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Boundary dependence of the coupling constant and the mass in the vector N-component $(\lambda\varphi^4)_D$ theory

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Abstract

Using the Matsubara formalism, we consider the massive $(\lambda\varphi^4)_D$ vector N-component model in the large-N limit, the system being confined between two infinite parallel planes. We investigate the behaviour of the coupling constant as a function of the separation L between the planes. For the Wick-ordered model in $D = 3$ we are able to give an exact formula to the L dependence of the coupling constant. For the non-Wick-ordered model we indicate how expressions for the coupling constant and the mass can be obtained for arbitrary dimension D in the small-L regime. Closed exact formulae for the L-dependent renormalized coupling constant and mass are obtained in $D = 3$ and their behaviours as functions of L are displayed. We are also able to obtain, in generic dimension D, an equation for the critical value of L corresponding to a second-order phase transition in terms of the Riemann zeta-function. In $D = 3$ an implicit formula for the critical L is given.

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1. Introduction

Translationally non-invariant field theories are of great interest in several branches of theoretical physics, particularly in the study of confined systems or systems undergoing changes of state induced by space inhomogeneities. For instance, in the Ginzburg–Landau theory of phase transitions applied to the study of type-II superconductors, the external magnetic field, constant in some fixed spatial direction, breaks translational invariance introducing important changes in the Feynman rules [1–3]. Another important situation is found in the finite temperature field theoretical formalism of statistical mechanics, in particular the Matsubara imaginary time formalism, which introduces a breaking of translational invariance along the imaginary time (temperature) axis, leading to corresponding modifications in the Feynman rules. In this context, in recent papers ([4–6]) the behaviour of coupling constants with temperature has been investigated in connection with stability and phase transitions. In [5] the finite temperature behaviour of the $\lambda\varphi^4_D$ model was investigated at the one-loop approximation. One of the results is that the thermal squared mass (in the disordered phase) is a positive increasing function of the temperature, while the thermal correction to the renormalized coupling constant is negative and increasing in modulus with the temperature. This could be interpreted as the possibility of the vanishing of the thermal coupling constant at some temperature and its change of sign afterwards. Of course we cannot conclude from this one-loop result that there is a breaking of stability at some temperature, since we cannot be sure that this peculiar behaviour will be preserved when higher order loop contributions are considered (see below).

Questions concerning the stability and the existence of phase transitions may also be raised if we consider the behaviour of field theories as a function of spatial boundaries. In this situation, for Euclidean field theories, time (temperature) and the spatial coordinates are on exactly the same footing and the Matsubara formalism applies also for the breaking of invariance along any one of the spatial directions. The possibility of vanishing of the coupling constant as mentioned above for temperature, or more generally the existence of phase transitions, would be in this case associated with values of some spatial parameters describing the breaking of translational invariance, for instance the distance L between planes confining the system. In this case, in a very similar way as has been remarked in the context of finite temperature field theory in [6], two-loop corrections in the small-L regime are positive and tend to compensate the one-loop lowering of the value of the L-dependent renormalized coupling constant. Similar questions could be raised in general about the existence of phase transitions induced by spatial boundaries separating different states of the system. One of the most striking examples of this kind of situation is the spatial separation between the Abrikosov lattice of vortices and the other phases in type-II superconductors [7]. In another related domain of investigation, there are systems that present defects created, for instance, in the process of crystal growth by some prepared circumstances (domain walls). At the level of effective field theories, in many cases this can be modelled by considering a Dirac fermionic field whose mass changes sign as it crosses the defect, which means that the domain wall plays the role of a critical boundary, separating two different states of the system (see, for instance, [8, 9] and other references therein). In any case, the question of the behaviour of physical parameters in field theories (masses or coupling constants) on spatial boundaries or inhomogeneities, is still largely open and deserves further investigation.

In order to shed some light on the problem, in this paper we examine a model that allows a non-perturbative approach. We consider the Euclidean vector N-component $\lambda\varphi^4_D$ theory at leading order in $\frac{1}{N}$, the system being confined between two infinite parallel planes a distance L apart from one another. In this case, for the Wick-ordered model we are able to obtain an exact formulae for the renormalized coupling constant behaviour as a function of the given spatial boundaries. For the non-Wick-ordered model we obtain for the coupling constant and the mass, formulae valid in the small-L regime for arbitrary dimension D. In dimension $D = 3$ we obtain closed formulae for the renormalized L-dependent coupling constant and mass, and plots of these parameters as functions of L are given. We are also able to obtain in generic dimension D, an equation for the critical value of L corresponding to a second-order phase transition in terms of the Riemann zeta-function. In $D = 3$ a renormalization is done and an explicit formulae for the critical L is given.

