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Gauge symmetry in Fokker–Planck dynamics

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Abstract

Using a Galilean metric approach, based on an embedding of the Euclidean space into a $(4+1)$ -Minkowski space, we analyze a gauge invariant Lagrangian associated with a Riemannian manifold \mathcal{R} , with metric g . With a specific choice of the gauge condition, the Euler–Lagrange equations are written covariantly in \mathcal{R} , and then the Fokker–Planck equation is derived, such that the drift and the diffusion terms are obtained from g . The analysis is carried out for both, Abelian and non-Abelian symmetries, and an example with the $su(2)$ symmetry is presented.

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1. Introduction

In this paper, we show that the Fokker–Planck equation can be derived via a gauge invariant theory. The basic ingredient in the derivation is Galilean covariance, which has been recently developed in different perspectives, providing a metric, and thus a tensor, structure for non-relativistic theory based in a $4+1$ Minkowski space [1–8]. As a consequence, a geometric unification of the non-relativistic and relativistic physics is accomplished [4,6]. One interesting result is that the possibility to use ideas and

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concepts of particles physics in transport theory, such as topological terms, symmetry breaking, gauge symmetries, and so on [1,2,9], can be investigated in a systematic and covariant way paralleling the relativistic physics [4,10]. In this context, it would be of interest to analyze typical stochastic processes such as those described by the Fokker–Planck dynamics.

The Fokker–Planck equation is often derived in the analysis of Markov processes. From a physical standpoint, it can be introduced either as the distribution of probability version of the Langevin equation, describing a classical particle under the influence of dissipative and stochastic forces [11–13], or as an approximation of the Boltzmann equation [14]. In this latter case, the collision term is approximated to consider the transition rate, say $W(p_1, k)$, where $p_1 = p + k$, terms up to the second order in k , resulting then in the drift and the diffusion terms of the Fokker–Planck equation [15]. Here we proceed in a different way, by analyzing (first) a $U(1)$ gauge invariant Lagrangians, in the $(4+1)$ -dimensional Minkowski space (to be referred to as \mathcal{G}). Using a suitable gauge condition and a proper definition of each component for the gauge field, the Euler–Lagrange equations result in the Fokker–Planck equation. The definition of the gauge field is based on the existence of a Riemannian manifold, say $\mathcal{R}(\mathcal{G})$, with metric g , in which \mathcal{G} is taken as a local flat space. Taking the five-dimensional equations covariantly written in $\mathcal{R}(\mathcal{G})$, the gauge field is defined with the use of the metric tensor, which gives rise to the drift and diffusion terms of the Fokker–Planck equation. The analysis of the connection, defined by g , establishes whether the diffusion tensor is a constant or not by a proper coordinate transformation. These results, in addition to improving the study of symmetries of the Fokker–Planck systems [16], opens the possibility to include in the description of stochastic processes non-Abelian gauge symmetry. This aspect is developed here by using, in particular, the $SU(2)$ gauge symmetry, following the methods of field theory, rather than the generalization of symplectic structures and Liouville equation [17].

The presentation is organized as it follows. In Section 2, to make the presentation self-contained and to fix the notation, a brief outline on the Galilei covariance is presented. The Fokker–Planck equation is derived from an Abelian gauge invariant Lagrangian in Section 3, and in Section 4 the non-Abelian situation is addressed. Concluding remarks are presented in Section 5.

2. Outline on the Galilei covariance

Let us begin with a brief outline of the Galilean covariant methods (for more details see for instance Refs. [5–7]). Let \mathcal{G} be a five-dimensional metric space, with an arbitrary vector denoted by $x = (x^1, x^2, x^3, x^4, x^5) = (\mathbf{x}, x^4, x^5)$. The inner product in \mathcal{G} is then defined by

$$(x|y) = \eta_{\mu\nu} x^\mu y^\nu = \sum_{i=1}^{n=3} x^i y^i - x^4 y^5 - x^5 y^4 \quad (1)$$

with $x, y \in \mathcal{G}$ and $\eta_{\mu\nu}$ being given by

$$\eta = \delta_{ij} dx^j \otimes dx^i - dx^4 \otimes dx^5 - dx^5 \otimes dx^4. \quad (2)$$

The set of linear transformations in \mathcal{G} of the type $\bar{x}^\mu = G^\mu_\nu x^\nu + a^\mu$ (that leaves $(dx|dy)$ invariant), such that $|G| = 1$, with $G^\mu_\nu = \delta^\mu_\nu + \varepsilon^\mu_\nu$, admits 15 generators of transformations, and 11 of them provide the Lie Galilei algebra with the usual central extension, describing the mass of a particle, being a generator of the group.

