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MULTIFRACTAL ASPECTS FOR A SELF ORGANIZED CRITICAL MODEL

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We investigate multifractal properties for an Abelian directed model of self-organized criticality that describes the growth of droplets inside a cloud and the subsequent rainfall. The probability distribution of events of the model satisfies finite-size scaling. We obtain the singularity spectra $f(\alpha)$ associated with temporal records for avalanche size and for the potential energy, defined by the total sum of the product between mass and height of each site. The measure defined by avalanche size has a clear cut multifractal character, while the obtained $f(\alpha)$ for potential energy may include a spurious branch.

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1. Introduction

Self Organized Criticality (SOC) theory[1] seeks to provide an explanation for the occurrence of scaling phenomena observed for open systems in the stationary out-of-equilibrium regime. The power law behavior of the probability distributions of event[a] size (e.g. Gutemberg–Richter's law for distribution of earthquakes) has been the most important feature of SOC systems. The usual approach within this framework is based on very simple models encompassing the essential physical mechanisms underlining the investigated phenomena, which are expected to be responsible for the observed scaling properties. A model which takes into account the details of single events, whatever it may be, corresponds to a much more complex issue, that can hardly be accomplished by SOC models.

Since only a small number of SOC models has shown to be amendable to exact analytical solutions,[2] actual investigation relies mainly on numerical integration of

[a]Usually called avalanche.

the equations of motion and/or of cellular automata that describe their time evolution. As the so obtained results depend on the size (L) of the used system, it is important to infer the behavior of an infinite system in the $L \to \infty$ thermodynamical limit. The most simple way to reach this task is to carry on a finite size scaling[3] analysis of the data, that turns out to be successful for some models, like the (exact solvable) Dhar and Ramaswamy[2] and the Manna[4] models. However, some SOC models, e.g. the very first one proposed by Bak, Tang and Wiesenfeld (BTW model),[5] do not satisfy finite size scaling,[3] but require a multi-scaling formalism[6] for an accurate description of the system when $L \to \infty$. Thus, multifractal approach is presented as a valuable tool within SOC framework.

The knowledge of the probability distribution of events does not exhaust the statistical information of a SOC model, since it does not take into account the dynamic properties encoded in the time evolution records, whose structure might be uncovered by suitable correlation functions. Recent results for the time correlation $C(t, L)$[7] of successive waves within avalanches indicate that this function is sensitive to the presence of multi-scaling in the probability distribution of this measure. For BTW model, $C(t, L)$ decreases slowly with time, while the models which obey finite size scaling are uncorrelated. It is important to observe that, for BTW model, $C(t, L)$ does not obey finite size scaling.

The above discussion suggests that finite size scaling method alone is sufficient to accurately describe some basic properties of a set of models with the special property that each site can topple only once during an avalanche, e.g. the set directed Abelian models. In this work we investigate a directed Abelian model,[8] looking for possible occurrence of other properties or measures that, in contrast to those analyzed so far, require multi-scaling and/or multifractal approach for its characterization.

Instead of looking at the time correlations, we will evaluate the singularity spectrum $f(\alpha)$[9] for two related measures associated with avalanches triggered from random perturbations, which are extracted from the time evolution records. If the related properties are single scaling, one expects the spectrum to converge to a single isolated point. On the other hand, if $f(\alpha)$ consistently converges to a concave shape with robust, distinct extreme values, $\alpha_{\min}$ and $\alpha_{\max}$, we can identify the presence of a multifractal measure.

The rest of this paper is so organized: In Section 2 we briefly describe the model as well as its main properties. In Section 3 we discuss the choice of the quantities whose multifractal properties are investigated, while the results and concluding remarks are presented in Section 4.