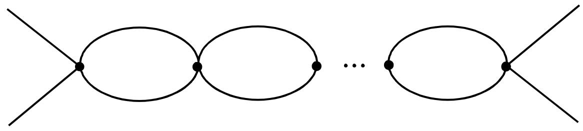

Figure 1. Typical diagram contributing to the four-point function at leading order in $\frac{1}{N}$. For each vertex there is a factor $\frac{\lambda}{N}$ and for each single bubble a colour circulation factor N.

2. The Wick-ordered model

We consider the model described by the Lagrangian density,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{1}{2}m^2 : \varphi_a\varphi_a : + \frac{\lambda}{N}\,(: \varphi_a\varphi_a :)^2 \tag{1}$$

in Euclidean D-dimensional space, where λ is the coupling constant, m is the physical mass (see below) and summation over repeated 'colour' indices a is assumed. We note that Wick-ordering of products of fields makes it unnecessary to perform an explicit mass renormalization at the order $\frac{1}{N}$ at the one-loop approximation. The tadpoles are suppressed by Wick-ordering. Thus we consider in the following only the zero external momenta four-point function, which we take as the basic object for our definition of the renormalized coupling constant. To simplify the notation in the following we drop the colour indices, summation over them being understood in field products.

The four-point function at leading order in $\frac{1}{N}$ is given by the sum of all diagrams of the type depicted in figure 1. We consider the system confined between two parallel planes a distance L apart from one another, normal to the x-axis. We use Cartesian coordinates $\boldsymbol{r} = (x, \boldsymbol{z})$, where $\boldsymbol{z}$ is a $(D-1)$-dimensional vector, with corresponding momenta $\boldsymbol{k} = (k_x, \boldsymbol{q})$, $\boldsymbol{q}$ being a $(D-1)$-dimensional vector in momenta space. Then since the x-dependence of the field is confined to a segment of length L, we are allowed to perform the Matsubara replacements,

$$\int \frac{\mathrm{d}k_x}{2\pi} \to \frac{1}{L}\sum_{n=-\infty}^{+\infty} \qquad k_x \to \frac{2n\pi}{L} \equiv \omega_n. \tag{2}$$

Performing the sum over all diagrams indicated in figure 1 we get for the L-dependent four-point function at zero external momenta the formal expression

$$\Gamma_D^{(4)}(0, L) = \frac{1}{N}\frac{\lambda}{1 - \lambda\Sigma(D, L)} \tag{3}$$

where $\Sigma(D, L)$ corresponds to the single bubble diagram in figure 1.

To obtain an expression for $\Sigma(D, L)$, we briefly recall some one-loop results described in [5] adapted to our present situation. These results have been obtained by the concurrent use of dimensional and zeta-function analytic regularizations, to evaluate formally the integral over the continuous momenta and the summation over the Matsubara frequencies. We get sums of polar (L-independent) terms and L-dependent analytic corrections. Renormalized quantities are obtained by the subtraction of the divergent (polar) terms appearing in the quantities obtained by the application of the finite temperature Feynman rules (Matsubara prescriptions) and dimensional regularization formulae. These polar terms are proportional to Γ-functions having the dimension D in the argument and correspond to the introduction of counterterms in the original Lagrangian density. In order to have a coherent procedure in any dimension, these subtractions should be performed even in the case of odd dimension D, where no poles of Γ-functions are present. In the following, to deal with dimensionless quantities in the regularization procedures, we introduce parameters $c^2 = m^2/4\pi^2\mu^2$, $(L\mu)^2 = a^{-1}$, $g = (\lambda/8\pi^2)$ and $(\varphi_0/\mu) = \phi_0$, where φ_0 is the normalized vacuum expectation value of the field (the classical field) and μ is a mass scale. In terms of these parameters, we start from the well-known expression for the one-loop contribution to the zero-temperature effective potential adapted to the situation under study,

$$U_1(\phi, L = \infty) = \mu^D \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s} g^s \phi_0^{2s} \int \frac{\mathrm{d}^D k}{(k^2 + c^2)^s}. \tag{4}$$

