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# Exact solutions of Brans–Dicke cosmology with decaying vacuum density

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**Abstract**

We investigate cosmological solutions of Brans–Dicke theory with both the vacuum energy density and the gravitational constant decaying linearly with the Hubble parameter. A particular class of them, with a constant deceleration factor, sheds light on the cosmological constant problems, leading to a presently small vacuum term, and to a constant ratio between the vacuum and matter energy densities. By fixing the only free parameter of these solutions, we obtain cosmological parameters in accordance with observations of both the relative matter density and the universe age. In addition, we have three other solutions, with Brans–Dicke parameter $\omega = -1$ and negative cosmological term, two of them with a future singularity of big-rip type. Although interesting from the theoretical point of view, two of them are not in agreement with the observed universe. The third one leads, in the limit of large times, to a constant relative matter density, being also a possible solution to the cosmic coincidence problem.

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## 1. Introduction

Recent observations suggest that the total energy density in the universe is greater than the (barionic + dark) matter density. On one hand, dynamical estimations lead to a ratio between the matter and critical densities around one third [1]. On the other hand, measurements of anisotropies in the microwave background radiation indicate that our universe is spatially flat [2], suggesting the existence of another, unknown form of energy, usually called dark energy. Its presence is also corroborated by age estimations [3] and by the observed distance–redshift

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relation for supernovae Ia, which suggests that, in the present phase of universe evolution, the deceleration parameter $q$ is negative, and therefore the universe performs an accelerated expansion [4].

Several models have been proposed in order to explain those observational data. There are some candidates for dark energy, as, for example, the cosmological constant, the so-called quintessence, or the generalized Chaplygin gas. Among them, the simplest and oldest one is the cosmological constant, also associated with the energy density of vacuum.

We can contextualize the study of dark energy in different gravitational theories. For instance, the theory of general relativity, where the Einstein field equations are used, and where the gravitation constant $G$ is a universal constant. Another one is the Brans–Dicke theory [5, 6], a scalar–tensor theory in which the gravitational constant is a function of space-time, and where a new parameter, $\omega$, is introduced. More recently the interest on this kind of theory was renewed, owing to its association with superstring theories, extra-dimensional theories and models with inflation or accelerated expansion [7–11].

It is generally assumed that general relativity is recovered in the limit $\omega \to +\infty$ (despite the existence of Brans–Dicke solutions for which this is not true [12]). Astronomical observations in the realm of solar system impose a very high inferior limit for $\omega$. Nevertheless, such a result corresponds to the weak field limit, and applies only in the simplest case of constant $\omega$. Therefore, it is possible that general relativity is not adequate to describe the universe at early times, or needs corrections in the cosmological limit.

In this paper we consider the Brans–Dicke theory and associate with dark energy the equation of state of vacuum. We investigate models in which the vacuum energy density decreases with the universe expansion, a hypothesis that has been considered as a possible solution to the cosmological constant problem, that is, to the question of why the presently observed value of $\Lambda$ is about 120 orders of magnitude below the value predicted by quantum field theories [13, 14].

Our goal is to find solutions of Brans–Dicke theory which satisfy a particular variation law for $G$. We shall use the Eddington–Dirac relation, based on the large number coincidence, $G \approx H\big/m_\pi^3$, where $H = \dot{a}/a$ is the Hubble parameter and $m_\pi$ is the pion mass [15, 16]. We will then take $G = H/8\pi\lambda$, where $\lambda$ has the order of $m_\pi^3$. In addition, as usual, we will relate the Brans–Dicke scalar field to the gravitational constant through $\phi = G_0/G$, where $G_0$ is a positive constant of the order of unity.

Together with that variation law for $G$, we shall consider two different ansätze. The first one is given by $\rho = 3\alpha H^2/8\pi G$, where $\rho = \rho_m + \rho_\Lambda$ is the total density, and $\alpha$ is an adimensional constant of the order of unity. This ansatz is suggested by observations, which show that $\rho_m \approx \rho_c/3$, where $\rho_c = 3H^2/8\pi G$ is the critical density. On the other hand, we know that $\rho_\Lambda$ has, at most, the same order of magnitude as $\rho_m$, otherwise its presence would be more evident. This ansatz was already considered in [17, 18].

The second ansatz will be given by $\Lambda = \beta H^2$, where $\beta$ is a constant of the order of unity. We are, in this case, inferring a variation law for the cosmological term, which has already been considered in the literature on the basis of different arguments [16–23] (for other variation laws for the cosmological term, see, for instance, [24–27]). We will show that this ansatz leads to a set of solutions larger than the first one, containing its solutions as a particular case.

We will look for solutions for recent times, that is, we shall consider a spatially flat ($k = 0$) Friedmann–Robertson–Walker space-time, filled with a perfect fluid whose matter component is pressureless ($p_m = 0$). For the cosmological term, we will take the equation of state of vacuum, $p_\Lambda = -\rho_\Lambda$.

