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QUASITRIANGULAR HOPF ALGEBRAS, BRAID GROUPS AND QUANTUM ENTANGLEMENT

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The aim of the paper is to provide a method to obtain representations of the braid group through a set of quasitriangular Hopf algebras. In particular, these algebras may be derived from group algebras of cyclic groups with additional algebraic structures. In this context, by using the flip operator, it is possible to construct R-matrices that can be regarded as quantum logic gates capable of preserving quantum entanglement.

Keywords: Quantum groups; R-matrices; entanglement; cyclic groups.

1. Introduction

The discovery of quantum entanglement has its origins in the seminal article by Einstein, Podolsky and Rosen in 1935.[1] In this work, they proposed a thought experiment that attempted to show that quantum mechanical theory was incomplete. Currently, the quantum entanglement plays a key role in the quantum information and quantum computation theory[2] and has been widely exploited in quantum teleportation,[3] quantum algorithms[4] and quantum cryptography.[5,6] An interesting proposal in quantum computing is the topological quantum computation that employs two-dimensional (2D) quasiparticles called anyons,[7] whose world lines cross over one another to form braids in a three-dimensional (3D) spacetime. These braids form the logic gates that make up the quantum computer. One advantage of this proposal is the fact that it allows a fault-tolerant computing. Small perturbations can cause decoherence and introduce errors in computing, however these small perturbations do not change the topological properties of braids. Experimental evidences of non-Abelian anyons appear in quantum Hall systems in 2D electron gases subject to high magnetic fields.[7]

From a mathematical perspective, the anyons are described by representations of the braid groups, an algebraic structure that was explicitly introduced by Artin in 1925.[8] In algebraic topology and knot theory, they can be recognized as the fundamental group of a configuration space, by using the homotopy concept.[9] A bridge between knot theory and quantum information can be found in Refs. 10–15. In particular, in Ref. 16, the quantum teleportation has been described by the group of braids and Temperley–Lieb algebra, providing diagrammatic representations for teleportation. In this line, no explicit link is established with the anyons, with a general algebraic-topological perspective. An approach for non-Abelian anyons through quasitriangular Hopf algebras[17] was performed by Kitaev.[18,19] The quasitriangular structure[20,21] provides a unified description of the braiding properties, by using the Yang–Baxter equations. The final piece is the universal R-matrix that can be used to define representations of the braid group on fusion spaces, also called topological Hilbert spaces.

Hopf algebras[22–25] appear naturally in algebraic topology, where they are related to the H-space concept. Its origin is in the axiomatizations of the works of Hopf on topological properties of Lie groups. The notion of quasitriangular Hopf algebras, or quantum groups, in its turn, is due to Drinfeld[17] as an abstraction of structures implicit in the studies of Sklyanin,[26,27] Jimbo[28] and others.[25] There are many applications of these structures in physics, especially related to quantum gravity.[24,25] A relationship between quantum groups and quantum entanglement can be found in Refs. 29 and 30. Trindade and Vianna[29] performed a study to understand a possible connection between quantum groups, nonextensive statistical mechanics and entanglement through the entropic parameter q. Korbicz *et al.*[30] addressed the problem of separability in terms of compact quantum groups, resulting in an analog of positivity of partial transpose criterion in quantum information theory. In Ref. 20, it was shown that quasitriangular Hopf algebras can generate R-matrices. This result is particularly interesting because it allows to obtain representations of braid groups, since we have a quasitriangular structure.

In this paper, we developed a general method to obtain representations of the braid groups from a set of quasitriangular Hopf algebras. We applied these results to Hopf algebras derived of cyclic group. In particular, we investigated our general results for the $CZ_{/2}$ group and obtained a quantum gate leading to entangled states in themselves. We performed a comparative analysis with other work emphasizing the differences, advantages, and necessity of symmetry considerations.

The paper is organized as follows: Section 2 presents some basic concepts about quasitriangular Hopf algebras and braid groups. Section 3 contains a general method for obtaining the representations of braid groups. In Sec. 4, we derive a quantum logic gate that turns Bell states into Bell states. Section 5 is devoted to concluding remarks and outlooks.