Then performing the Matsubara replacements (2), the boundary-dependent (*L*-dependent) one-loop contribution to the effective potential can be written in the form

$$U_1(\phi, L) = \mu^D \sqrt{a} \sum_{s=1}^{\infty} \frac{(-1)^{s+1}}{2s} g^s \phi_0^{2s} \sum_{n=-\infty}^{+\infty} \int \frac{\mathrm{d}^{D-1} q}{(an^2 + c^2 + \boldsymbol{q}^2)^s} \tag{5}$$

or, using a well-known dimensional regularization formula [10],

$$U_1(\phi, L) = \mu^D \sqrt{a} \sum_{s=1}^{\infty} f(D, s) g^s \phi_0^{2s} Z_1^{c^2} \left(s - \frac{D-1}{2}; a \right) \tag{6}$$

where $f(D, s)$ is a function proportional to $\Gamma\big(s - \frac{D-1}{2}\big)$ and $Z_1^{c^2}\big(s - \frac{D-1}{2}; a\big)$ is one of the Epstein–Hurwitz zeta-functions defined by

$$Z_K^{c^2}(u; a_1, \ldots, a_K) = \sum_{n_1, \ldots, n_K = -\infty}^{+\infty} \big(a_1 n_1^2 + \cdots + a_K n_K^2 + c^2\big)^{-u} \tag{7}$$

valid for $\mathrm{Re}(u) > K/2$ (in our case $\mathrm{Re}(s) > D/2$). The Epstein–Hurwitz zeta-function can be extended to the whole complex s-plane and we obtain, after some manipulations [11], the one-loop correction to the effective potential,

$$\begin{aligned} U_1(D, L) = \mu^D \sum_{s=1}^{\infty} g^s \phi_0^{2s} h(D, s) \Bigg[& 2^{-(\frac{D}{2} - s + 2)} \Gamma \left(s - \frac{D}{2} \right) (m/\mu)^{D-2s} \\ & + \sum_{n=1}^{\infty} \left(\frac{m}{\mu^2 n L} \right)^{\frac{D}{2} - s} K_{\frac{D}{2} - s}(nmL) \Bigg] \end{aligned} \tag{8}$$

where

$$h(D, s) = \frac{1}{2^{D/2 - s - 1} \pi^{D/2 - 2s}} \frac{(-1)^{s+1}}{s \Gamma(s)} \tag{9}$$

and K_ν are the Bessel functions of the third kind.

Note that since we are using dimensional regularization techniques, a factor μ^{4-D} in the definition of the coupling constant is implicit in the above formulae. In what follows we make this factor explicit, the symbol λ stands for the dimensionless coupling parameter (which coincides with the physical coupling constant in $D = 4$). To proceed we use the renormalization conditions

$$\frac{\partial^2}{\partial \phi^2} U_1(D, L)|_{\phi_0 = 0} = m^2 \mu^2 \tag{10}$$

and

$$\frac{\partial^4}{\partial \phi^4} U_1(D, L)_{\phi_0 = 0} = \lambda \mu^4 \tag{11}$$

from which we deduce that formally the single bubble function $\Sigma(D, L)$ is the coefficient of the fourth power of the field ($s = 2$) in equation (8). Of course such a coefficient is in general ultraviolet divergent and a renormalization procedure is needed. Then using equation (11) we can write $\Sigma(D, L)$ in the form

$$\Sigma(D, L) = H(D) - G(D, L) \tag{12}$$

where the L-dependent term $G(D, L)$ comes from the second term between the brackets in equation (8),

$$G(D, L) = \frac{3}{2}\frac{\mu^{4-D}}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left[\frac{m}{nL}\right]^{(D-4)/2} K_{\frac{D-4}{2}}(nLm) \tag{13}$$

and $H(D)$ is a polar term coming from the first term between the brackets in equation (8)

