Results. Math. 60 (2011), 287–291
© 2011 Springer Basel AG
1422-6383/11/010287-5
published online June 7, 2011
DOI 10.1007/s00025-011-0144-7

Results in Mathematics

On Hypersurfaces of Spheres with Two Principal Curvatures

Aldir Brasil Jr and Ezio de Araujo Costa

Dedicated to Keti Tenenblat on the occasion of her 65th anniversary

Abstract. In this paper we obtain a classification of hypersurfaces in the Euclidean sphere having two principal curvatures; for some of the results we impose that the sectional curvature (Ricci curvature, resp.) is non-negative Ricci.

Mathematics Subject Classification (2010). 53C40, 53C42.

Keywords. Hypersurfaces in spheres, principal curvatures, ricci curvature, sectional curvature.

1. Introduction and Statements of Results

Let $M^n, n \geq 3$, be an oriented Riemannian n-manifold and *Ric* its Ricci curvature. Let $f\colon M^n \longrightarrow \mathbb{S}^{n+1}$ be a hypersurface, where $\mathbb{S}^{n+1}$ is the Euclidean unit sphere. Let $\mathbb{S}^k_c$ be the sphere with constant sectional curvature c and let $\widetilde{M}^n$ be the universal covering of M^n. Consider

$$\Lambda := \{r \in \mathbb{R} : \exists\, x \in M^n,\ \exists \lambda \ \text{such that}\ \ \lambda(x) = r\},$$

where λ is a principal curvature of f and $\Lambda^{\pm} := \Lambda \cap \mathbb{R}^{\pm}$. In [9] (see also [10]) the authors (see Theorem 1.3) classified compact hypersurfaces f with constant scalar curvature and two distinct principal curvatures λ and μ of multiplicity 1 and $n-1$, resp., such that $1+\lambda\mu \leq -1$ in M^n. In [4] the authors obtained the classification of minimal hypersurfaces f with two principal curvatures such that $S \geq n$, where S is the square of the norm of the second fundamental form of f. In particular, the condition $S \geq n$ in [4] implies that $1+\lambda\mu \leq -1$, thus $0 \notin \Lambda$. In the following we apply the condition $(*)$:

Birkhäuser

If there exist $x \in M^n$ and two principal curvatures λ, μ of f such that $\lambda(x) \in \Lambda^+$ and $\mu(x) \in \Lambda^-$ then $1 + \lambda(x)\mu(x) \leq 0$.

In what follows we restrict the dimension to $n \geq 3$. Our first result is the following theorem.

Theorem 1. *Let $f\colon M^n \hookrightarrow \mathbb{S}^{n+1}$ be a complete oriented hypersurface such that the condition $(*)$ holds. If $0 \notin \Lambda$ then $\widetilde{M}^n$ $(\widetilde{f(M^n)}$, resp.) is homeomorphic (isometric, resp.) to one of the following manifolds:*

$$\mathbb{S}^n, \ \mathbb{R}^n, \ f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}, \ \widetilde{f(M^n)} = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}.$$

Notice that Theorem 1 should be compared with Theorem B of [1]. An consequence of the Theorem B of [1] is the following.

Theorem 2. *Let $f\colon M^n \longrightarrow \mathbb{S}^{n+1}$ be complete and oriented, where M^n has Ricci curvature $Ric \geq 0$. If M^n is compact and has infinite fundamental group then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$. If M^n is non compact and has at least two ends then $\widetilde{f(M^n)} = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.*

Let us consider hypersurfaces with two principal curvatures. In view of the Classification Theorem in [7] (p. 438), the Corollaries 3.3 and 3.6 in [7], and finally Theorem 2.2 in [5], we have:

Let $f\colon M^n \longrightarrow \mathbb{S}^{n+1}$ be a hypersurface, where M^n is a n-dimensional, oriented and connected Riemannian manifold. If f has two distinct principal curvatures then there exists a diffeomorphism $\beta : M^n \longrightarrow A$, where A is an open part of one the following manifolds

$$\mathbb{S}^r \times \mathbb{S}^{n-r}, \quad \mathbb{S}^r \times \mathbb{R}^{n-r}, \quad \mathbb{S}^r \times \mathbb{H}^{n-r},$$

and where $\mathbb{H}^{n-r}$ is the hyperbolic space.

With this we have our *main result*:

Theorem 3. *Let $f\colon M^n \longrightarrow \mathbb{S}^{n+1}$ be a complete and oriented hypersurface such that f has two principal curvatures λ, μ. Let K be the sectional curvature of M^n and Ric its Ricci curvature.*

a) *If M^n is compact, $K \geq 0$ in M^n and $\lambda \neq \mu$ in M^n then $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.*
b) *If M^n is compact and for all $x \in M^n$ and if there exists a two plane $P \subset T_xM$ such that $K(P) \leq 0$ then $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.*
c) *If M^n is compact with $Ric \geq 0$ in M^n and λ and μ have multiplicities 1 and $n-1$, respectively, then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$.*
d) *If $Ric \geq 0$ in M^n and if, for all $x \in M^n$, there exists $v \in T_xM, |v| = 1$, such that $Ric(v) = 0$ then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$ or $\widetilde{f(M^n)} = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.*
e) *If $Ric \geq 0$ and f has constant mth mean curvature H_m, if furthermore λ and μ have multiplicities 1 and $n-1$, resp., then $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$ or $\widetilde{f(M^n)} = \mathbb{R} \times \mathbb{S}^{n-1}_{c_2}$.*