2. Basic Concepts

In this section, we are going to review some basic concepts[21] that we shall need later.

Definition 1. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We call it quasi-cocomutative, if there exists an invertible element R of the algebra $H \otimes H$ such that for all $x \in H$ we have

$$\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}, \tag{1}$$

where $\Delta^{\mathrm{op}} = \tau_{H,H} \circ \Delta$ denotes the opposite coproduct on H, μ and η are linear maps that express the multiplication and unit, respectively; Δ is a product and ε is counity. An element R satisfying this condition is called a universal R-matrix.

Definition 2. A quasi-cocomutative Hopf algebra $(H, \mu, \eta, \Delta, \varepsilon, S, S^{-1}, R)$ is quasitriangular, if the universal R-matrix R satisfies the two relations

$$(\Delta \otimes id_H)(R) = R_{13}R_{23}, \tag{2}$$

and

$$(id_H \otimes \Delta)(R) = R_{13}R_{12}, \tag{3}$$

by using Sweedler's notation[25] for R_{ij}.

It is possible to show that universal R-matrix R satisfies the equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \tag{4}$$

denoted by algebraic Yang–Baxter equation.

Definition 3. Let V be a vector space over a field k. A linear automorphism R' of $V \otimes V$ is said to be an R-matrix of it is a solution of the Yang–Baxter equation

$$(R' \otimes id_V)(id_V \otimes R')(R' \otimes id_V) = (id_V \otimes R')(R' \otimes id_V)(id_V \otimes R'), \tag{5}$$

that holds in the automorphism group of $V \otimes V$.

The latter relation has fundamental importance because by identification

$$R'_i = \mathbb{I}^{\otimes(i-1)} \otimes R' \otimes \mathbb{I}^{\otimes(N-i)}, \tag{6}$$

where $\mathbb{I}$ is an identity operator and allows to build representations of braid groups with N strands that have to satisfy

$$R'_i R'_j = R'_j R'_i, \quad |i-j| \geq 2, \tag{7}$$

and

$$R'_i R'_{i+1} R'_i = R'_{i+1} R'_i R'_{i+1}, \quad i = 1, \dots, N-2. \tag{8}$$

3. Results

We now perform generalization about some results in Ref. 20 and we explore these expressions in the context of cyclic groups.

Lemma 1. *Let* $(H_1, \mu_1, \eta_1, \Delta_1, \varepsilon_1, S_1, S_1^{-1}, R_1), \dots, (H_n, \mu_n, \eta_n, \Delta_n, \varepsilon_n, S_n, S_n^{-1}, R_n)$ *quasitriangular Hopf algebras. Hence, there is an invertible element* R *such that for all* $x \in H_1 \otimes \cdots \otimes H_n$, *we have*

$$\Delta^{\text{op}}(x) R = R \Delta(x), \tag{9}$$

with $R_1 = \sum_{i_1} s_{i_1} \otimes t_{i_1}, \dots, R_n = \sum_{i_n} s_{i_n} \otimes t_{i_n}$, *and*

$$R = \sum_{i_n, \dots, i_n} s_{i_1} \otimes \cdots \otimes s_{i_n} \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}. \tag{10}$$

Moreover, the following relations

$$(\Delta \otimes id_{H_1 \otimes \cdots \otimes H_n})(R) = R_{13} R_{23}, \tag{11}$$

$$(id_{H_1 \otimes \cdots \otimes H_n} \otimes \Delta)(R) = R_{13} R_{12}, \tag{12}$$

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \tag{13}$$

with $R_{12} = \sum_{i_1 \dots i_n} s_{i_1} \otimes \dots s_{i_n} \otimes t_{i_1} \otimes \cdots \otimes t_{i_n} \otimes 1 \otimes \cdots \otimes 1$, $R_{13} = \sum_{i_1 \dots i_n} s_{i_1} \otimes \cdots s_{i_n} \otimes 1 \otimes \cdots \otimes 1 \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}$ *and* $R_{23} = \sum_{i_1 \dots i_n} 1 \otimes \cdots \otimes 1 \otimes s_{i_1} \otimes \cdots \otimes s_{i_n} \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}$ *are satisfied.*