2. Abelian SOC Models

The present investigation considers a cellular automata two-dimensional model,[8] proposed to describe certain aspects of the hyperbolic statistical distributions of rain events and suggested from the analysis of historical records from meteorological stations.[10,11] It is important also to recall that many other scaling properties of rain fields have been reported by many authors.[12,13] We point out that this simple model is based on a very simple description of the essential aspects of condensation, coalescence, and break-up of water droplets inside a rain cloud. The whole description of these aspects requires much more complex theories of vapor condensation, drop downward motion in the Stokes regime, stochastic coagulation (Smoluschovsky equation) and drop breakup.[14]

The model is defined on a rectangular lattice of $N = H$ (height) $\times L$ (length) sites with horizontal periodic boundary conditions and vertical non-periodic ones. An integer variable m_i is defined on each site $i = (h, l)$, which represent the liquid water content around a cloud condensation nuclei. The addition of mass on a randomly chosen site i, $m_i \to m_i + 1$, mimics the growth of cloud droplets, by condensation of water vapor. An internal avalanche (flip), usually called avalanche, is triggered when the mass m_i of the perturbed site i reaches a threshold value m_{th}. In this case the site topples, distributing the excess mass m_{th} to the three nearest neighbors in the row $h-1 : (h-1, l-1)$, $(h-1, l)$, $(h-1, l+1)$. Thus, it belongs to the class of directed models. The original two-dimensional directed model proposed by Dhar and Ramaswamy[2] is obtained if the amount of mass that would topple to site $(h-1, l)$ is set to zero. The avalanche continues until all sites of a given row become stable, or when it reaches the open lower edge of the lattice. In this case, there is a rain event or an external avalanche whose size is defined as the total mass that falls out the lattice. The avalanche size a is the number of critical sites $(m_i > m_{th})$ during the avalanche process. This model can also be exactly solved, leading to the probability distribution of the number a of flips that characterizes the avalanche, $\rho(a) \sim a^{-\beta}$, with the same exponent $\beta = 4/3$ found for the quoted model.[2] This has been corroborated by numerical simulations.[8] Thus, it is a SOC model.

It belongs to the class of Abelian models since its final state (configuration of the lattice) after two successive perturbations (addition of mass and relaxation) on any randomly chosen sites (say i and j) is the same if the order of the perturbations is inverted. Some of these models are amenable to exact analytical solutions, from which the scaling exponent of the probability distribution of avalanche sizes can be evaluated. Even though Abelian models with toppling conditions which depend on a critical height are not a realistic representation of actual sandpiles (where the toppling criterion depends on a critical gradient), due to its mathematical properties, they have been investigated in many different aspects. This "rainfall" model is an interesting application of model for which the toppling depends on a critical value of the automaton variable (the water content).

3. Multifractal Measures

It is well known that strange attractors, the limit sets of chaotic dynamical systems, present a multifractal behavior. Though less explored, multifractal properties of some discrete lattice models, such as cellular automata, have already been considered in the literature (see Ref. 15 and references therein for the multifractal aspects of space patterns for simple cellular automata). More recently, in the context of sandpile cellular automata, investigations on multi-scaling properties for BTW and Manna models[6,7] have been reported. Within this approach, the measure associated with the $f(\alpha)$ spectra is the number of toppling sites in the waves into which a given avalanche is split. The probability distribution of waves counts the relative number of occurrences of waves of a given size, and the resulting $f(\alpha)$ spectrum is linear for the system following finite size scaling. For the rainfall model, each avalanche of size a is formed by a single wave, so that its wave probability distribution multifractal spectrum must be a linear function. Moreover, since its avalanche probability distribution of avalanches $\rho(a, L)$ satisfies finite size scaling, it can be written as

$$\rho(a, L) = a^{-\beta} g(a/L^b) \tag{3.1}$$

where the parameter b is the scaling factor associated with the lattice size L.

Now we turn our attention to other measures, and ask if they require multifractal analysis even for $\rho(a, L)$. As the projection of the time distribution of events into $\rho(a, L)$ destroys any possible avalanche clustering, which could display a multifractal character, it is wise to consider the actual dynamics of events. Consider the time evolution record of length N_t produced by the model and define the record $a(\tau)$ by the magnitude of successive avalanches. Note that the value assumed by τ is different from the usual time t that count the time steps for the evolution of the automata, as τ increases only after a relaxation event, so that the total number of events of $a(\tau)$ record is $N_\tau < N_t$. Now we define a new measure as