Consider now the embedding of the Euclidean space \mathcal{E} in \mathcal{G} , given by

$$\mathbf{A} \mapsto A = \left(\mathbf{A}, A_4, \frac{\mathbf{A}^2}{2A_4} \right), \tag{3}$$

where $\mathbf{A} = (A^1, A^2, A^3) \in \mathcal{E}, A \in \mathcal{G}$. It follows that A is a null-like vector, since

$$\begin{aligned} (A|A) &= \eta_{\mu\nu} A^\mu A^\nu \\ &= \sum_{i=1}^3 A^i A^i - 2A^4 A^5 = 0. \end{aligned}$$

In other words, according to Eq. (3), each vector in \mathcal{E} is in homomorphic correspondence with null-like vectors in \mathcal{G} . As an example, consider $x = (\mathbf{x}, kt, \mathbf{x}^2/2kt)$, where k is a constant with units of velocity (we consider $k = 1$). Under the subgroup of linear transformations in \mathcal{G} , given by the generators

$$K_i = e^{-v^i B_i}, \quad R_{ij} = e^{\varepsilon_{ijk} L_k}, \quad T_i = e^{a^i P_i}, T_4 = e^{bH},$$

where $a^5 = 0, H = P^4$, the vector $x = (\mathbf{x}, t, \mathbf{x}^2/2t)$ transforms as a Galilean vector; that is

$$\bar{x}^i = R^i_j x^j - v^i x^4 + \mathbf{a}, \tag{4}$$

$$\bar{x}^4 = x^4 + b, \tag{5}$$

$$\bar{x}^5 = x^5 - v^i (R^i_j x^j) + \frac{1}{2} \mathbf{v}^2 x^4. \tag{6}$$

There are at least two other types of embeddings which will be useful here:

$$\mathbf{A} \mapsto A = (\mathbf{A}, A_4, 0) \tag{7}$$

and

$$\mathbf{A} \mapsto A = \left(\mathbf{A}, \frac{1}{\sqrt{2}} A_4, \frac{1}{\sqrt{2}} A_4 \right). \tag{8}$$

We take \mathcal{G} to be a frame of the following form:

$$F = dJ + \lambda J \wedge J \tag{9}$$

satisfying the Bianchi identities $dF + \lambda[J, F] = 0$ and the equation regarding the sources which are considered to be non-existent, that is $d * F + \lambda[J, *F] = 0$.

3. Abelian gauge symmetry

We consider $\lambda = 0$ in Eq. (9) and write the following $U(1)$ -gauge invariant Lagrangian, in terms of the components of F (say $F^{\mu\nu}$),

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}, \tag{10}$$

where $F^{\mu\nu}$ is written in terms of the Abelian gauge fields as

$$F_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu, \quad (11)$$

where J remains to be specified. Following the usual procedure, we can also write down the Lagrangian as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu J_\nu \partial^\nu J^\mu - \partial_\mu J_\nu \partial^\mu J^\nu) \quad (12)$$

resulting in the Euler–Lagrange equations

$$\partial^\mu \partial_\mu J^\nu - \partial^\nu \partial_\mu J^\mu = 0. \quad (13)$$

The Lagrangian \mathcal{L} is invariant under the gauge transformation $J^\mu \rightarrow \bar{J}^\mu = J^\mu + \partial^\mu h(x)$, and in the ordinary procedure, we take $\partial_\mu J^\mu = 0$ as the (Lorentz) gauge condition in order to derive, from Eq. (13), the wave equation for the electromagnetic field, that is $\partial^\mu \partial_\mu J^\nu = 0$. Here, we are not interested in interpreting J^ν as a vector potential, so that we have the freedom to explore a different gauge condition. We take then the gauge condition to be $\partial^\mu \partial_\mu J^\nu = 0$, such that $h(x)$ fulfills the constraint equation $\partial^\mu \partial_\mu h(x) = \beta$, where β is an arbitrary constant. As a result $\partial_\mu J^\mu = \alpha$, where α is another arbitrary constant, which can be assumed to be zero. The Euler–Lagrange equations can then be written as