Proof. Let the coproduct:

$$\Delta(x) = \sum_{(x)} x' \otimes x'',$$

in Sweedler's notation

$$\begin{aligned} \Delta(x_1 \otimes \cdots \otimes x_n) &= \sum_{(x_1 \otimes \cdots \otimes x_n)} (x_1 \otimes x_2 \otimes \cdots \otimes x_n)' \otimes (x_1 \otimes x_2 \otimes \cdots \otimes x_n)'' \\ &= \sum_{(x_1 \otimes \cdots \otimes x_n)} x'_1 \otimes \cdots \otimes x'_n \otimes x''_1 \otimes \cdots \otimes x''_n. \end{aligned}$$

Then, in a general case, we have

$$\Delta^{\rm op}(x_1 \otimes \cdots \otimes x_n) = \sum_{(x_1\otimes\cdots\otimes x_n)} x_1'' \otimes \cdots \otimes x_n'' \otimes x_1' \otimes \cdots \otimes x_n'.$$

Consequently

$$\begin{aligned}
\Delta^{\rm op}(x_1 \otimes \cdots \otimes x_n)R &= \sum_{(x_1\otimes\cdots\otimes x_n)} (x_1'' \otimes \cdots \otimes x_n'' \otimes x_1' \otimes \cdots \otimes x_n') \\
&\quad \times \sum_{i_1\ldots i_n} s_{i_1} \otimes \cdots \otimes s_{i_n} \otimes t_{i_1} \otimes \cdots \otimes t_{i_n} \\
&= \sum_{(x_1,\ldots,x_n;i_1,\ldots,i_n)} x_1''s_{i_1} \otimes x_2''s_{i_2} \otimes \cdots \otimes x_n''s_{i_n} \\
&\quad \otimes x_1't_{i_1} \otimes \cdots \otimes x_n't_{i_n} \\
&= \left(\sum_{(x_1,\ldots,x_n;i_1,\ldots,i_n)} x_1''s_{i_1} \otimes 1 \otimes \cdots \otimes x_1't_{i_1} \otimes \cdots \otimes 1\right)\ldots \\
&\quad \left(\sum_{(x_1,\ldots,x_n;i_1,\ldots,i_n)} 1 \otimes \cdots \otimes x_{i_n}''s_{i_n} \otimes 1 \otimes \cdots \otimes x_n't_{i_n}\right) \\
&= \left(\sum_{(x_1,\ldots,x_n;i_1,\ldots,i_n)} s_{i_1}x_1' \otimes 1 \otimes \cdots \otimes t_{i_1x_1''} \otimes \cdots \otimes 1\right)\ldots \\
&\quad \left(\sum_{(x_1,\ldots,x_n;i_1,\ldots,i_n)} 1 \otimes \cdots \otimes s_{i_n}x_{i_n}' \otimes 1 \otimes \cdots \otimes t_{i_n}x_n''\right) \\
&= \left(\sum_{(x_1,\ldots,x_n;i_1,\ldots,i_n)} s_{i_1}x_1' \otimes s_{i_2}x_2' \otimes \cdots \otimes s_{i_n}x_n' \otimes t_{i_1}x_1''\right. \\
&\quad \left. \otimes \cdots \otimes t_{i_n}x_n''\right) \\
&= R\Delta(x).
\end{aligned}$$