$$\mu_\tau^a(\epsilon) = \sum_{i=\tau-\epsilon}^{\tau+\epsilon} a(i)$$

and inquire whether it shows multifractal properties. It means that the singularity spectrum $f(\alpha)$ for the measure $\mu_\tau^a(\epsilon)$, in an interval ϵ around the point a, has a non-trivial (non-single scaling) behavior, when considering time series of the internal avalanche size a when $a \neq 0$. This procedure is equivalent to that developed for the analysis of multifractal properties for the local magnetization of spin models in suitably defined lattices,[16,17] where the time τ now plays the role of the position of the spins on the lattice, $\tau \in [1, N_\tau]$. In those analyses, $f(\alpha)$ is able to detect the presence of sites where the magnetizations vanish with different Holder exponents at the critical temperature, while in the ferromagnetic phase the local magnetization is single scaling. Analogously, the present analysis is able to detect different avalanche clustering within the time record. In other words, the probability $p_i(\epsilon)$ of $a(\tau)$ in the hyper-sphere of radius ϵ with center at τ, diverges with different values of exponents when $\epsilon \to 0$.

We also observe that, after each avalanche of intensity $a(\tau)$, the total potential energy associated with the whole lattice, defined by

$$U(\tau) = \sum_{(h,l)} m_{(h,l)} h \,, \tag{3.2}$$

is decreased by the amount $a(\tau)$. So, we have also scrutinized the time record for the potential energy, looking for further evidences of multifractal behavior for the measure $\mu_\tau^U(\epsilon)$ defined by $U(\tau)$ in an analogous way as $\mu_\tau^a(\epsilon)$:

$$\mu_\tau^U(\epsilon) = \sum_{i=\tau-\epsilon}^{\tau+\epsilon} U(i) \,,$$

where $U(\tau)$ is given by (3.2). The potential energy provides global information of water content through the lattice, which is not provided by the avalanche size.

4. Results and Concluding Remarks

The singularity spectra $f(\alpha)$ for the two measures $\mu_\tau^z(\epsilon)$, $z = a$ or $z = U$, discussed above were analyzed with a routine based on the Chabbra–Jensen algorithm,[18] which amounts to evaluate α and the spectrum $f(\alpha)$ in terms of a parameter q as:

$$\begin{aligned} \alpha(q) &= -\lim_{N_\tau \to \infty} \frac{1}{\ln N_\tau} \sum_{\tau=1}^{N_\tau} \xi_\tau(q,\epsilon) \ln[\mu_\tau(q,\epsilon)] \,, \\ f(\alpha(q)) &= -\lim_{N_\tau \to \infty} \frac{1}{\ln N_\tau} \sum_{\tau=1}^{N_\tau} \xi_\tau(q,\epsilon) \ln[\xi_\tau(q,\epsilon)] \,, \end{aligned} \tag{4.1}$$

where

$$\xi_\tau(q,\epsilon) = \frac{\mu_\tau(\epsilon)^q}{\sum_\tau [\mu_\tau(\epsilon)]^q} \,, \tag{4.2}$$

and the superscripts for a and U were omitted for the sake of simplicity. For the same reason, the explicit dependence of the quantities $f(\alpha(q))$ and $\alpha(q)$ on the length N_τ of the record and on the size L of the lattice has been left out.

The Chhabra–Jensen method has proven to be quite reliable for the investigation of the multifractal properties of multi-scaling sets.[19] It is a precise and easily performed procedure that converges faster than the one based on the directed identification of points for which the measure $\mu_\tau(\epsilon)$ scales. In certain sense it is equivalent to evaluating $f(\alpha)$ from a Legendre transform of the generalized dimensions $D(q)$, i.e. $f(\alpha) = \alpha q - (q-1)D_q$.

We considered lattices with several different sizes, all of them with the same aspect ratio $r = H/L$. Time records were prepared in two different ways:

(i) R_1: starting with random values for $m_i < m_{th}$, $\forall i$;
(ii) R_2: starting with the completely charged lattice.