$$H(D) \propto \Gamma\left(2 - \frac{D}{2}\right)(m/\mu)^{D-4}. \tag{14}$$

We see from equation (14) that for even dimensions $D \geqslant 4$, $H(D)$ is divergent due to the pole of the Γ-function. Accordingly this term must be subtracted to give the renormalized single bubble function $\Sigma_R(D, L)$. We simply get

$$\Sigma_R(D, L) = -G(D, L). \tag{15}$$

In order to have a coherent procedure for generic dimension D, the subtraction of the term $H(D)$ should be performed even in the case of odd dimension D, where no poles of Γ-functions are present. In these cases we perform a finite renormalization. From the properties of Bessel functions it can be seen from equation (13) that for any dimension D, $G(D, L)$ satisfies the conditions,

$$\lim_{L\to\infty} G(D, L) = 0 \qquad \lim_{L\to 0} G(D, L) \to \infty. \tag{16}$$

We also conclude from Bessel function properties that $G(D, L)$ is always positive for any values of D and L.

Let us define the L-dependent renormalized coupling constant $\lambda_R(D, L)$ at the leading order in $1/N$ as

$$\Gamma_D^{(4)}(0, L) \equiv \frac{1}{N}\lambda_R(D, L) = \frac{1}{N}\frac{\lambda}{1 - \lambda\Sigma_R(D, L)} \tag{17}$$

and the renormalized coupling constant in the absence of boundaries as

$$\frac{1}{N}\lambda_R = \lim_{L\to\infty} \Gamma_{D,R}^{(4)}(0, L). \tag{18}$$

From equations (18), (17) and (16) we simply get $\lambda_R = \lambda$. In other words we have a choice of renormalization scheme such that the constant λ introduced in the Lagrangian corresponds to the *renormalized* coupling constant. From equations (17) and (15) we obtain the L-dependent renormalized coupling constant,

$$\lambda_R(D, L) = \frac{\lambda}{1 + \lambda G(D, L)}. \tag{19}$$

The above procedure corresponds, on perturbative grounds, to summing up all the chains of bubble graphs in figure 1. It is just the resummation of all the perturbative contributions including the counterterms from the chains of bubbles and the subtraction of the divergent (polar) parts, written in compact form. These subtractions are to be performed even in the case of odd dimension, where no poles of Γ-functions are present.

Note that we have used above a modified minimal subtraction scheme where the mass and coupling constant counterterms are poles at the physical values of s. The L-dependent correction to the coupling constant is proportional to the regular part of the analytical extension

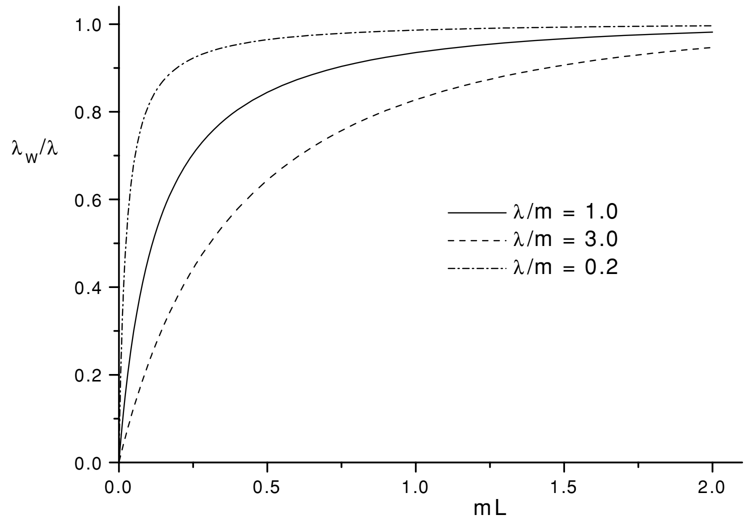

Figure 2. Renormalized coupling constant (in units of $\frac{1}{N}$) for the Wick-ordered model as a function of the separation between the planes in dimension $D = 3$. In the plot, we take $\mu = 1$.

of the inhomogeneous Epstein zeta-function in the neighbourhood of the pole at $s = 2$. The same argument applies to the renormalized mass, the L-dependent renormalized mass at one-loop approximation is given by

$$m_R^2(L) = m^2 + \frac{4\mu^{4-D}\lambda}{(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left[\frac{m}{nL}\right]^{(D-2)/2} K_{\frac{D-2}{2}}(nmL). \tag{20}$$

For the purposes of this section, we simply take $m_R^2(L) = m^2$ since Wick-ordering suppressing the tadpoles makes one-loop corrections to the mass disappear.