## 2. Solutions with a varying cosmological term

### *2.1. The first ansatz*

Taking $p_m = 0$, $k = 0$ and $p_\Lambda = -\rho_\Lambda$, the Brans–Dicke equations [5] can be written as

$$\frac{\mathrm{d}(\dot{\phi} a^3)}{\mathrm{d}t} = \frac{8\pi}{3+2\omega}\,(\rho + 3\rho_\Lambda)\,a^3, \tag{1}$$

$$\dot{\rho} = -3H\rho_m, \tag{2}$$

$$H^2 = \frac{8\pi\rho}{3\phi} - \frac{\dot{\phi}}{\phi}H + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2}. \tag{3}$$

We then have a system of three ordinary differential equations, with four unknown functions of time: $a$, $\rho_m$, $\rho_\Lambda$ and $\phi$. The system becomes solvable if we add the Eddington–Dirac relation, $G = H/8\pi\lambda$, which relates $a$ and $\phi$. In order to restrict our class of solutions, we will take in addition our first ansatz, given by $\rho = 3\alpha H^2/8\pi G$. In this way, we obtain

$$\rho = 3\alpha\lambda H, \tag{4}$$

$$\phi = \frac{8\pi\lambda G_0}{H}, \tag{5}$$

and

$$\dot{\phi} = 8\pi\lambda G_0(1+q), \tag{6}$$

where $q = -\ddot{a}a/\dot{a}^2$ is the deceleration factor.

With the help of equations (4)–(6), we can rewrite (1)–(3) in the form

$$(3+2\omega)\lambda G_0[\dot{q} + (1+q)3H] = 3\alpha\lambda H + 3\rho_\Lambda, \tag{7}$$

$$\rho_m = \alpha\lambda H(1+q), \tag{8}$$

$$\frac{\alpha}{G_0} = 2 + q - \frac{\omega}{6}(1+q)^2. \tag{9}$$

Equation (9) tells us that $q$ is constant, since $\alpha$, $G_0$ and $\omega$ also are. Therefore, $\dot{q} = 0$, and equation (7) reduces to

$$(3+2\omega)\lambda G_0(1+q)H = \alpha\lambda H + \rho_\Lambda. \tag{10}$$

By using (4) and (8), we can obtain the vacuum density,

$$\rho_\Lambda = \alpha\lambda H(2-q). \tag{11}$$

Substituting $\lambda = H/8\pi G$, we obtain

$$\rho_\Lambda = \frac{\alpha(2-q)H^2}{8\pi G}.$$

Since $\rho_\Lambda = \Lambda/8\pi G$, we conclude that $\Lambda = \alpha(2-q)H^2$, which suggests our second ansatz, with $\beta = \alpha(2-q)$, to be used later.

Leading (4) and (8) into (2), one obtains

$$\frac{1}{H} = (1+q)t + C,$$

where $C$ is an integration constant. Let us take $C = 0$, in such a way that $H \to \infty$ for $t \to 0$. We then have

$$H = \frac{1}{1+q}\frac{1}{t}. \tag{12}$$

Substituting $\dot{a}/a$ for $H$ in (12), we also have

$$a = At^{\frac{1}{1+q}}, \tag{13}$$

where $A$ is another integration constant.

The relative density of matter, defined with respect to the critical density, can be obtained by using $G = H/8\pi\lambda$. Then, $\rho_c = 3\lambda H$, and, using (8), one obtains

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{\alpha(1+q)}{3}. \tag{14}$$

Substituting $\rho_\Lambda$ from (11) into (10), we have as well

$$\frac{\alpha}{G_0} = \frac{(3+2\omega)(1+q)}{3-q}. \tag{15}$$

Comparing $\alpha/G_0$ given by equations (9) and (15), one can derive a relation between $\omega$ and $q$, given by

$$(3+2\omega)(1+q) = \left[2+q-\frac{\omega}{6}(1+q)^2\right](3-q). \tag{16}$$

Eliminating $\omega$ from equations (9) and (15), we can also obtain $\alpha/G_0$ as a function of $q$ only,

$$\frac{\alpha}{G_0} = \frac{12(2+q)+3(1+q)^2}{(1+q)(3-q)+12}. \tag{17}$$

With these relations, it is easy to derive some results to compare with current observations. For example, if $q = 0$, from equations (16) and (17) we obtain $\omega = 6/5$ and $\alpha/G_0 = 9/5$. From (12) we have $Ht = 1$. From equation (13) it follows that $a = At$. And, from (14), one has $\Omega_m/\alpha = 1/3$. Since $\alpha \approx 1$, we can see that $\Omega_m \approx 1/3$, in agreement with astronomical estimations [1]. The age parameter $Ht$ is also in good accordance with globular clusters observations [3].

If, on the other hand, we would take $q = -1$, we would obtain, instead of equation (12), the result $H =$ constant, that is, the de Sitter universe, with $\rho_m = 0$ and a constant $\rho_\Lambda$. Note, however, that $q = -1$ does not satisfy equation (16), that is, the de Sitter universe is not a solution of Brans–Dicke equations for this ansatz.