For relations (11) and (12)

$$\begin{aligned}
&(\Delta \otimes id_H)\left(\sum_{i_1\ldots i_n} s_{i_1} \otimes \cdots \otimes s_{i_n} \otimes t_{i_1} \otimes \cdots \otimes t_{i_n}\right) \\
&\quad = \sum_{i_1\ldots i_n} \Delta(s_{i_1} \otimes \cdots \otimes s_{i_n}) \otimes id_H(t_{i_1} \otimes \cdots \otimes t_{i_n})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i_1\dots i_n} s'_{i_1} \otimes s'_{i_2} \otimes \cdots s'_{i_n} \otimes s''_{i_1} \otimes \cdots \otimes s''_{i_n} \otimes t_{i_1} \otimes \cdots \otimes t_{i_n} \\
&= \left(\sum_{i_1 s_1} s'_{i_1} \otimes 1 \otimes \cdots \otimes s''_{i_1} \otimes \cdots \otimes t'_{i_1} \otimes \cdots \otimes 1 \right) \dots \\
&\quad \left(\sum_{i_n s_n} 1 \otimes \cdots \otimes s'_{i_n} \otimes \cdots \otimes s''_{i_n} \otimes \cdots \otimes t_{i_n} \right) \\
&= \left(\sum_{i_1 j_1} s_{i_1} \otimes 1 \otimes \cdots \otimes s_{j_1} \otimes 1 \otimes \cdots \otimes t_{i_1} t_{j_1} \otimes \cdots \otimes 1 \right) \dots \\
&\quad \left(\sum_{i_n j_n} 1 \otimes \cdots \otimes s_{i_n} \otimes \cdots \otimes s_{j_n} \otimes 1 \otimes \cdots \otimes t_{i_n} t_{j_n} \right) \\
&= R_{13} R_{23}.
\end{aligned}
$$

Similarly

$$
(id_{H_1 \otimes \cdots \otimes H_n} \otimes \Delta) = R_{13} R_{12}.
$$

For the last expression, we have

$$
\begin{aligned}
R_{12} R_{13} R_{23} &= \sum_{i_1 \dots i_n, j_1 \dots j_n, k_1 \dots k_n} s_{k_1} s_{j_1} \otimes \cdots \otimes s_{k_n} s_{j_n} \otimes t_{k_1} s_{i_1} \otimes \cdots \otimes t_{k_n} s_{i_n} \otimes t_{j_1} t_{i_1} \\
&\quad \otimes \cdots \otimes t_{j_n} t_{i_n} \\
&= \left(\sum_{i_1, j_1, k_1} s_{k_1} s_{j_1} \otimes \cdots \otimes 1 \otimes t_{k_1} s_{i_1} \otimes \cdots \otimes 1 \otimes t_{j_1} t_{i_1} \otimes \cdots \otimes 1 \right) \dots \\
&\quad \left(\sum_{i_n, j_n, k_n} 1 \otimes \cdots \otimes s_{k_n} s_{j_n} \otimes \cdots \otimes 1 \otimes t_{k_n} s_{i_n} \otimes \cdots \otimes t_{j_n} t_{i_n} \right) \\
&= \sum_{i_1 \dots i_n} (s_{j_1} s_{i_1} \otimes \cdots \otimes 1 \otimes s_{k_1} t_{k_1} \otimes \cdots \otimes t_{k_1} t_{j_1} \otimes \cdots \otimes 1) \dots \\
&\quad \left(\sum_{i_n, j_n, k_n} 1 \otimes \cdots \otimes s_{j_n} s_{i_n} \otimes \cdots \otimes 1 \otimes s_{k_n} t_{i_n} \otimes \cdots \otimes t_{k_n} t_{j_n} \right) \\
&= \sum_{i_1 \dots i_n, j_1 \dots j_n, k_1 \dots k_n} s_{j_1} s_{i_1} \otimes \cdots \otimes s_{j_n} s_{i_n} \otimes \cdots \otimes s_{k_n} t_{i_n} \otimes t_{k_1} t_{j_1} \\
&\quad \otimes \cdots \otimes t_{k_n} t_{j_n} \\
&= R_{23} R_{13} R_{12}.
\end{aligned}
$$

□

Let now $V_1, \dots, V_n$ and $W_1, \dots, W_n$ H-modules. We can build a isomorphism $C^R_{V_1 \dots V_n, W_1 \dots W_n}$ of $H_1, \dots, H_n$ modules between $V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_n$ and

$W_1 \otimes \cdots \otimes W_n \otimes V_1 \otimes \cdots \otimes V_n$, defined by

$$\begin{aligned} &C^R_{V_1\ldots V_n, W_1\ldots W_n}(v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n) \\ &\quad = \tau_{V_1\ldots V_n, W_1\ldots W_n}(R \triangleright (v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n)) \\ &\quad = \sum_{i_1\ldots i_n} t_{i_1} \triangleright w_1 \otimes \cdots \otimes t_{i_n} \triangleright w_n \otimes s_{i_1} \triangleright v_1 \otimes \cdots \otimes s_{i_n} \triangleright v_n, \end{aligned}$$

where $\triangleright$ denotes the action of H on U, V and W.