$$\partial_\mu J^\mu = 0. \quad (14)$$

In order to specify the five-dimensional vector field theory, we assume the existence of a Riemannian manifold, $\mathcal{R}(\mathcal{G})$, with metric $g^{\mu\nu}(x)$, such that at each point of $\mathcal{R}(\mathcal{G})$ there is a flat space \mathcal{G} . The covariant form of Eq. (14) is

$$\partial_\mu (g^{1/2} J^\mu) = 0, \quad (15)$$

such that J^μ is considered as a covariant current density in $\mathcal{R}(\mathcal{G})$. We can construct J as an explicit derivative of a tensor of the theory; and the natural candidate for such a proposal is $g^{\mu\nu}(x)$. In this way, the physical content of J as a current can be emphasized. Using a general expression for a (covariant-like) derivative, say $\partial_\nu + f_\nu(x)$, we define $J^\mu = g^{-1/2} S^\mu$, where $S^\mu = (f_\nu(x) + \partial_\nu) g^{\mu\nu}(x)$, with f_ν being a five-vector given by $f^\nu(x) = (f^i(\mathbf{x}, t), f^4(\mathbf{x}, t), 0)$ (we have taken $f^5(\mathbf{x}, t) = 0$ for sake of convenience). The five-vector S plays the role of the current density in the Minkowski space, and J the covariant gravitational counterpart. Nevertheless, it is worth noting that here we are working in the five-dimensional space \mathcal{G} without use of the equivalence principle of the general relativity. The physical meaning of the Riemannian manifold $\mathcal{R}(\mathcal{G})$ will be discussed in the following:

Using Eq. (15) we find

$$\begin{aligned} \partial_\mu S^\mu &= \partial_\mu f^\mu + \partial_\mu \partial_\nu g^{\mu\nu} \\ &= \partial_4 f^4 + \partial_i f^i + \partial_\mu \partial_\nu g^{\mu\nu} = 0. \end{aligned} \quad (16)$$

This equation can be converted into a Fokker–Planck equation if we define

$$\begin{aligned} f^i(\mathbf{x}, t) &= D^i(\mathbf{x}, t) P(\mathbf{x}, t), \\ f^4(\mathbf{x}, t) &= P(\mathbf{x}, t) \end{aligned}$$

and the metric tensor as

$$g = P(x)D_{ij}(x) dx^j \otimes dx^i - dx^4 \otimes dx^5 - dx^5 \otimes dx^4, \tag{17}$$

where $P(x)=P(\mathbf{x}, t)$ is a scalar function and $D^{ij}(x)$ are the components of a Riemannian metric associated with the Euclidean space. The components of S are given by

$$S^i = D^i(x)P(x) + \partial_j P(x)D^{ij}(x),$$

$$S^4 = P(x),$$

$$S^5 = 0.$$

Using Eq. (16), with the embedding $x = (\mathbf{x}, t, \mathbf{x}^2/2t)$, we obtain

$$\partial_t P(\mathbf{x}, t) = \frac{\partial}{\partial x^i} \left[-D^i(\mathbf{x}, t)P(\mathbf{x}, t) + \frac{\partial}{\partial x^j} D^{ij}(\mathbf{x}, t)P(\mathbf{x}, t) \right]. \tag{18}$$

This is the Fokker–Planck equation with $D^i(\mathbf{x}, t)$ standing for the drift term and $D^{ij}(\mathbf{x}, t)$ the diffusion tensor, since we can take $P(\mathbf{x}, t)$ as a real positive and normalized function, such that it can be interpreted as a (covariant) probability density. This probability attribute of $P(x)$ is consistent with the fact that $P(x)$ cannot be zero, providing then that $g^{\mu\nu}(x)$ has an inverse, say $g_{\mu\nu}(x)$.