### *2.2. The second ansatz*

Taking now our second ansatz, $\Lambda = \beta H^2$ and $G = H/8\pi\lambda$, and recalling that $\rho_\Lambda = \Lambda/8\pi G$, we obtain

$$\rho_\Lambda = \beta\lambda H. \tag{18}$$

Furthermore, as well as in the first ansatz, we have

$$\phi = \frac{8\pi\lambda G_0}{H} \tag{19}$$

and

$$\dot{\phi} = 8\pi\lambda G_0(1+q). \tag{20}$$

With the help of (18)–(20), we can put (1)–(3) in the form

$$(3+2\omega)\lambda G_0[\dot{q}+3H(1+q)] = \rho + 3\beta\lambda H, \tag{21}$$

$$\dot{\rho} + 3H\rho - 3\beta\lambda H^2 = 0, \tag{22}$$

$$\rho = 3\lambda G_0 H\left[2+q-\frac{\omega}{6}(1+q)^2\right]. \tag{23}$$

Here we have a solvable system, with three differential equations for three unknown functions, $H$, $\rho$ and $q$. By finding $\rho$ one can, using (18), determine $\rho_\Lambda$ and $\rho_m$.

Leading $\rho$ given by (23) into (21), we derive

$$\frac{\beta}{G_0} = \frac{(3+2\omega)[\dot{q}+3H(1+q)] - 3H\left[2+q-\frac{\omega}{6}(1+q)^2\right]}{3H}. \tag{24}$$

In this way, since $\beta/G_0$ is constant, there are two possibilities: either $\dot{q}=0$, in which case we have a simple relation between $q$ and $\beta/G_0$, or (24) is an evolution equation, with $\dot{q}\neq 0$.

*2.2.1. The case* $\dot{q}=0$. In this case, equation (24) becomes

$$\frac{\beta}{G_0} = (3+2\omega)(1+q) - \left[2+q-\frac{\omega}{6}(1+q)^2\right]. \tag{25}$$

Using (23) into (22), one obtains, by integration,

$$\frac{1}{H} = \frac{3G_0\left[2+q-\frac{\omega}{6}(1+q)^2\right]-\beta}{G_0\left[2+q-\frac{\omega}{6}(1+q)^2\right]}t + C.$$

Let us choose $C=0$, such that $H\to\infty$ for $t\to 0$. So we have

$$H = \frac{n}{t}, \tag{26}$$

where we have defined

$$n = \frac{G_0\left[2+q-\frac{\omega}{6}(1+q)^2\right]}{3G_0\left[2+q-\frac{\omega}{6}(1+q)^2\right]-\beta}. \tag{27}$$

Substituting $\dot{a}/a$ for $H$ in equation (26), we find

$$a = At^n, \tag{28}$$

where $A$ is an integration constant. On the other hand, from (28) we can obtain $q=(1-n)/n$, or

$$n = \frac{1}{1+q}. \tag{29}$$

Leading $n$ from equation (27) into (29), one obtains

$$\frac{\beta}{G_0} = \left[2+q-\frac{\omega}{6}(1+q)^2\right](2-q), \tag{30}$$

and we can write

$$\rho_\Lambda = \beta\lambda H = \lambda G_0 H\left[2+q-\frac{\omega}{6}(1+q)^2\right](2-q). \tag{31}$$

By using (23) and (31), we also have

$$\rho_m = \rho - \rho_\Lambda = \lambda G_0 H\left[2+q-\frac{\omega}{6}(1+q)^2\right](1+q). \tag{32}$$

We can also obtain the relative density of matter from equations (30) and (32). Recalling that $\rho_c = 3\lambda H$, we have

$$\Omega_m = \frac{\rho_m}{\rho_c} = \frac{\beta}{3}\left(\frac{1+q}{2-q}\right). \tag{33}$$

Eliminating $\omega$ from (25) and (30), one can obtain $\beta/G_0$ as a function of $q$ only,

$$\frac{\beta}{G_0} = \frac{12(2+q)+3(1+q)^2}{(1+q)(3-q)+12}(2-q). \tag{34}$$

On the other hand, comparing $\beta/G_0$ given by (25) with that given by (30), we obtain the same relation between $\omega$ and $q$ we have obtained with the first ansatz, equation (16).

This is not a mere coincidence. If we compare $\rho$ given by (23) with that obtained from equation (4) of the first ansatz, we obtain equation (9) of the first ansatz. Then, (31) can be reduced to equation (11) of the first ansatz. Equation (32), on the other hand, is reduced to equation (8) of the first ansatz.

Equation (30) can be put in the form

$$\beta = \alpha(2-q), \tag{35}$$

already anticipated in the first ansatz. By using it, it is possible to verify that equations (33) and (34) are the same as (14) and (17) of the first ansatz. Finally, one can also verify, with the help of (29), that equations (26) and (28) are identical to equations (12) and (13) of the first ansatz, respectively.