An exact result for the coupling constant can be obtained in dimension $D = 3$. From the relationships [12]

$$K_{n+\frac{1}{2}}(z) = K_{-n-\frac{1}{2}}(z) \qquad K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}}\,\mathrm{e}^{-z} \tag{21}$$

we obtain from equations (19) and (13), after summing a geometric series, the exact expression for the coupling constant in the large-N limit,

$$\lambda_W(D = 3, L) = \frac{8\pi m\lambda(\mathrm{e}^{mL} - 1)}{8\pi m(\mathrm{e}^{mL} - 1) + 3\lambda\mu} \tag{22}$$

where the subscript W is used to indicate Wick-ordering explicitly. A plot of $\lambda_W(D = 3, L)$ is given in figure 2.

3. The non-Wick-ordered model

The effect of the suppression of Wick-ordering is that the renormalized mass cannot be taken as the coefficient of the term $\varphi_a\varphi_a$ in the Lagrangian, as we have done in the preceding section. We must take an L-corrected renormalized mass which for a large number of field components N satisfies, at leading order in $1/N$, the gap equation in the ordered phase,

$$m_R^2(L) = m^2 + \frac{4\mu^{4-D}\lambda(N+2)}{N(2\pi)^{D/2}} \sum_{n=1}^{\infty} \left[\frac{m_R(L)}{nL}\right]^{D/2-1} K_{\frac{D}{2}-1}(nm_R(L)L) \tag{23}$$

where m stands for the constant mass parameter in the Lagrangian density (1), which corresponds to the renormalized mass in the absence of boundaries (free space) for the Wick-ordered model. To obtain the L-dependent coupling constant the constant mass parameter m should be replaced in equations (19) and (13) by the L-corrected mass $m_R(L)$ and the resulting system of equations should be solved with respect to L. Under these conditions exact closed expressions are almost impossible to obtain, since it would require a procedure equivalent to solving exactly the Dyson–Schwinger equations.

Nevertheless results can be obtained, both analytic and numerical. In this sense we later indicate how to generalize to our case, in arbitrary space dimension D, some results already obtained in [13, 14] for $D = 4$ in the context of thermal field theory. We take an integral representation for the Bessel function in equation (23) [12] as

$$K_\nu(z) = \frac{\sqrt{\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}\left(\frac{z}{2}\right)^\nu \int_1^\infty \mathrm{e}^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,\mathrm{d}t \tag{24}$$

valid for $\mathrm{Re}(\nu) > -\frac{1}{2}$ and $|\arg(z)| < \frac{\pi}{2}$. Using this representation in equation (23), after some straightforward calculations the L-dependent renormalized mass can be written in the form

$$\begin{aligned} m_R^2(L) = m^2 + \frac{4\lambda\mu^{4-D}(N+2)}{N} F(D) m_R^{D-2}(L) \int_{m_R(L)L}^\infty \frac{\mathrm{d}\tau}{m_R(L)L} \\ \times\left(\left(\frac{\tau}{m_R(L)L}\right)^2 - 1\right)^{\frac{D-3}{2}} \frac{1}{\mathrm{e}^\tau - 1} \end{aligned} \tag{25}$$

where $F(D) = \frac{1}{2^D}\frac{1}{\pi^{\frac{D-1}{2}}}\frac{1}{\Gamma\left(\frac{D-1}{2}\right)}$. When D is odd, the power $\frac{D-3}{2}$ is an integer and Newton binomial theorem gives a direct way of evaluating the integral in equation (25), implying an algebraic equation for $m_R^2(L)$. When D is even the expansion of $\left(\left(\frac{\tau}{m_R(L)L}\right)^2 - 1\right)^{\frac{D-3}{2}}$ yields an infinite power series. We obtain

$$m_R^2(L) = m^2 + \frac{4\lambda\mu^{4-D}(N+2)}{NL^{D-2}} \sum_{k=0}^\infty g(D,k)(m_R(L)L)^{2k} \int_{m_R(L)L}^\infty \mathrm{d}\tau \frac{\tau^{D-3-2k}}{\mathrm{e}^\tau - 1} \tag{26}$$