The Riemannian space $\mathcal{R}(\mathcal{G})$ has been used to introduce the drift and the diffusion tensor in Fokker–Planck dynamics. Regarding the metric $D^{ij}(\mathbf{x}, t)$, the connection in the Euclidean part of $\mathcal{R}(\mathcal{G})$ is given by

$$\Gamma_k^{ij} = \frac{1}{2} \left(\frac{\partial D^{jk}}{\partial x^i} + \frac{D^{ik}}{\partial x^j} - \frac{D^{ij}}{\partial x^k} \right).$$

If $\Gamma_k^{ij}=0$, we recover the Euclidean flat space, and therefore there exists a transformation $U(x)$ such that

$$UgU^{-1} = P(x)D\delta_{ij} dx^j \otimes dx^i - dx^4 \otimes dx^5 - dx^5 \otimes dx^4, \tag{19}$$

where D is a constant. Hence, the diffusion tensor can be diagonalized. This is a result derived by Graham [18] in a work analyzing the invariance properties of the Fokker–Planck equation. In our case, the invariance has been used, from the beginning, as a central ingredient to write the Lagrangian given in Eq. (10) and the corresponding covariant Eq. (15).

Let us briefly discuss the case of relativistic Fokker–Planck equation. This can be obtained if we use Eq. (8) with $x^v = (x^i, ct/\sqrt{2}, ct/\sqrt{2})$, where c is the speed of light, and $f^v(x)=(f^i(\mathbf{x}, t), P/\sqrt{2}, P/\sqrt{2})$. In this case, for instance, we have $x^\mu x_\mu = x^i x_i - (ct)^2$, which is a vector in Minkowski space. Then we have the following correspondence of five-tensors in \mathcal{G} into four-tensors in the Minkowski space,

$$\partial_\mu \rightarrow \hat{\partial}_\mu = (\hat{\partial}_0 = \partial_{ct}, \partial_i),$$

$$S^\mu \rightarrow S^\mu = (S^0 = P(x), S^i = D^i(x)P(x) + \partial_j P(x)D^{ij}(x)),$$

$$\eta \rightarrow \hat{\eta} = \delta_{ij} dx^j \otimes dx^i - dx^0 \otimes dx^0.$$

Using these definitions and Eq. (14) (but now in the Minkowski space), we derive a relativistic Fokker–Planck equation which has the same form as Eq. (18) for $t \rightarrow ct$ (we consider $c = 1$). Therefore, the usual Fokker–Planck equation can be taken as Lorentz invariant, provided there exists a drift four-vector given by $f^v(x) = (P(\mathbf{x}, t), f^i(\mathbf{x}, t))$, and a Riemannian metric given by

$$g = P(x)D_{ij}(x) dx^j \otimes dx^i - dx^0 \otimes dx^0 .$$

In the next section we consider non-Abelian symmetries.

4. Non-Abelian gauge symmetry

Generalization of these results for non-Abelian gauge fields can be addressed as well. (From this point on, the covariant notation means relativistic or non-relativistic theory.) In five dimensions a pure Yang–Mills field can be described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^{a\mu\nu} F_{a\mu\nu} , \tag{20}$$

where the Latin index, a , stands for the gauge group, with generators t^a , $a = 1, 2, \dots, n$, satisfying the Lie algebra $[t^a, t^b] = C_c^{ab} t^c$, where C_c^{ab} are structure constants of the gauge group (sum over repeated Latin indices is assumed). The field strength tensor $F_{\mu\nu}^a$ is given by

$$F_{\mu\nu}^a = \partial_\mu J_{\nu a} - \partial_\nu J_{\mu a} - \lambda C_a^{bc} J_{\mu b} J_{\nu c}$$

for which the equation of motion is written as

$$D_a^{\mu b} F_{\mu\nu}^b = 0 ,$$

where $D_a^{\mu b}$ is the covariant derivative given by $D_a^{\mu b} = \partial^\mu \delta_a^b + \lambda C_a^{bc} J_c^\mu$. The equations of motion for each component of J are

$$\begin{aligned} \partial_\nu \partial_\mu J_a^\mu &= \lambda C_a^{bc} \partial_\mu (J_c^\mu J_{\nu b}) + \lambda C_a^{bc} J_c^\mu \partial_\mu J_{\nu b} \\ &+ \lambda C_a^{cb} J_c^\mu \partial_\nu J_{\mu b} + \lambda^2 C_a^{cb} C_b^{de} J_c^\mu J_{\mu d} J_{\nu e} , \end{aligned} \tag{21}$$