We thus conclude that, in the case of a constant $q$, the two ansätze are equivalent.

*2.2.2. The case $\dot{q} \neq 0$.* In the differential equation (24), substituting $\dot{a}/a$ for $H$ and separating the variables, we obtain

$$\frac{\mathrm{d}q}{\frac{\omega}{6}\left[\left(\frac{G_0(6\omega+6)^2+6\omega(G_0+\beta)}{G_0\omega^2}\right) - \left(q+\frac{6+7\omega}{\omega}\right)^2\right]} = \frac{3}{3+2\omega}\frac{\mathrm{d}a}{a}.$$

We will initially analyse the case in which the quantity

$$\kappa^2 = \frac{G_0(6\omega+6)^2+6\omega(G_0+\beta)}{G_0\omega^2}$$

is positive. Later on we shall analyse the cases in which it is negative or zero, respectively.

With $\kappa^2 > 0$, let us integrate the above equation by doing

$$z = q + \frac{6+7\omega}{\omega}.$$

Then, we obtain

$$\frac{6}{\omega}\int\frac{\mathrm{d}z}{\kappa^2 - z^2} = \frac{3}{3+2\omega}\int\frac{\mathrm{d}a}{a}.$$

Its solution is given by

$$a = A\left|\frac{\kappa + q + \frac{6+7\omega}{\omega}}{\kappa - q - \frac{6+7\omega}{\omega}}\right|^{\frac{3+2\omega}{\omega\kappa}},$$

where $A$ is an integration constant.

By defining $B = \kappa + (6+7\omega)/\omega$, $C = \kappa - (6+7\omega)/\omega$ and $D = (3+2\omega)/\omega\kappa$, we have

$$a = A\left|\frac{B+q}{C-q}\right|^{D}. \tag{36}$$

Then,

$$\frac{B+q}{C-q}<0 \quad \Rightarrow \quad a=A\left(\frac{B+q}{q-C}\right)^{D}, \tag{37}$$

while

$$\frac{B+q}{C-q}>0 \quad \Rightarrow \quad a=A\left(\frac{B+q}{C-q}\right)^{D}. \tag{38}$$

Let us solve equations (37) and (38), in order to find the functions $q$, $H$ and $\rho$.

*The solution of equation (37).* Introducing $x=a/A$ and inverting equation (37), we obtain

$$q=\frac{B+Cx^{\frac{1}{D}}}{x^{\frac{1}{D}}-1}. \tag{39}$$

With the definition of $q$, it becomes

$$\dot{x}^2\left(B+Cx^{\frac{1}{D}}\right)+x\ddot{x}\left(x^{\frac{1}{D}}-1\right)=0.$$

By taking $y=\dot{x}$ and $y'=\mathrm{d}y/\mathrm{d}x$, one has

$$y\left(B+Cx^{\frac{1}{D}}\right)+y'x\left(x^{\frac{1}{D}}-1\right)=0,$$

which solution is

$$y(x)=\frac{C_1x^B}{\left(x^{\frac{1}{D}}-1\right)^{D(B+C)}},$$

where $C_1$ is an integration constant.

It is easy to see that $H=y/x$, leading to

$$H(x)=\frac{C_1x^{B-1}}{\left(x^{\frac{1}{D}}-1\right)^{D(B+C)}}. \tag{40}$$

On the other hand, we have $\mathrm{d}t=\mathrm{d}x/y$, and so

$$t=\int\frac{\left(x^{\frac{1}{D}}-1\right)^{D(B+C)}}{C_1x^B}\,\mathrm{d}x. \tag{41}$$

Equations (39)–(41) are solutions of the Brans–Dicke equations (21) and (23). Let us now verify whether they satisfy the third Brans–Dicke equation (22). Introducing $\rho'=\mathrm{d}\rho/\mathrm{d}x$, and using $\rho$ given by (23), equation (22) becomes

$$3\lambda G_0\frac{\mathrm{d}H}{\mathrm{d}x}\left[2+q-\frac{\omega}{6}(1+q)^2\right]+3\lambda G_0H\frac{\mathrm{d}q}{\mathrm{d}x}\left[\frac{3-\omega(1+q)}{3}\right]$$
$$+\frac{3}{x}\left\{3\lambda G_0H\left[2+q-\frac{\omega}{6}(1+q)^2\right]-\beta\lambda H\right\}=0.$$

By using (39) and (40), we obtain

$$\frac{\beta}{G_0}=-\frac{(C+1)x^{\frac{1}{D}}+(B-1)}{x^{\frac{1}{D}}-1}\left[2+q-\frac{\omega}{6}(1+q)^2\right]$$
$$-\frac{(B+C)x^{\frac{1}{D}}}{D\left(x^{\frac{1}{D}}-1\right)^2}\left[\frac{3-\omega(1+q)}{3}\right]+3\left[2+q-\frac{\omega}{6}(1+q)^2\right].$$