Theorem 1. *For any triple* $(U_1 \otimes \cdots \otimes U_n, V_1 \otimes \cdots \otimes V_n, W_1 \otimes \cdots \otimes W_n)$ *of* $H_1 \otimes \cdots \otimes H_n$*-module we have*

(a) *the map* $C^R_{V_1\ldots V_n, W_1\ldots Wn}$ *is an isomorphism of* $H_1 \otimes \cdots \otimes H_n$*-module.*

(b) $(C^R_{V_1\ldots V_n, W_1\ldots W_n} \otimes id_{U_1\ldots U_n})(id_{V_1\ldots V_n} \otimes C^R_{U_1\ldots U_n, W_1 W_n})(C^R_{U_1\ldots U_n, V_1\ldots V_n} \otimes id_{W_1\ldots W_n})$
$= (id_{W_1\ldots W_n} \otimes C^R_{U_1\ldots U_n, V_1\ldots V_n})(C^R_{U_1\ldots U_n, W_1\ldots W_n} \otimes id_{V_1\ldots V_n})(id_{U_1\ldots U_n} \otimes C^R_{V_1\ldots V_n, W_1\ldots W_n})$.

Proof. (a) For any $x_1 \otimes \cdots \otimes x_n \in H_1 \otimes \cdots \otimes H_n$-module by using definition

$$\begin{aligned} &C^R_{V_1\ldots V_n, W_1\ldots W_n}(x_1 \otimes \cdots \otimes x_n) \triangleright (v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n) \\ &\quad = \tau_{v_1\ldots v_n, w_1\ldots w_n}(R \triangleright \Delta(x_1 \otimes \cdots \otimes x_n) \triangleright (v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n)). \end{aligned}$$

By using Lemma 1 and the notation $C^R_{V_1\ldots V_n, W_1\ldots W_n}(x_1 \otimes \cdots \otimes x_n) \triangleright (v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n) = \mathcal{C}$, we get

$$\begin{aligned} \mathcal{C} &= \tau_{v_1\ldots v_n, w_1\ldots w_n}(\Delta^{op}(x_1 \otimes \cdots \otimes x_n) R(v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n)) \\ &= \tau_{v_1\ldots v_n, w_1\ldots w_n}\Bigg(\sum_{x_1\ldots x_n, i_1\ldots i_n} x''_1 s_{i_1} \triangleright v_1 \otimes x''_2 s_{i_2} \triangleright v_2 \otimes \cdots \otimes x''_n s_{i_n} \triangleright v_n \\ &\quad \otimes x'_1 t_{i_1} \triangleright w_1 \otimes x'_2 t_{i_2} \triangleright w_2 \otimes \cdots \otimes x'_n t_{i_n} \triangleright w_n\Bigg) \\ &= \sum_{x_1\ldots x_n, i_1\ldots i_n} x'_1 t_{i_1} \triangleright w_1 \otimes x'_2 t_{i_2} \triangleright w_2 \otimes \cdots \otimes x'_n t_{i_n} \triangleright w_n \\ &\quad \otimes x''_1 s_{i_1} \triangleright v_1 \otimes x''_2 s_{i_2} \triangleright v_2 \otimes \cdots \otimes x''_n s_{i_n} \triangleright v_n \\ &= \Delta(x_1 \otimes \cdots \otimes x_n) \sum_{i_1\ldots i_n} t_{i_1} \triangleright w_1 \otimes \cdots \otimes t_{i_n} \triangleright w_n \otimes x''_1 s_{i_1} \triangleright v_1 \otimes \cdots \otimes x''_n s_{i_n} \triangleright v_n \\ &= \Delta(x_1 \otimes \cdots \otimes x_n)\tau_{v_1\ldots v_n, w_1\ldots w_n}(R \triangleright [v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n]) \\ &= (x_1 \otimes \cdots \otimes x_n)(C^R_{V_1\ldots V_n, W_1\ldots W_n}[v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_n]). \end{aligned}$$