where use has been made of the aforementioned gauge condition $\partial_\mu \partial^\mu J_{\nu a} = 0$. Despite the non-linear structure of these equations, a Fokker–Planck system can be recognized, if we assume J is defined as before (in terms of S), and discard all the non-linear terms in Eq. (21), such that $\partial^\mu \partial_\nu J_{\mu a} = 0$. As a consequence

$$\partial_\mu J_a^\mu = \alpha , \tag{22}$$

where α is a constant. Taking $\alpha = 0$, we obtain Eq. (14), and so a Fokker–Planck equation, for each gauge index a . On the other hand, consider $\alpha \ll 1$, then Eq. (21) reduces, up to second order terms in $\lambda\alpha$, to

$$\partial_\nu (\partial_\mu J_a^\mu + \lambda C_a^{bc} J_c^\mu J_{\mu b}) = 2\lambda C_a^{bc} J_c^\mu \partial_\mu J_{\nu b} + \lambda C_a^{bc} (\partial_\nu J_c^\mu) J_{\mu b} . \tag{23}$$

The left-hand side of this equation can be integrated for each $\nu = 1, \dots, 5$, so that the right-hand side results in a non-local term along each direction. Discarding this

non-local term we obtain the following non-linear equation:

$$\partial_\mu J_a^\mu + \lambda C_a^{bc} J_c^\mu J_{\mu b} = 0 . \tag{24}$$

Let us consider as an example the $su(2)$ symmetry with J_a^μ defined by

$$\begin{aligned} J_a^i &= \varepsilon_{aij}(D_j^k P_k + D_j^{nk} \partial_k P_n) , \\ J_a^4 &= P_a , \\ J_a^5 &= 0 , \end{aligned}$$

where both gauge and tensor indices are of the same tensor nature ($i, j, k, a, b, c = 1, 2, 3$), $D_j^k = D_j^k(x)$ describes the drift term (which is now a second rank tensor, taking into account the vector and the gauge index), whilst D_j^{nk} , independent of x , stands for the diffusion term. Notice that this definition can be developed with the reasoning used in the Abelian case. With Eq. (24), we see that $\varepsilon_{abc} J_c^\mu J_{\mu b} = \varepsilon_{abc} J_{ic} J_{ib} = 0$. Hence,

$$\partial_t P_a = \varepsilon_{aji} [\partial_i (D_j^b P_b) + D_j^{cb} \partial_i \partial_b P_c] . \tag{25}$$

The content of this Fokker–Planck-like equation can be analyzed in a simple particular situation. Consider the stationary situation $\partial_t P_a = 0$, and define

$$\begin{aligned} P_2 &= P_3 = P , \\ D_2^1 &= kx^3 = kz , \\ D_3^1 &= kx^2 = ky , \\ D_2^{13} &= D_3^{12} = \frac{D}{2} , \end{aligned}$$

where P, D and k are constant and the other components of D_j^b and D_j^{cb} are zero. (The expressions for D_2^1 and D_3^1 , the drift terms, and $D_2^{13} = D_3^{12}$, the diffusion tensor assure we have an Ornstein–Uhlenbeck-like process [11,13] for this color theory.) Writing $P_1(y, z) = \varphi(y)\phi(z)$, we derive

$$\frac{1}{\phi(z)} \left[\frac{D}{2} \frac{d^2}{dz^2} \phi(z) + \frac{d}{dz} (kz\phi(z)) \right] = \frac{1}{\varphi(y)} \left[\frac{D}{2} \frac{d^2}{dy^2} \varphi(y) + \frac{d}{dy} [ky\varphi(y)] \right] .$$

Therefore, we can write

$$\frac{D}{2} \frac{d^2}{dy^2} \varphi(y) + \frac{d}{dy} [ky\varphi(y)] = \varphi(y)F ,$$

where F is a constant. If $F \neq 0$, a solution is given with $F = 3k$, such that

$$\varphi(y) = a_0 \left(y^2 + \frac{D}{k} \right) + a_1 \exp(-ky^2/D) ,$$

where a_0, a_1 are constant; and similarly for $\phi(z)$. However, these type of solution diverges for $y, z \rightarrow \infty$. A non-divergent solution is found for $F = 0$. In this case,

we recover a known result for $P_1(y, z)$, that is

$$\begin{aligned} P_1(y, z) &= \varphi(y)\phi(z) \\ &= \frac{1}{N} \exp(-k(y^2 + z^2)/D) \end{aligned}$$

with N being a normalization constant.