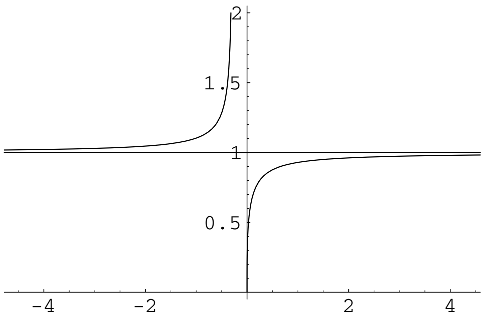


**Figure 1.** Solution (45): $a/A$ versus $C_1 t$.

Substituting in this equation $x$ given by (37), we then have

$$\frac{\beta}{G_0} = 4 - \frac{\omega}{3} - \frac{BC(\omega-3)}{3D(B+C)} + \left[\frac{(B-C)(\omega-3) - BC\omega}{3D(B+C)} - \frac{\omega}{2}\right] q + \left[\frac{\omega(B-C)+\omega-3}{3D(B+C)} - 1\right] q^2 + \left[\frac{\omega}{3D(B+C)} + \frac{\omega}{6}\right] q^3. \quad (42)$$

In this equation, $\omega$, $\beta/G_0$, $B$, $C$ and $D$ are constants. Therefore, for a varying $q$, the coefficients of $q, q^2$ and $q^3$ must be identically zero, simultaneously. This is only possible for $\omega = -1$ and $\beta/G_0 = -3$. As $G_0$ is positive, we conclude that $\beta$ is negative, that is, the cosmological term is negative.

We then have $B = \sqrt{12} + 1, C = \sqrt{12} - 1$ and $D = -1/\sqrt{12}$. In this way, equations (39)–(41) may be written as

$$q = \frac{\sqrt{12} + 1 + (\sqrt{12} - 1)x^{-\sqrt{12}}}{x^{-\sqrt{12}} - 1}, \quad (43)$$

$$H = C_1 x^{\sqrt{12}} \left(x^{-\sqrt{12}} - 1\right)^2, \quad (44)$$

$$C_1 t = \frac{1}{\sqrt{12}\left(x^{-\sqrt{12}} - 1\right)}. \quad (45)$$

In the last one, we have taken a second integration constant in such a way that $a \to 0$ when $t \to 0$.

Solution (45) is plotted in figure 1. From it we note that equation (37) originates two different universes. In one of them, we have $a = 0$ at $t = 0$ and, when $t \to +\infty$, $a/A \to 1$ asymptotically. On the other hand, $q = \sqrt{12} - 1$ for $a = 0$, tending to $+\infty$ as $t \to +\infty$. Since $q$ is positive, the expansion is decelerated, with its velocity tending to zero as $a/A \to 1$.

In the second universe, we have the origin of time in $-\infty$, with $a$ expanding from its asymptotic value $a/A = 1$ to $+\infty$, when $C_1 t \to -1/\sqrt{12}$. On the other hand, $q$ is initially $-\infty$, when $a/A \to 1$, and increases approaching $-(\sqrt{12}+1)$ asymptotically, as $a/A \to +\infty$. As $q$ is negative, the expansion is always accelerated.

*The solution of equation (38).* By performing the same transformations and the same steps used to solve (37), we obtain the following equations:

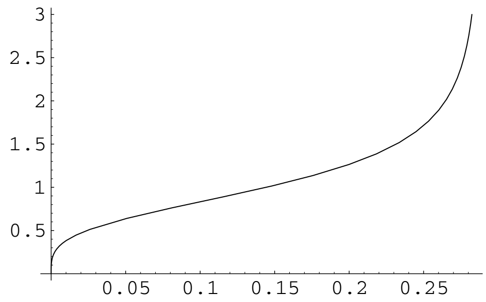


**Figure 2.** Solution (52): $a/A$ versus $C_1 t$.

$$q = \frac{Cx^{\frac{1}{D}} - B}{x^{\frac{1}{D}} + 1}, \tag{46}$$

$$H = \frac{C_1 x^{B-1}}{\left(x^{\frac{1}{D}} + 1\right)^{D(B+C)}}, \tag{47}$$

and

$$t = \int \frac{\left(x^{\frac{1}{D}} + 1\right)^{D(B+C)}}{C_1 x^B}\, \mathrm{d}x, \tag{48}$$

with $B$, $C$ and $D$ defined as before.

Equations (46)–(48) are solutions of (21) and (23). Let us verify whether they satisfy the third field equation (22). Once more, performing the same transformations used to solve (37), we arrive at the same equation (42) derived before. As we have seen, only with $\omega = -1$ and $\beta/G_0 = -3$ we have the coefficients of $q$, $q^2$ and $q^3$ identically zero, simultaneously. Then, we have again $B = \sqrt{12} + 1$, $C = \sqrt{12} - 1$ and $D = -1/\sqrt{12}$.