where $g(D,k) = F(D)(-1)^k C^k_{\frac{D-3}{2}}$ and the C are the generalized binomial coefficients for arbitrary power. For small values of k the integral that appears in the above equation is the well-known Debye integral

$$I(x,n) = \int_x^\infty \mathrm{d}\tau \frac{\tau^n}{\mathrm{e}^\tau - 1} = \sum_{q=1}^\infty \mathrm{e}^{-qx} x^n \left(\frac{1}{q} + \frac{n}{q^2} + \cdots + \frac{n!}{q^{n+1}}\right) \tag{27}$$

which is valid for $x > 0$ and $n \geqslant 1$. For $k > \frac{D-3}{2}$, which corresponds to $n < 1$ in the preceding equation, the exponent of τ in equation (27) becomes negative and the integral is undefined. In this case, for small values of L, a generalization to negative odd powers of the argument of the integrand in the Debye integral can be done using the methods in [15] and the integral has the expansion ($u = m_R(L)L$ and $n = D - 3 - 2k$),

$$J(u,n) = \int_u^\infty \mathrm{d}\tau \frac{\tau^{-n}}{\mathrm{e}^\tau - 1} = -\sum_{q=0, q\neq n}^\infty \frac{B_q}{q!}\frac{u^{q-n}}{q-n} - \frac{1}{n!}B_n \ln u + \alpha_n \tag{28}$$

where B_k are the Bernoulli numbers and α_n is a constant. Replacing the above equations in equation (26) we have the following expression in the small-L regime:

$$m_R^2(L) = m^2 + \frac{4\lambda\mu^{4-D}(N+2)}{NL^{D-2}}[A(L,D) + B(L,D)] \tag{29}$$

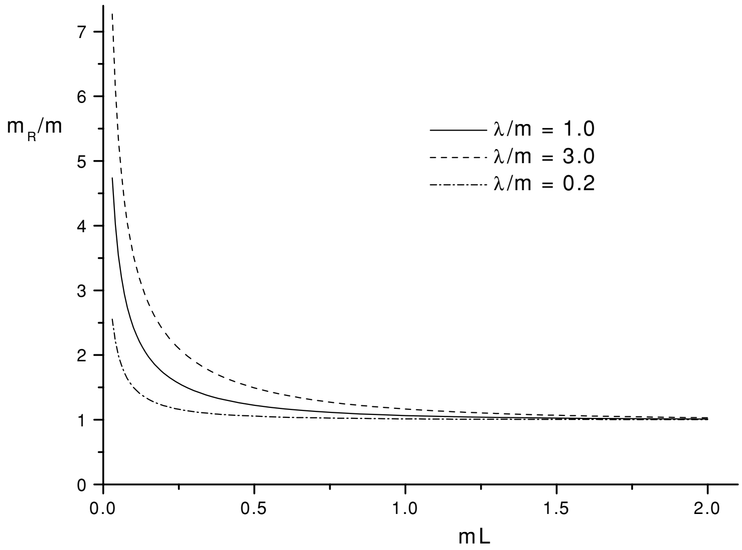

Figure 3. Renormalized mass for the non-Wick-ordered model as a function of the spacing L between the planes; we fix $\mu = 1$.

where

$$A(L, D) = \sum_{k=0}^{k\leqslant \frac{D-3}{2}} g(D, k)(m_R(L)L)^{2k} I(m_R(L)L, D - 3 - 2k) \tag{30}$$

and

$$B(L, D) = \sum_{k>\frac{D-3}{2}}^{k=\infty} g(D, k) m_R(L)L)^{2k} J(m_R(L)L, D - 3 - 2k). \tag{31}$$

The above equations give a non-perturbative expression for the L-corrected renormalized mass in the small-L regime, for even dimensional Euclidean space. Closed expressions are very difficult to obtain in arbitrary dimension, but for L sufficiently small, the series in equation (31) and the representations (27) and (28) for the functions $I(m_R(L)L, D-3-2k)$ and $J(m_R(L)L, D-3-2k)$ could be truncated, giving an approximate algebraic equation for the L-corrected mass. The resulting expression should be injected into equations (13) and (19) to give the L-corrected coupling constant in the small-L regime. We will not perform these manipulations here. Instead, we will obtain some exact results in $D = 3$ in the following.