(b) It is easy to verify that

$$
\begin{aligned}
&(C^R_{V_1\ldots V_n,W_1\ldots W_n} \otimes id_{U_1\ldots U_n})(id_{V_1\ldots V_n} \otimes C^R_{U_1\ldots U_n,W_1\ldots W_n})(C^R_{U_1\ldots U_n,V_1\ldots V_n} \otimes id_{W_1\ldots W_n})\\
&\quad = \sum_{i_1\ldots i_n,j_1\ldots j_n,k_1\ldots k_n} t_{k_1}t_{j_1} \triangleright w_1 \otimes \cdots \otimes t_{k_n}t_{j_n} \triangleright w_n \otimes s_{k_1}t_{i_1} \triangleright v_1 \cdots\\
&\qquad \otimes s_{k_n}t_{i_n} \triangleright v_n \otimes s_{j_1}s_{i_1} \triangleright u_1 \otimes \cdots \otimes s_{j_n}s_{i_n} \triangleright u_n\\
&\quad = \left(\sum_{i_1,j_1,k_1} t_{k_1}t_{j_1} \triangleright w_1 \otimes \cdots \otimes s_{k_1}t_{i_1} \triangleright v_1 \otimes \cdots \otimes s_{j_1}s_{i_1} \triangleright u_1 \otimes \cdots \otimes 1\right)\ldots\\
&\qquad \left(\sum_{i_n,j_n,k_n} 1 \otimes t_{k_n}t_{j_n} \triangleright w_n \otimes \cdots \otimes s_{k_n}t_{i_n} \triangleright v_n \otimes \cdots \otimes 1 \otimes s_{j_n}s_{i_n} \triangleright u_n\right)\\
&\quad = \left(\sum_{i_1,j_1,k_1} t_{j_1}t_{i_1} \triangleright w_1 \otimes \cdots \otimes t_{k_1}s_{i_1} \triangleright v_1 \otimes \cdots \otimes s_{j_1}s_{i_1} \triangleright u_1 \otimes \cdots \otimes 1\right)\ldots\\
&\qquad \left(\sum_{i_n,j_n,k_n} 1 \otimes t_{j_n}t_{i_n} \triangleright w_n \otimes \cdots \otimes t_{k_n}s_{i_n} \triangleright v_n \otimes \cdots \otimes 1 \otimes s_{k_n}s_{j_n} \triangleright u_n\right)\\
&\quad = (id_{W_1\ldots W_n} \otimes C^R_{U_1\ldots U_n,V_1\ldots V_n})(C^R_{U_1\ldots U_n,W_1\ldots W_n} \otimes id_{V_1\ldots V_n})\\
&\qquad \times (id_{U_1\ldots U_n} \otimes C^R_{V_1\ldots V_n,W_1\ldots W_n}),
\end{aligned}
$$

by using of Lemma 1. □

Note that the setting, is $U_1 = V_1 = W_1, \ldots, U_n = V_n = W_n$. From this setting, we conclude that $C^R_{V_1\ldots V_n,W_1\ldots W_n}$ is a solution of the Yang–Baxter equation and therefore can be used to generate representation of braid groups.