Therefore, equations (38) and (46)–(48) can be written as

$$x = \left(\frac{\sqrt{12} + 1 + q}{\sqrt{12} - 1 - q}\right)^{-\frac{1}{\sqrt{12}}}, \tag{49}$$

$$q = \frac{(\sqrt{12} - 1)x^{-\sqrt{12}} - \sqrt{12} - 1}{x^{-\sqrt{12}} + 1}, \tag{50}$$

$$H = C_1 x^{\sqrt{12}}\left(x^{-\sqrt{12}} + 1\right)^2, \tag{51}$$

$$C_1 t = -\frac{1}{\sqrt{12}\left(x^{\sqrt{12}} + 1\right)} + \frac{1}{\sqrt{12}}. \tag{52}$$

In the last one, we have chosen the second integration constant in such a way that $a \to 0$ when $t \to 0$.

Solution (52) is plotted in figure 2. We have $a = 0$ at $t = 0$, and $a \to +\infty$ when $C_1 t \to 1/\sqrt{12}$. On the other hand, $q = \sqrt{12} - 1$ for $a = 0$ decreases with the expansion, becomes negative, and tends to $-(\sqrt{12} + 1)$ when $a \to +\infty$.

With the help of (49), we can express equations (51) and (52) as functions of $q$:

$$H = -\frac{48 C_1}{q^2 + 2q - 11}, \tag{53}$$

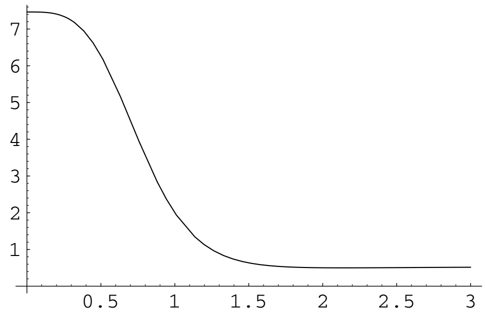


**Figure 3.** Solution (52): $\Omega_m/G_0$ versus $a/A$.

$$C_1 t = \frac{\sqrt{12} - 1 - q}{24}, \tag{54}$$

with $q$ in the interval $(-\sqrt{12} - 1, \sqrt{12} - 1]$.

The age parameter, on the other hand, can be obtained from (53) and (54), leading to

$$Ht = \frac{2}{q + \sqrt{12} + 1}. \tag{55}$$

By using (18), (23) and (53), we obtain

$$\rho_m = -24\lambda G_0 C_1 \left(\frac{q^2 + 8q + 19}{q^2 + 2q - 11}\right). \tag{56}$$

The relative density of matter can then be obtained with the help of equations (53) and (56), and is given by

$$\frac{\Omega_m}{G_0} = \frac{1}{6}(q^2 + 8q + 19). \tag{57}$$

Equations (56) and (57) can be expressed in terms of $x$, by using (50). We have

$$\frac{\rho_m}{3\lambda G_0 C_1} = (4 - \sqrt{12})x^{\sqrt{12}} + (4 + \sqrt{12})x^{-\sqrt{12}} \tag{58}$$

and

$$\frac{\Omega_m}{G_0} = \frac{(4 + \sqrt{12})x^{-2\sqrt{12}} + 4 - \sqrt{12}}{\left(x^{-\sqrt{12}} + 1\right)^2}, \tag{59}$$

which is plotted in figure 3.

*The case* $\kappa^2 < 0$. The above solutions were derived from the differential equation (24) by assuming $\kappa^2 > 0$. Let us now suppose that it is negative. We can solve equation (24) by doing

$$\kappa_1^2 = -\frac{G_0(6\omega + 6)^2 + 6\omega(G_0 + \beta)}{G_0\omega^2}$$

and

$$z = q + \frac{6 + 7\omega}{\omega}.$$

In this way, we obtain

$$-\frac{6}{\omega}\int\frac{\mathrm{d}z}{\kappa_1^2+z^2}=\frac{3}{3+2\omega}\int\frac{\mathrm{d}a}{a},$$

which solution is

$$a=C_1\exp\left(-\frac{6+4\omega}{\omega\kappa_1}\arctan\frac{q+\frac{6+7\omega}{\omega}}{\kappa_1}\right),$$

where $C_1$ is an integration constant.

Introducing $D_1=\omega\kappa_1/(6+4\omega)$ and $E=(6+7\omega)/\omega$, one has

$$a=C_1\exp\left(-\frac{1}{D_1}\arctan\frac{q+E}{\kappa_1}\right). \tag{60}$$

Taking its inverse function and defining $x=\ln(a/C_1)$, we obtain

$$q=-\kappa_1\tan(D_1x)-E, \tag{61}$$

or, by using the definition of $q$,

$$\ddot{x}+\dot{x}^2[1-\kappa_1\tan(D_1x)-E]=0.$$

After doing $y=\dot{x}$ and $y'=\mathrm{d}y/\mathrm{d}x$, we obtain

$$y'+y[1-\kappa_1\tan(D_1x)-E]=0,$$

whose solution is

$$y(x)=H=C_2\,\mathrm{e}^{(E-1)x}[\cos(D_1x)]^{-\frac{\kappa_1}{D_1}}, \tag{62}$$

where $C_2$ is a second integration constant.