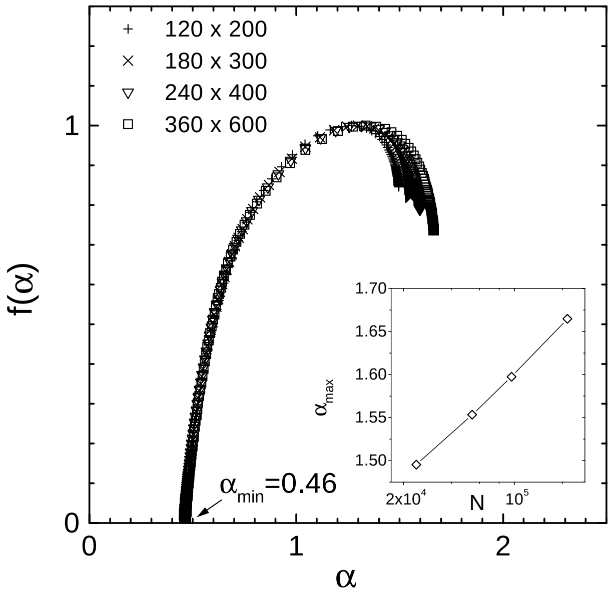
In the first procedure we run the program keeping track of the moment when the system reaches the stationary SOC state where the total mass of the lattice fluctuates around an average value. Then we discard the transient part of the record, and analyze the multifractal aspects of its stationary regime. In the second approach, we keep in the record the very large avalanches produced by the first perturbations. This procedure is necessary in order to obtain well defined extreme values, $\alpha_{\rm min}$ and $\alpha_{\rm max}$, for the $f(\alpha)$ spectrum, which correspond to the largest and the smallest avalanches respectively. The stationary state encompasses avalanches of all sizes, including those of the size of the system itself. However, as such events are very rare in the first procedure, they were not likely to be observed for our available CPU computing time. Using records prepared the second way guarantees that these events will indeed be taken into account.

Our results are illustrated on Figs. 1 and 2. The spectrum for μ_τ^a becomes very well defined for Record R_2 (see Fig. 1b). The value $\alpha_{\rm min} = 0.46 \pm 0.01$ does not depend on the lattice size L, since Record R_2 includes the largest avalanches for that specific lattice size. On the other hand, $\alpha_{\rm max} \sim \log(N)$ is associated to the relative frequency of smaller avalanches, increasing logarithmically with the size L (see inset of Fig. 1b). For Record R_1 (see Fig. 1a), we observe the same behavior for $\alpha_{\rm max}$, but the left branch, associated with $\alpha_{\rm min}$ becomes size dependent and fluctuates, as we cannot be sure of the presence of the very large events.

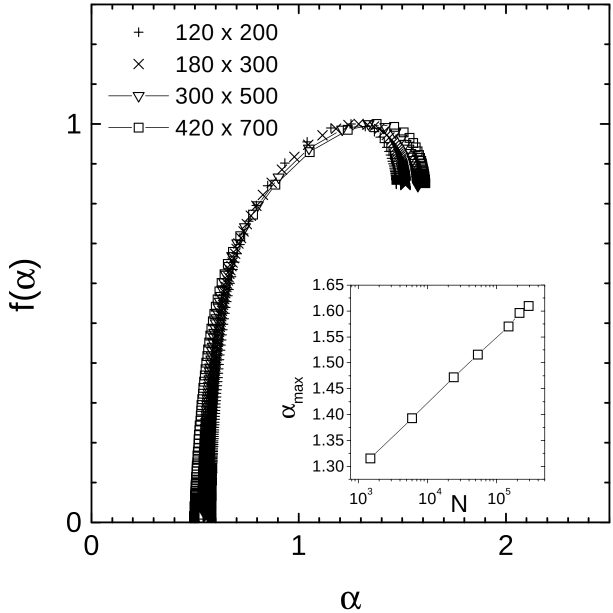
For the measure μ_τ^U, we observe the inverse effect. The spectrum is better defined for Record R_1 (see Fig. 2a), with a size independent $\alpha_{\rm min} = 0.92 \pm 0.02$, which is related to the high populated lattice states. The size dependent right branch, expressed by $\alpha_{\rm max} \sim \log(N)$, is related to the states with sparse site occupation. For Record R_2 (see Fig. 2b), the left branch shows slight size dependent fluctuations, as the very high initial populated states influence the spectrum in different ways. The value for $\alpha_{\rm max}$ now decreases with the lattice size.

A closer observation of the spectra for μ_τ^a reveals that the points of the singularity spectrum are regularly spaced and frequent with respect to the q parameter in the Chhabra–Jensen algorithm, indicating in a very convincing way the measure's multifractal character. On the other hand, the same feature is absent in the spectra for μ_τ^U, which may give rise to questions whether this measure is actually multifractal. Indeed, the points become irregularly spaced in the right branch of the spectra, much in the way of the spectra obtained within Chhabra–Jensen method for actual single scaling, smooth input data.[19] Furthermore, we note that the value for $\alpha_{\rm min}$ is very close to 1 (see Fig. 2), the value for the dimension of the support of the data. So it is wise to collect further evidences before making definite statements on the multifractality of U.