Consider now $Z_{/\eta_1}, Z_{/\eta_2}, \ldots, Z_{/\eta_n}$ be the finite cyclic groups of order $\eta_1, \eta_2, \ldots, \eta_n$ and $CZ_{/\eta_1}, CZ_{/\eta_2}, \ldots, CZ_{/\eta_n}$ be its group algebras respectively, so that we can build Hopf algebras[25] with the quasitriangular structures

$$R_1 = \frac{1}{\eta_1} \sum_{a_1,b_1=0}^{\eta_1-1} \mathrm{e}^{\frac{-2\pi I a_1 b_1}{\eta_1}} g^{a_1} \otimes g^{b_1}, \tag{14}$$

$$R_2 = \frac{1}{\eta_2} \sum_{a_2,b_2=0}^{\eta_2-1} \mathrm{e}^{\frac{-2\pi I a_2 b_2}{\eta_2}} g^{a_2} \otimes g^{b_2}, \tag{15}$$

$$\vdots$$

$$R_n = \frac{1}{\eta_n} \sum_{a_n,b_n=0}^{\eta_n-1} \mathrm{e}^{\frac{-2\pi I a_n b_n}{\eta_n}} g^{a_n} \otimes g^{b_n}, \tag{16}$$

and has coproduct $\Delta g^{a_1} = g^{a_1} \otimes g^{a_1}, \Delta g^{a_2} = g^{a_2} \otimes g^{a_2}, \ldots, \Delta g^{a_n} = g^{a_n} \otimes g^{a_n}$. The counit is given by $\epsilon g^{a_1} = \epsilon g^{a_2} = \cdots = \epsilon g^{a_n} = 1$ and the antipode $S g^{a_1} = (g^{a_1})^{-1}$, $S g^{a_2} = (g^{a_2})^{-1}, \ldots, S g^{a_n} = (g^{a_n})^{-1}$.

According to our formulation, we have

$$R = \frac{1}{\eta_1 \eta_2 \ldots \eta_n} \sum_{a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n = 0}^{\eta_1 - 1, \eta_2 - 1, \ldots, \eta_n - 1} \mathrm{e}^{\frac{-2\pi I a_1 b_1 \ldots a_n b_n}{\eta_1 \eta_2 \cdots \eta_n}} g^{a_1} \otimes g^{a_2} \otimes \cdots \otimes g^{a_n} \otimes g^{b_1} \otimes g^{b_2} \otimes \cdots \otimes g^{b_n}. \tag{17}$$

Using the notation $C^R_{U_1 \ldots U_n, V_1 \ldots V_n}(u_1 \otimes u_2 \otimes \cdots \otimes u_n \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \mathcal{C}_1$, we can show that

$$\mathcal{C}_1 = \frac{1}{\eta_1 \eta_2 \ldots \eta_n} \sum_{a_1, a_2, \ldots, a_n b_1, b_2, \ldots, b_n}^{\eta_1 - 1, \eta_2 - 1, \ldots, \eta_n - 1} \mathrm{e}^{\frac{-2\pi I a_1 b_1 a_2 b_2 \ldots a_n b_n}{\eta_1 \eta_2 \cdots \eta_n}} g^{b_1} \triangleright v_1 \otimes g^{b_2} \triangleright v_2 \otimes \cdots \otimes g^{b_n} \triangleright v_n \otimes g^{a_1} \triangleright u_1 \otimes g^{a_2} \triangleright u_2 \otimes \cdots \otimes g^{a_n} \triangleright u_n. \tag{18}$$

4. Applications

In order to illustrate our formalism, we consider a simple case $CZ_{/2}$ with group $G = \{\epsilon, x\}$, where ϵ is the identity. In this case

$$\begin{aligned} R &= \frac{1}{2} \sum_{a,b=0}^{1} \mathrm{e}^{-\pi I ab} g^a \otimes g^b \\ &= \frac{1}{2}(\epsilon \otimes \epsilon + x \otimes \epsilon + \epsilon \otimes x - x \otimes x). \end{aligned} \tag{19}$$

Using the regular representation Γ of the algebra, we have

$$\Gamma(\epsilon \otimes \epsilon) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad \Gamma(x \otimes \epsilon) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\Gamma(\epsilon \otimes x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Gamma(x \otimes x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

resulting in

$$\Gamma(R) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{20}$$

We next consider $\Gamma(R) \equiv R$. The associated flip operator is given by

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Consequently,

$$R' = \tau R = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}. \tag{22}$$

This matrix satisfies the braid relation

$$(\mathbb{I} \otimes R')(R' \otimes \mathbb{I})(\mathbb{I} \otimes R') = (R' \otimes \mathbb{I})(\mathbb{I} \otimes R')(R' \otimes \mathbb{I}), \tag{23}$$

where $\mathbb{I}$ is a identity matrix 2×2, and it can be visualized as quantum logic gate.