As $\mathrm{d}t=\mathrm{d}x/y$, we also have

$$t=\int\frac{[\cos(D_1x)]^{\frac{\kappa_1}{D_1}}}{C_2\,\mathrm{e}^{(E-1)x}}\,\mathrm{d}x. \tag{63}$$

Equations (61)–(63) are solutions of Brans–Dicke equations (21) and (23). As before, let us verify whether they also satisfy the remaining equation (22). Introducing $\rho'=\mathrm{d}\rho/\mathrm{d}x$ and using $\rho$ given by (23), equation (22) becomes

$$3\lambda G_0\frac{\mathrm{d}H}{\mathrm{d}x}\left[2+q-\frac{\omega}{6}(1+q)^2\right]+3\lambda G_0H\frac{\mathrm{d}q}{\mathrm{d}x}\left[\frac{3-\omega(1+q)}{3}\right]$$
$$+3\left\{3\lambda G_0H\left[1+q-\frac{\omega}{6}(1+q)^2\right]-\beta\lambda H\right\}=0. \tag{64}$$

By using (62), one obtains

$$\frac{\beta}{G_0}=[E+2+\kappa_1\tan(D_1x)]\left[2+q-\frac{\omega}{6}(1+q)^2\right]+\frac{\mathrm{d}q}{\mathrm{d}x}\left[\frac{3-\omega(1+q)}{3}\right].$$

Now, with the help of (61), we arrive at

$$\frac{\beta}{G_0}=(2-q)\left[2+q-\frac{\omega}{6}(1+q)^2\right]+\frac{\dot{q}}{3H}[3-\omega(1+q)], \tag{65}$$

since $\mathrm{d}q/\mathrm{d}x=\dot{q}/H$.

On the other hand, from (24) one can obtain

$$\frac{\dot{q}}{3H}=\frac{2+q-\frac{\omega}{6}(1+q)^2-(3+2\omega)(1+q)+\frac{\beta}{G_0}}{3+2\omega}.$$

Leading this expression into (65), we have

$$9 + \frac{3\omega}{2} + \frac{3\omega^2}{2} - 3\omega\frac{\beta}{G_0} + \left(-6 - \frac{11\omega}{2} + \frac{7\omega^2}{2} - \omega\frac{\beta}{G_0}\right)q + \left(-3 - \frac{\omega}{2} + \frac{5\omega^2}{2}\right)q^2 + \left(\frac{\omega}{2} + \frac{\omega^2}{2}\right)q^3 = 0. \qquad (66)$$

It is possible to verify that (42) and (66) are identical. In the latter, $\omega$ and $\beta/G_0$ are constants. Therefore, for a varying $q$, the coefficients of $q$, $q^2$ and $q^3$ must be simultaneously zero, which is only possible if $\omega = -1$ and $\beta/G_0 = -3$. But, in this case, $\kappa_1^2 = -12$, contrary to our initial supposition that $\kappa_1^2 > 0$.

Therefore, in the case $\dot{q} \neq 0$, equation (60) satisfies the Brans–Dicke equations (21) and (23), but not (22). The later is satisfied only if $\dot{q} = 0$, in which case equation (65) reduces to (30), already studied.

*The case* $\kappa^2 = 0$. In order to fulfil all the possible cases (and solutions) let us suppose that $\kappa^2 = 0$. From (24) we have

$$-\frac{6}{\omega}\int\frac{\mathrm{d}z}{z^2} = \frac{3}{3+2\omega}\int\frac{\mathrm{d}a}{a},$$

where we have done, as before,

$$z = q + \frac{6+7\omega}{\omega}.$$

Its solution is

$$\frac{1}{q + \frac{6+7\omega}{\omega}} = \frac{\omega}{2(3+2\omega)}\ln\frac{a}{C_1},$$

where $C_1$ is an integration constant.

Taking $D_2 = \omega/[2(3+2\omega)]$ and, as before, $E = (6+7\omega)/\omega$, one obtains

$$\frac{1}{q+E} = D_2\ln\frac{a}{C_1}.$$

Introducing the new variable $x = \ln(a/C_1)$, we then have

$$q = \frac{1}{D_2 x} - E. \qquad (67)$$

With the definition of $q$, this becomes

$$\dot{x}^2 + D_2(1-E)x\dot{x}^2 + D_2 x\ddot{x} = 0,$$

or, by doing $F = D_2(1-E)$,

$$\dot{x}^2 + Fx\dot{x}^2 + D_2 x\ddot{x} = 0.$$

Taking now $y = \dot{x}$ and $y' = \mathrm{d}y/\mathrm{d}x$, we obtain

$$y + Fxy + D_2 x y' = 0,$$

which solution is

$$y(x) = H = C_2\exp\left(-\frac{Fx + \ln x}{D_2}\right), \qquad (68)$$

where $C_2$ is another integration constant.