In dimension $D = 3$, using equations (21) the sum over n in equation (23) can be performed exactly. We obtain a closed transcendent equation satisfied by the L-corrected renormalized mass

$$m_R^2(L) = m^2 - \frac{\lambda\mu(N+2)}{N\pi L} \log\left(1 - \mathrm{e}^{-m_R(L)L}\right) \tag{32}$$

or in the large-N limit

$$m_R^2(L) = m^2 - \frac{\lambda\mu}{\pi L} \log\left(1 - \mathrm{e}^{-m_R(L)L}\right). \tag{33}$$

The large-N renormalized mass is plotted as a function of L in figure 3.

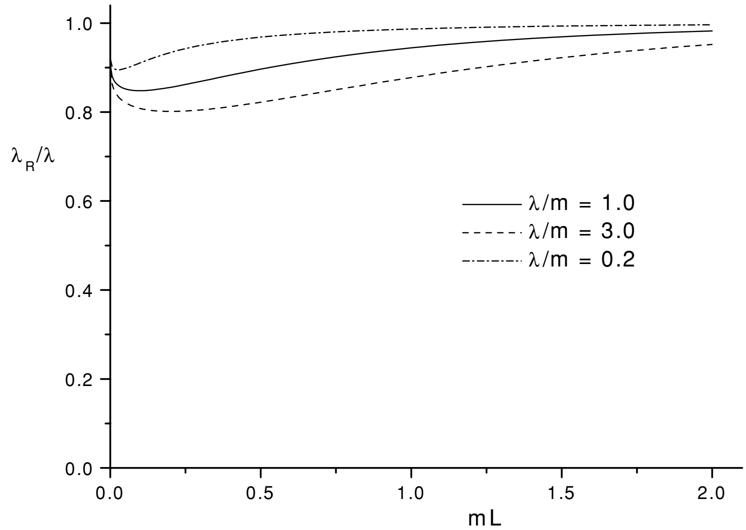

Figure 4. Renormalized coupling constant (in units of $\frac{1}{N}$) for the non-Wick-ordered model as a function of the distance between the planes in dimension $D = 3$; we take $\mu = 1$.

Also, as in the preceding section, we have an exact expression in $D = 3$ for the coupling constant in the large-N limit as a function of the renormalized L-dependent mass,

$$\lambda_R(D = 3, L) = \frac{8\pi m_R(L)\lambda\left(\mathrm{e}^{m_R(L)L} - 1\right)}{8\pi m_R(L)\left(\mathrm{e}^{m_R(L)L} - 1\right) + 3\lambda\mu}. \tag{34}$$

In figure 4 we plot the L-corrected coupling constant in dimension $D = 3$ for the non-Wick-ordered model using jointly equations (33) and (34) on the same scale we have used to plot the Wick-ordered coupling constant. We see that the behaviour of the coupling constant is quite different from the monotonically increasing $\frac{1}{L}$ behaviour of the renormalized squared mass and also from the behaviour of the coupling constant in the Wick-ordered case. The non-Wick-ordered coupling constant slightly decreases for decreasing values of L until some minimum value and then starts to increase. In the Wick-ordered model the coupling constant tends to zero monotonically as L goes to zero, while in the non-Wick-ordered model it has a non-vanishing value even for very small values of L. In fact, up to the precision (figure 4) numerical estimates have been made, the L-corrected non-Wick-ordered coupling constant has a non-vanishing value at $L = 0$, which is equal to the free space value λ. As a general conclusion it can be said that for the non-Wick-ordered model the L-dependent renormalized coupling constant slightly departs to lower values, from the free space coupling constant. Furthermore this departure is smaller for smaller values of λ (in units of m).