Its action on the Bell states is given by

$$R' \left[\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \right] = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) = |\Psi^+\rangle, \tag{24}$$

$$R' \left[\frac{1}{\sqrt{2}} (|01\rangle + |10\rangle) \right] = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) = |\Phi^+\rangle, \tag{25}$$

$$R' \left[\frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) \right] = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle) = |\Phi^-\rangle, \tag{26}$$

$$R' \left[\frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \right] = \frac{1}{\sqrt{2}} (|10\rangle - |01\rangle) = -|\Psi^-\rangle, \tag{27}$$

i.e. the entanglement is preserved under the action of this gate. Importantly, the symmetry groups in 3D are cyclic as abstract group. Therefore, the cyclic groups may indirectly reflect symmetries of physical systems transforming maximally entangled states in themselves. Interestingly, starting from an extremely simple case, it is possible to generate a nontrivial structure.

In a seminal work that established a connection between quantum entanglement and topological entanglement, Kauffman and Lomanaco Jr.[12] introduced the following matrix solution to the Yang–Baxter equation

$$R = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & b \end{pmatrix},$$

where a, b, c and d are any scalars on the unit circle in the complex plane. It was shown that, if R is chosen so that $ab \neq cd$, then the state $R(\psi \otimes \psi)$, with $\psi = |0\rangle + |1\rangle$, is entangled. All the 4×4 unitary matrix solutions to the braided Yang-Baxter equation was obtained by Dye[31]. For this dimension, the relationship between quantum entanglement and topological entanglement was analyzed and the families of solutions have been classified. A solution explored by Zhang[32] is given by:

$$B = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix},$$

called Bell matrix. The action of this matrix on the basis state results in Bell states:

$$\begin{aligned} B|00\rangle &= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\Phi^-\rangle, \\ B|01\rangle &= \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Psi^+\rangle, \\ B|10\rangle &= -\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = -|\Psi^-\rangle, \\ B|11\rangle &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = |\Phi^+\rangle. \end{aligned}$$

In this context, equations for teleportation and their diagrammatical representations were presented. Here, unlike the above mentioned approaches, our purpose is to obtain representations of braid groups in a systematic way for arbitrary dimensions, exploiting the structure of underlying symmetries. This is interesting because anyonic physics, for example, Monchon[33] showed that for anyons obtained from a finite gauge theory, the computational power depends on the symmetry group. Besides that, for cyclic groups, according to the recent investigations about anyons in integer quantum Hall magnets,[34] the nontrivial fundamental homotopy group $\pi_1(O(3)) = Z_2$ guarantees the existence of the Z_2 vortices. It is noteworthy that, although our analysis have been done for cyclic groups, any group or algebra could have been used, since our method is general.

5. Conclusions

The main objective of this paper has been to present a systematic method to derive representations of braid groups through a set of quasitriangular Hopf algebras. This approach should be related to the topological quantum computation. In Ref. 35, possible experimental implementations of lattice models based on non-Abelian discrete symmetry groups have been proposed. The group elements can be viewed as transformations between the states of the sites of a superconducting Josephson-junction array. The algebraic structure employed was the dihedral group that can be expressed in terms of cyclic groups, using the semidirect product. In this paper, we generalize some results obtained in Ref. 20 and we explore the structure of a quasitriangular Hopf algebra derived from a cyclic group. In particular, we show how to obtain a quantum logic gate of a simple Abelian structure generated by $CZ_{/2}$ group algebra. This gate becomes entangled states in themselves. Furthermore, we compared our method with some related works, highlighting differences and possible advantages, and the necessity of symmetry considerations. As perspectives, it seems interesting to investigate other cyclic groups, as well as, possible associated topological structure.

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