Summarizing, we have found robust indications for the existence of (at least one) multifractal measure for an Abelian directed SOC model, for which the probability distribution $\rho(a, L)$, associated with avalanche size a for a lattice size LH, satisfies finite size scaling. This behavior is related to a time dependent clustering of avalanches that is destroyed when we collapse the data to yield $\rho(a, L)$. The

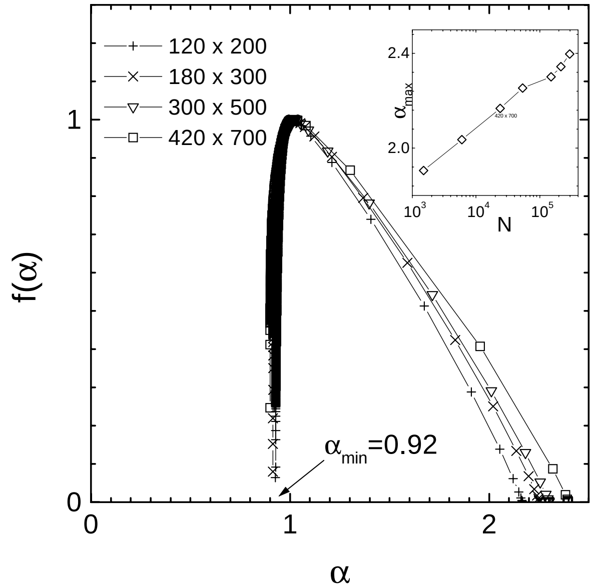


(a)

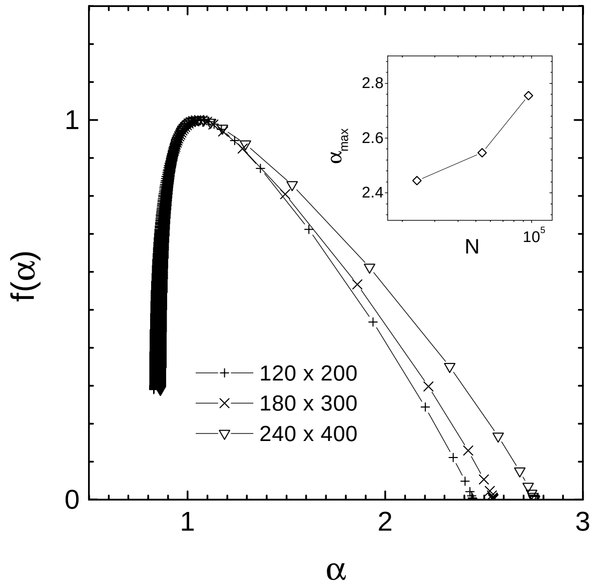


(b)

Fig. 1. Singularity spectra for the measure $\mu_a(\tau)$, with initial conditions R_1 (case a) and R_2 (case b). In the first case, the spectra depend on the size of the systems for both right and left size branches. In the second case the left side of the spectra is well-defined and does not depend on the size of the lattice. The right side moves to larger α regions when the size increases. The inset indicates that $\alpha_{\max} \sim \log(N)$ for both cases.



(a)



(b)

Fig. 2. The same as in Fig. 1 for the measure $\mu_U(\tau)$. The left branch of the spectra becomes better defined for initial conditions R_1, while for R_2 it is possible to note a weak dependence on the size of the system. In the right branch, the points of the spectra get sparse, and they move to the right logarithmically with size.

identification of a multifractal measure for this directed model is rather interesting, as it shows a more complex behavior not revealed by the single scaling $\rho(a, L)$.

The approach used in this work is different from that one used for the multifractal properties of wave distribution for non-directed models. We expect that it will indicate multifractal clustering avalanches for other directed models as well as for the Manna model, for which the wave correlation function also proved to obey finite size scaling.

Finally we stress that finding multifractal properties for this SOC model is important as it relates two distinct properties observed for rain fields:

(i) The distribution of events seems to follow hyperbolic laws, as predicted by the SOC framework;[10,11]

(ii) There are strong evidences that several meteorological properties present multiscaling behavior.[12]

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