In what concerns the mass behaviour, let us refer to equation (23) and remember that the Bessel function K is positive for real positive values of the argument and decreases for increasing values of the argument. For space dimension $D > 2$ the correction in L to the squared mass is positive and the L-dependent squared mass is a monotonically increasing function of $\frac{1}{L}$. If we start in the disordered phase with a negative squared mass, the model exhibits spontaneous symmetry breaking of the $O(N)$ symmetry to $O(N - 1)$, but for a sufficiently small critical value of L the symmetry is restored. The critical value of L, L_c, is defined as the value of L for which the inverse squared correlation length, $\xi^{-2}(L, \varphi_0)$, vanishes in the gap equation,

$$\xi^{-2}(L, \varphi_0) = m^2 + 2\mu^{4-D}\left[\lambda\varphi_0^2 + \frac{\lambda(N+2)}{NL}\sum_n \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \frac{1}{\boldsymbol{k}^2 + \omega_n^2 + \xi^{-2}(L, \varphi_0)}\right] \tag{35}$$

where φ_0 is the normalized vacuum expectation value of the field (different from zero in the ordered phase). In the neighbourhood of the critical point φ_0 vanishes and the thermal gap equation reduces to equation (23). If we limit ourselves to the neighbourhood of criticality, $m_R^2(L) \approx 0$, we may use an asymptotic formula for small values of the argument of the Bessel functions,

$$K_\nu(z) \approx \frac{1}{2}\Gamma(\nu)\left(\frac{z}{2}\right)^{-\nu} \qquad (z \sim 0) \tag{36}$$

and equation (23) becomes

$$m_R^2(L) \approx m^2 + \frac{\lambda\mu^{4-D}(N+2)}{N(2\pi)^{D/2}}\Gamma\left(\frac{D}{2} - 1\right)L^{2-D}\zeta(D-2) \tag{37}$$

where $\zeta(D-2)$ is the Riemann zeta-function, $\zeta(D-2) = \sum_{n=1}^{\infty}(1/n^{D-2})$.

Taking $m_R(L) = 0$ in the above equation we obtain the critical value of L in Euclidean space dimension D $(D > 2)$,

$$(L_c)^{D-2} = -\frac{\lambda\mu^{4-D}(N+2)}{m^2N}\frac{1}{\pi^{\frac{D}{2}}}\Gamma\left(\frac{D}{2} - 1\right)\zeta(D-2). \tag{38}$$

For $D = 3$ the zeta-function has a pole and equations (37) and (38) become meaningless. We cannot obtain a critical value for L in dimension $D = 3$ by a limiting procedure from equations (36) to (38). To obtain a critical value for L in this case, we must take directly $D = 3$ and perform as in the finite temperature case in [16]. Then we can adapt to our case the reasoning and results obtained in the case of finite temperature in [16, 17]. According to the results in dimension $D = 3$ from [16] adapted to our case, the state of the system and the validity of perturbation theory are controlled by the magnitude of the quantity $\kappa(L)$, given by

$$\kappa(L) = (L^2\mu^2)\exp\left[-\frac{8\pi L m_R^2(L)}{\lambda}\right]. \tag{39}$$

Perturbation theory is valid for small values of the parameter κ. But the critical value of L should correspond to a regime where perturbation theory fails, i.e. to a value of the parameter $\kappa \approx 1$. Therefore, we obtain an implicit equation for the critical value of L from equation (39),

$$\frac{8\pi m_R^2(L_c)}{\lambda} = L_c^{-1}\ln\left(L_c^2\mu^2\right). \tag{40}$$

The above results generalize to a phase transition associated with a spatial boundary, estimates and numerical simulations in [16–18].

4. Concluding remarks

Taking the Wick-ordering, which eliminates all contributions from the tadpoles, we decouple in some sense the boundary behaviour of the coupling constant from the mass boundary behaviour. Wick-ordering is a useful and simplifying procedure in the applications of field theory to particle physics, but the same is not necessarily true in applications of field theory to investigate critical phenomena, where the contribution from tadpoles could be physically significant. As a consequence of the suppression of Wick-ordering the boundary behaviour of the coupling constant is sensibly modified with respect to the monotonic behaviour in the Wick-ordered case (see comments following equation (34)). As a general qualitative conclusion it can be said that, for the non-Wick-ordered model in the perturbative regime in λ, the L-dependent renormalized coupling constant only slightly departs from the free space value λ and even so, for small values of L. Our results, particularly the almost L-independent behaviour of the effective coupling constant, should be compared with recent publications on the behaviour of the vacuum stress tensor and the one-loop vacuum fluctuations, associated with a scalar field confined in an infinitely long rectangular waveguide [19, 20]. This will be the subject of future study.

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