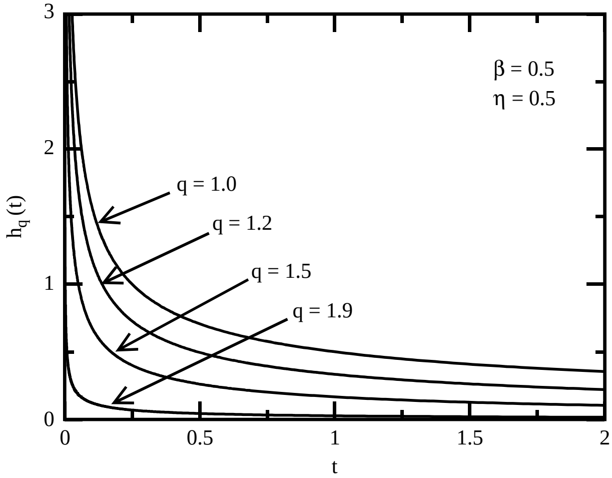


Note: Inset shows maximum of failure rate as a function of q (equation (13))

Figure 7.
q-Weibull failure rate for unimodal case, with $\beta = 2$ and $\eta = 1$, and different values of $q > 1$ (indicated)

the table). All 438 sample values were used to estimate the unreliability, according to the median rank.

Table IV shows the fitting parameters for each example, and also the coefficient of determination and the MSE. There are two examples with $\beta < 1$ and $q < 1$, and one



Note: $h_q(t)$ is a monotonically decreasing function for $\beta < 1$ and $1 \leq q < 2$

Figure 8. q-Weibull failure rate, given by equation (8), with $\beta = 0.5$, $\eta = 1$ and different values of $q > 1$

Table I. Pump time to failure in days in ascending order (days)

Pump time to failure in ascending order (days)						
8	38	42	59	71	146	184
185	199	204	214	379	457	457
494	515	568	680	684	808	964

Table II. Time to failure of pumping rods in days

i	Time interval			Failures	i	Time interval			Failures
1	1	$\leq t \leq$	93.1	625	11	922	$< t \leq$	1,014.1	8
2	93.1	$< t \leq$	185.2	320	12	1,014.1	$< t \leq$	1,106.2	4
3	185.2	$< t \leq$	277.3	170	13	1,106.2	$< t \leq$	1,198.3	4
4	277.3	$< t \leq$	369.4	65	14	1,198.3	$< t \leq$	1,290.4	4
5	369.4	$< t \leq$	461.5	65	15	1,290.4	$< t \leq$	1,382.5	8
6	461.5	$< t \leq$	553.6	67	16	1,382.5	$< t \leq$	1,474.6	4
7	553.6	$< t \leq$	645.7	12	17	1,474.6	$< t \leq$	1,566.7	1
8	645.7	$< t \leq$	737.8	33	18	1,566.7	$< t \leq$	1,658.8	1
9	737.8	$< t \leq$	829.9	33	19	1,658.8	$< t \leq$	1,750.9	1
10	829.9	$< t \leq$	922	22	20	1,750.9	$< t \leq$	1,843	1

with $\beta > 1$ and $q > 1$, thus covering different behaviors of the failure rate. In all the cases, the q-Weibull presented a greater coefficient of determination and a smaller mean square error.

Figures 9-11 shows the reliability and failure rate curves for the three examples. It is to be noted that the usual Weibull model (with $q = 1$) systematically departures from the experimental data (circles) for large times, in the three examples considered, while the q-Weibull model is able to fit the whole range of the data.

Table III.
Time to failure of production tubing in days

Time to failure of production tubing (in days)														
1(2)	4(2)	6(10)	7(8)	8(7)	9(2)	10(3)	12(2)	14(4)	15(2)	17(9)	19(8)	20(3)	22(3)	23(3)
24(5)	25(1)	26(10)	27(5)	29(6)	30(5)	31(2)	32(1)	34(3)	35(5)	36(12)	38(16)	41(2)	42(3)	43(11)
44(2)	46(2)	47(5)	48(3)	51(2)	53(6)	55(6)	56(3)	60(2)	61(4)	63(9)	64(8)	65(6)	68(2)	69(3)
70(4)	73(6)	74(3)	75(2)	78(3)	79(3)	80(2)	81(4)	82(2)	83(4)	86(4)	87(4)	88(2)	89(6)	91(5)
92(3)	93(4)	94(3)	101(4)	106(3)	107(2)	108(3)	111(4)	117(3)	119(3)	121(3)	123(3)	124(3)	126(3)	133(4)
136(2)	142(3)	143(1)	148(2)	150(2)	154(2)	157(3)	161(4)	163(2)	167(3)	168(2)	170(5)	172(3)	177(2)	178(4)
185(4)	189(3)	194(3)	203(3)	207(6)	210(4)	219(3)	220(3)	222(3)	226(3)	227(2)	233(3)	234(3)	238(3)	245(3)
248(4)	265(2)	277(3)	347(4)	393(2)	425(3)	432(4)	488(2)	688(2)	691(3)					