5. Concluding remarks

Summarizing, our main goal in this work has been to derive the Fokker–Planck dynamics via a variational principle, considering gauge invariant Lagrangians written in a $(4 + 1)$ Minkowski space \mathcal{G} . First we consider the $U(1)$ symmetry with a suitable choice for the gauge condition, and a Riemannian manifold $\mathcal{R}(\mathcal{G})$ specified by the metric g given in Eq. (17). Then using Eq. (14) the Fokker–Planck equation has been derived. The physical meaning of the manifold $\mathcal{R}(\mathcal{G})$ is crucial for the definition of the diffusion and drift terms, which are derived from the metric tensor g . This is why the equivalence principle was no longer invoked.

The analysis has been extended to deal with non-Abelian symmetries and the relativistic case, taking the advantage of proper embeddings in this five-dimensional formalism. For non-Abelian groups, despite the difficulties imposed with the non-linearity, we have been able to recognize a Fokker–Planck dynamics and discuss, as an example, a solution for the $su(2)$ gauge symmetry. Other possibilities for the gauge group remain to be studied, and a situation involving the $su(3)$ symmetry will be addressed elsewhere.

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References

- [1] Y. Takahashi, Fortschr. Phys. 36 (1988) 63.
- [2] Y. Takahashi, Fortschr. Phys. 36 (1988) 83.
- [3] M. Omote, S. Kamefuchi, Y. Takahashi, Y. Ohnuki, Fortschr. Phys. 37 (1989) 933.
- [4] C. Duval, G. Burdet, H.P. Künzle, M. Perrin, Phys. Rev. D 31 (1985) 1841; H.P. Künzle, C. Duval, Relativistic and nonrelativistic physical theories on five-dimensional space time, in: U. Majer, H.J. Schmidt (Eds.), Semantical Aspects of Spacetime Theories, BI-Wissenschaftsverlag, Mannheim, 1994, p. 113.
- [5] A.E. Santana, F.C. Khanna, Y. Takahashi, Prog. Theor. Phys. 99 (1998) 327.
- [6] M. de Montigny, F.C. Khanna, A.E. Santana, E.S. Santos, J.D.M. Vianna, Ann. Phys. (NY) 277 (1999) 144.
- [7] M. de Montigny, F.C. Khanna, A.E. Santana, Galilean covariance and applications in physics, in: Recent Research Developments in Physics, Vol. 1, Transworld Research Network, Trivandrum, 2001, p. 45.

- [8] M. de Montigny, F.C. Khanna, A.E. Santana, J. Phys. A 34 (2001) 10921.
- [9] R. Jackiw (A particle field theorist's) Lectures on (supersymmetric, non-Abelian) fluid mechanics (and d-branes) 2000, Preprint physics/0010042.
- [10] L.M. Abreu, A.E. Santana, A. Ribeiro Filho, Ann. Phys. (NY) 297 (2002) 396.
- [11] H. Risken, The Fokker–Planck Equation: Methods and Solutions and Applications, Springer, Berlin, 1989.
- [12] T. Tomé, M.J. Oliveira, Dinâmica Estocástica e Irreversibilidade, EdUSP, São Paulo, 2001.
- [13] C.W. Gardiner, Handbook of Stochastic Methods, Springer, Berlin, 1985.
- [14] R.L. Liboff, Kinetic Theory: Classical, Quantum and Relativistic Descriptions, Wiley-Interscience, New York, 1998.
- [15] D. Brian, J. Rafelski, Phys. Rev. Lett. 84 (2000) 31.
- [16] J.A. Cardeal, A.E. Santana, T.M. Rocha Filho, Physica A 308 (2002) 292.
- [17] U. Heinz, Ann. Phys. (NY) 161 (1986) 48;
U. Heinz, Ann. Phys. (NY) 168 (1986) 148.
- [18] R. Graham, Z. Physik B 26 (1977) 397.