As in the previous case, equations (67) and (68) are solutions of (21) and (23), but we should also verify whether they satisfy (22). Introducing $\rho' = \mathrm{d}\rho/\mathrm{d}x$ and using $\rho$ given by (23), equation (22) reduces, as we have seen, to (64). Now, using equations (68) and (67) leads to the same equation (65) of the previous case, which, as already seen, leads to (66).

Therefore, solutions with varying $q$ are only possible if $\omega = -1$ and $\beta/G_0 = -3$. With these values, however, $\kappa^2 = 12$, contrary to our initial supposition that $\kappa^2 = 0$. We thus conclude, also in this case, that (22) is not satisfied, that is, there is no solution with varying $q$.

## 3. Conclusions

In this work we have found some exact solutions of Brans–Dicke cosmology, by using two different ansätze. We have shown that the first ansatz is a particular case of the second one, when the deceleration parameter $q$ is constant.

In the first ansatz, the ratio between the energy densities of matter and vacuum is constant, characterizing a possible solution for the cosmic coincidence problem, that is, the approximated coincidence presently observed between $\rho_m$ and $\rho_c$. This possibility survives to a quantitative analysis, since a relative matter density around $1/3$, as indicated by observations, leads to an age parameter $Ht \approx 1$, corresponding to a universe age around 14 Gyr, also in accordance with observational limits.

Nevertheless, this ansatz presents some problems as well. The most severe of them is the presence of a constant deceleration factor (equals to zero if $Ht = 1$). In spite of the claim of some authors (see, for example, [28, 29]) about the possibility of a uniform expansion along the whole universe evolution, a decelerated phase is usually considered necessary for large structure formation. For this reason, we should consider this ansatz valid only in the limit of late times, restricting in this way the predictive power of the model.

With the second ansatz, besides the case of constant $q$, we have found three other universes, with varying $q$, in which the dark energy density is negative and the Brans–Dicke parameter is $\omega = -1$. In one of them the deceleration parameter is always highly positive. In the second one, it is always highly negative. Therefore, these two cases are interesting just from a theoretical viewpoint.

In the third case, on the other hand, the deceleration parameter is initially positive, becoming negative at later times, but always finite. In this case (as well as in one of the previous cases) one has a future *big-rip*, with the scale factor, the matter density and the Hubble parameter diverging in a finite time, but with the relative matter density remaining finite. As one can see from equation (55), for an age parameter in the interval $0.8 < Ht < 1.3$, as defined by the observational limits, the deceleration parameter is $-2.0 > q > -2.9$. Whence, with the help of (57), it is possible to see that we have, for the relative matter density, $1.1 > \Omega_m/G_0 > 0.7$. As we know, different observations restrict the matter density parameter to $0.2 < \Omega_m < 0.4$. Therefore, this solution satisfies such observations, provided $G_0$ is in the interval $0.3 < G_0 < 0.4$. Furthermore, for the whole evolution we have $\Omega_m/G_0 < 7.5$, tending, for future times, to a constant value around 0.5 (see figure 3). This also characterizes a possible explanation for the cosmic coincidence.

It is interesting to observe that, in the three cases with varying $q$, the cosmological time varies linearly with $q$, which may, therefore, be used to define the time measurement. It is interesting to note as well that the total energy density can be negative, since the dark energy density is negative. In our last solution, for example, $\rho$ is positive until $q \simeq -2.3$, becoming negative since then.

There is a particular result which may seem a limitation of our solutions, namely the typical values found for the Brans–Dicke parameter $\omega$. As we know, observations in the realm of solar system impose very high inferior limits for it. Let us remember, however, that we are considering the simplest version of scalar–tensor theories, which plays just an effective role here. Corrections to general relativity, if exist, may be scale dependent, and, therefore, observations in the solar system cannot, in principle, impose limits to corrections at the cosmological scale. Particularly, we should not expect any time dependence of the Brans–Dicke scalar field (and so of $G$) in the solar system, where the metric is stationary. While no spatial dependence should exist in large scale, because of the cosmological principle.

Anyway, a generalization of the solutions studied here seems to be necessary, either by modifying our ansätze in the case of early times, or by considering more general scalar–tensor theories, with $\omega$ depending on the scale. The study performed here, though limited in its scope, shows the variability of solutions in these contexts.

We should also note that, for the solution of Brans–Dicke equations, it is enough to add, for instance, the Eddington–Dirac relation. The inclusion of additional constraints, even when empirically or theoretically justified, had the goal of limiting the set of solutions. Therefore, a possible line of investigation would be to relax our ansätze, imposing to the Brans–Dicke equations, for example, just the constraint given by the Eddington–Dirac relation, enlarging in this way the class of possible solutions. Such a generalization may also include the radiation phase, although we do not have any indication about the validity of the Eddington–Dirac relation at early times.

To conclude, let us remind that, despite the analysis we have made about the observational limits for the age and matter density parameters, a more detailed analysis of the whole set of current observational data is still in order. In particular, a careful study of the distance–redshift relation for Ia supernovae constitutes the subject of a forthcoming publication.

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