Note: Number of repetitions of values in parentheses

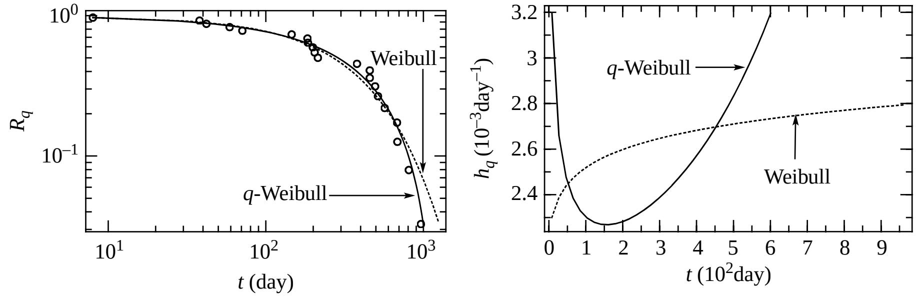
Two examples (oil pumps and pumping rods) present failure rates with a bathtub shape, and the last example (production tubings) exhibits the failure rate as a unimodal function. Of course the usual Weibull model (with q = 1) is unable to represent these cases.

5. Final remarks

Several models for failure rate function are found in the literature, many of them use Weibull (or Weibull-like) as a basis. These distributions share in common the exponential nature. The q-Weibull generalization uses a function that is exponential

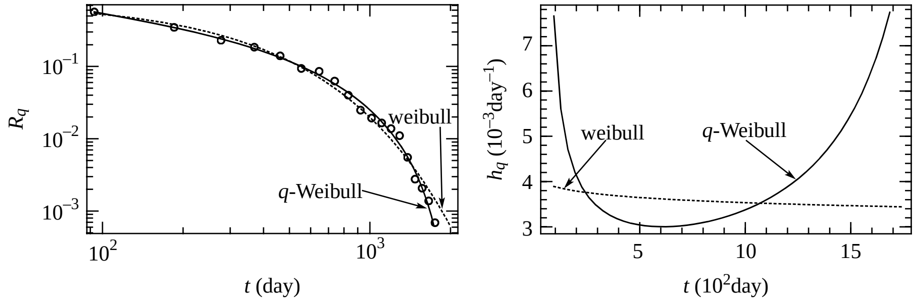
Table IV. Fitting results

	Pump		Pumping rod		Production tubing	
	Weibull	q-Weibull	Weibull	q-Weibull	Weibull	q-Weibull
β	1.05	0.82	0.95	0.42	0.12	1.31
η (day)	383	1,277	195	520	105	65
t_0 (day)	−7.66	−0.51	−56	59	0.61	0.24
Q	1.00	0.00	1.00	0.46	1.00	1.30
R^2	0.9761	0.9815	0.9904	0.9981	0.9818	0.9900
MSE	2.16×10^{-3}	1.59×10^{-3}	2.05×10^{-4}	2.74×10^{-5}	5.81×10^{-4}	2.52×10^{-4}



Notes: Left panel: log-log plot of reliability curves; right panel: failure rate curves; abscissas show time to failure of pumps

Figure 9. Weibull (dotted lines) and q-Weibull (solid lines) models, and experimental data (circles)



Notes: Left panel: log-log plot of reliability curves; right panel: failure rate curves; both plots show abscissas in time to failure of pumping rods

Figure 10. Weibull (dotted lines) and q-Weibull (solid lines) models, and experimental data (circles)

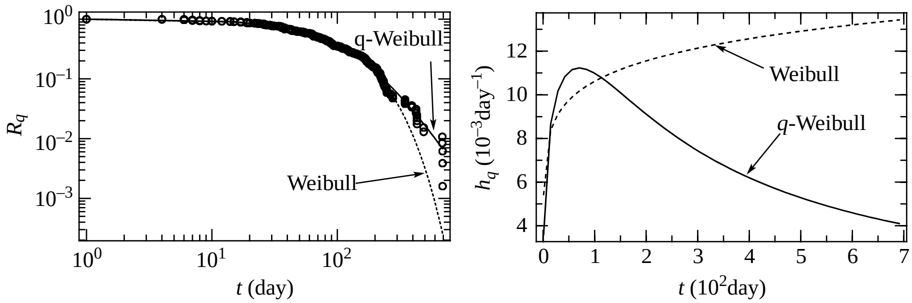


Figure 11. Weibull (dotted lines) and q-Weibull (solid lines) models, and experimental data (circles)

Notes: Left panel: log-log plot of reliability curves; right panel: failure rate curves; the independent variable is time to failure of production tubing for both plots

only as a limiting case, and may yield asymptotic power-laws. The q-Weibull model is able to describe four types of failure rate function, namely monotonically decreasing, monotonically increasing, unimodal and U-shaped curves, with a single parsimonious set of three parameters, representing a unification of various models, including the versatile Burr XII distribution. Table V summarizes the possibilities with the corresponding ranges of parameters.

Usual ($q = 1$) Weibull model is unable to represent the whole bathtub curve, once $h_1(t)$ is monotonically decreasing or monotonically increasing, depending on the value of parameter β. Modeling of U-shaped bathtub curve with Weibull requires a piecewise description, with $\beta < 1$ for the warm in phase, then $\beta = 1$ for the intermediary random failure phase and finally $\beta > 1$ for the wear out phase. In the present work we have shown the q-Weibull capacity of continuously reproducing the whole bathtub curve with the same set of constant parameters and without need of introducing *ad hoc* hypotheses.

We compare the usual Weibull with the q- Weibull models by means of three examples which present different behaviors: failure rate as a bathtub shape and as a unimodal curve. In all cases, the performance of the q-Weibull was superior to that of the usual Weibull. Of course such a result was to be expected, due to the extra parameter q, but it is important to remark that the improvements of the fittings are not merely quantitative (as it should be, due to the additional parameter), but also qualitative, once the q-Weibull model can describe behaviors (bathtub shape, and unimodal shape in the failure rate curve) that are impossible to be described by the usual Weibull model.

q-Weibull is a natural extension of usual Weibull, and it has the advantage of being originated from a theoretical background rooted in nonextensive statistical physics. Of course the introduction of additional (empirically or theoretically based) generalizations, like the use of linear or nonlinear transformation of time, use of multiple distributions, time dependence of parameters, etc. as it was done with Weibull, will further enhance flexibility and accuracy of q-Weibull model.

Table V. Behavior of q-Weibull failure rate according to the range of parameters q and β

	$0 < \beta < 1$	$\beta = 1$	$\beta > 1$
$q < 1$	Bathtub curve	Monotonically increasing	Monotonically increasing
$q = 1$	Monotonically decreasing	Constant	Monotonically increasing
$1 < q < 2$	Monotonically decreasing	Monotonically decreasing	Unimodal

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Appendix. Some mathematical properties of the q-Weibull distribution

A probability distribution is better characterized when its moments are known, and here we advance them. We consider the usual raw moments (moments about zero), and the usual central moments (moments about the mean). For a detailed analysis of the generalized moments of q-distributions, see Tsallis *et al.* (2009).

To evaluate the raw moments (moments about zero) of equation (3), $\mu'_n = \int_0^\infty t^n f_q(t)dt$, we shall consider separately the cases $q < 1$ and $q > 1$. It is not necessary to set $\eta = 1$ as shown by Vuorenmaa (2006). For the case $q < 1$, it is useful to consider the integral representation of the q-exponential given by Lenzi *et al.* (1999). For the case $q > 1$, it is necessary to use the integral representation proposed by Tsallis (1994). Straightforward calculations lead to the raw moments. For $q < 1$:

$$\mu'_n = \eta^n \Gamma\left(1 + \frac{n}{\beta}\right) \frac{\Gamma((3-2q)/(1-q))}{(1-q)^{n/\beta}\Gamma(((3-2q)/(1-q)) + (n/\beta))}, \quad \text{if } t_0 = 0, \tag{A1}$$

or:

$$\mu'_n = \sum_{j=0}^{n} \left\{ \binom{n}{j} t_0^{n-j} (\eta - t_0)^j \Gamma\left(1 + \frac{j}{\beta}\right) \times \frac{\Gamma((3-2q)/(1-q))}{(1-q)^{j/\beta}\Gamma(((3-2q)/(1-q)) + (j/\beta))} \right\}, \tag{A2}$$

with $t_0 \neq 0$

and for $q > 1$:

$$\mu'_n = \eta^n \Gamma\left(1 + \frac{n}{\beta}\right) \frac{\Gamma(((2-q)/(q-1)) - (n/\beta))}{(q-1)^{n/\beta}\Gamma((2-q)/(q-1))}, \quad \text{if } t_0 = 0, \tag{A3}$$

or:

$$\mu'_n = \sum_{j=0}^{n} \left\{ \binom{n}{j} t_0^{n-j} (\eta - t_0)^j \Gamma\left(1 + \frac{j}{\beta}\right) \times \frac{\Gamma(((2-q)/(q-1)) - (j/\beta))}{(q-1)^{j/\beta}\Gamma((2-q)/(q-1))} \right\}, \tag{A4}$$

with$t_0 \neq 0$

with $1 < q < q_{upper}$ and $q_{upper} = 1 + \beta/(n + \beta)$. Note that $q \rightarrow 1$ recovers the moments of usual Weibull pdf, $\mu'_n = \eta^n \Gamma((1+n)/\beta)$, for $t_0 = 0$ e $\mu'_n = \sum_{j=0}^{n} \left\{ \binom{n}{j} t_0^{n-j} (\eta - t_0)^j \Gamma((1+j)/\beta) \right\}$, for $t_0 \neq 0$. The upper limit *qupper* attains the values $\lim_{\beta\to 0} q_{upper} = 1$, $\lim_{\beta\to\infty} q_{upper} = 2$, and $\lim_{n\to\infty} q_{upper} = 1$. The latter limiting behavior means that it is not possible that q-Weibull pdf has all its moments for $q > 1$ (all moments are defined for $q \leq 1$). As q departs from unity from above (for constant β), q-Weibull loses its higher moments (normalizability, that is $\mu_0' = 1$, is preserved $\forall q < 2$). Note that t_{max} is the mean time between failures (MTBF). We remind the reader that there are many distributions that do not have all its moments. The Cauchy-Lorentz distribution, for instance, has no mean, variance or higher moments. Usual Weibull pdf has all moments, which is typical for distributions with exponential decay.

Central moments (moments about the mean) are found using the binomial transformation of the raw moments, as usual:

$$\mu_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} \mu_k' (\mu_1')^{n-k}, \quad \text{if } t_0 = 0, \tag{A5}$$

or, for $t_0 \neq 0$:

$$\mu_n = \sum_{j=0}^{n} \left\{ \binom{n}{j} (t_0 - \mu_1')^{n-j} (\eta - t_0)^j \times \Gamma\left(1 + \frac{j}{\beta}\right) \frac{\Gamma(((2-q)/(q-1)) - (j/\beta))}{(q-1)^{j/\beta}\Gamma((2-q)/(q-1))} \right\}, \quad 1 < q < 1 + \tfrac{\beta}{\beta+n}, \tag{A6}$$

and:

$$\mu_n = \sum_{j=0}^{n} \left\{ \binom{n}{j} (t_0 - \mu_1')^{n-j} (\eta - t_0)^j \times \Gamma\left(1 + \frac{j}{\beta}\right) \frac{\Gamma((3-2q)/(1-q))}{(1-q)^{j/\beta}\Gamma(((3-2q)/(1-q)) + (j/\beta))} \right\}, \quad q < 1. \tag{A7}$$

The median of q-Weibull pdf is $\mathrm{Md} = t_0 + (\eta - t_0)(2^{q'-1} q' \ln_{q'} 2)^{1/\beta}$, with the q-logarithm defined as Tsallis (1994) $\ln_q x = (x^{1-q} - 1)/(1-q)$, that is the inverse function of the q-exponential, $q' = 1/(2-q)$. Its mode is $\mathrm{Mo} = t_0 + (\eta - t_0)\{(\beta - 1)/[\beta + (1-q)(\beta - 1)]\}^{1/\beta}$, if $\beta > 1$.

An interesting mathematical feature is found by proper scaling of variables in the failure rate curve. The dimensionless failure rate may be defined as $\zeta(\tau) \equiv h_q(t)/h_q(t^*)$ ($h_q(t^*)$ is given by equation (13)), and the dimensionless time $\tau \equiv t/t^*$, with t^* given by equation (12), for the unimodal case ($1 < q < 2$ and $\beta > 1$), or $\tau \equiv t/t_{max}$, with t_{max} given by equation (10), for the bathtub shaped case ($q < 1$, $0 < \beta < 1$).

With this procedure, the dependence of the parameters q, η and t_0 is curiously absorbed by the dimensionless time, and the dimensionless failure rate ζ depends only on β and τ. For the unimodal case, $\zeta(\tau) = (\beta\tau^{\beta-1})/(1 + (\beta - 1)\tau^{\beta})$, and for the bathtub case, $\zeta(\tau) = (1 - \beta)^{(1-\beta)/\beta}(\beta\tau^{\beta-1})/(1 - \tau^{\beta})$. Data collapse yielded by proper scaling appears very frequently in the physics literature, and may be also useful within the context of reliability engineering.

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