Corollary 4. *There is no compact hypersurface $f\colon M^n \longrightarrow \mathbb{S}^{n+1}, n \geq 3$, with only two distint principal curvatures in each point of M^n such that M^n has scalar curvature $\tau \leq 0$ everywhere.*

Remark 5. i) The Cartan hypersurface in $\mathbb{S}^n$ (see [6]) has three distinct principal curvatures and has scalar curvature $\tau = 0$.

ii) In the proof of Theorem 3 we use several arguments of T. Otsuki [8]. The Codazzi-arguments used in the proof of the Theorem 3 appear in the papers [2] and [3] of A. Derdzinski.

iii) Taking into account Theorem 3 e), d), let $i : \mathbb{S}^1 \times \mathbb{S}^{n-1} \to \mathbb{S}^{n+1}$ be the inclusion, $M^n = \mathbb{R} \times \mathbb{S}^{n-1}$ and $\pi : M^n \to \mathbb{S}^1 \times \mathbb{S}^{n-1}$ the covering map. Then $f = i \circ \pi : M^n \to \mathbb{S}^1 \times \mathbb{S}^{n-1}$ is a complete non compact hypersurface with two principal curvatures and $\widetilde{f(M^n)} = \mathbb{R} \times \mathbb{S}^{n-1}$.

Proof of Theorem 1. Consider $f\colon M^n \longrightarrow \mathbb{S}^{n+1}$ complete oriented such $0 \notin \Lambda$ and the condition $(*)$ holds. Then the Gauss map $N\colon M^n \longrightarrow \mathbb{S}^{n+1}$ is a complete hypersurface with principal curvatures $1/\lambda_i, i = 1, \ldots, n$, where the λ_i are the principal curvatures of f. Let $i \neq j$ and $x \in M^n$. Then we have only the possibilities :

(i) $\lambda_i(x) > 0$ and $\lambda_j(x) > 0$,
(ii) $\lambda_i(x) < 0$ and $\lambda_j(x) < 0$,
(iii) $\lambda_i(x) < 0$ and $\lambda_j(x) > 0$.

Thus the condition $(*)$ implies $(\lambda_i(x)\lambda_j(x))^{-1} + 1 \geq 0$ and N has nonnegative sectional curvature. Then Theorem 1 follows from Theorem B of [1]. □

Proof of Theorem 3. a) Let M^n compact with sectional curvature $K \geq 0$. By Theorem B of [1] and in view of the Clasification theorem in [7] we have that $f(M^n)$ is isometric to $\mathbb{S}^r \times \mathbb{S}^{n-r}$.

b) Let M^n be compact such that $\forall x \in M^n$ there exists a two-plane $P \subset T_xM$ with $K(P) \leq 0$. Consider $x \in M^n$ and λ and μ the principal curvatures of f (in x) such that λ and μ have multiplicities r and $n-r$, resp. Then the sectional curvatures in corresponding 2-plane directions of T_xM are λ^2+1, $\lambda\mu+1$ and $\mu^2+1>0$. If $\lambda\mu+1>0$ then M^n has positive sectional curvature in x (contradiction). So, $\lambda\mu+1 \leq 0$ and condition (*) holds. Note that $\lambda \neq 0, \mu \neq 0$ and $\lambda \neq \mu$. Then the Gauss map $N\colon M^n \longrightarrow \mathbb{S}^{n+1}$ is a compact hypersurface with nonnegative sectional curvature. Using the same arguments as in the proof of Theorem 1 (a) we have that $f(M^n) = \mathbb{S}^r_{c_1} \times \mathbb{S}^{n-r}_{c_2}$.

c) Let M^n compact and assume that f has two principal curvatures of multiplicities 1 and $n-1$, resp. It is easy to see that M^n has nonnegative sectional curvature; by Theorem 1 we have $f(M^n) = \mathbb{S}^1_{c_1} \times \mathbb{S}^{n-1}_{c_2}$.

d) Let M^n be complete with $Ric \geq 0$ and assume that $\forall x \in M^n$, there exists $v \in T_xM$, $|v| = 1$ with $Ric\,(v) = 0$. Consider $x \in M^n$ and let λ and μ the principal curvatures of f (in x), of multiplicities n and $n-r$, respectively. Then the eigenvalues of the Ricci curvatures of M (in x) are:

$$(\lambda^2+1)(r-1)+(\lambda\mu+1)(n-r)\geq 0$$

and

$$(\mu^2+1)(n-r-1)+(\lambda\mu+1)r\geq 0.$$

Since there exists $v\in T_xM,\ |v|=1$ with $Ric\,(v)=0$, we have

$$(\lambda^2+1)(r-1)+(\lambda\mu+1)(n-r)=0$$

or

$$(\mu^2+1)(n-r-1)+(\lambda\mu+1)r=0.$$

Assume that $r>1$ and $(n-r)>1$ and consider the sets

$$M_1:=\{x\in M^n;(\lambda^2+1)(r-1)+(\lambda\mu+1)(n-r)=0\}$$

and

$$M_2:=\{x\in M^n;(\mu^2+1)(n-r-1)+(\lambda\mu+1)r=0\}.$$

Let A be the Weingarten operator of f, $X\in D_\lambda:=\{X\in TM\mid AX=\lambda X\}$ and $Y\in D_\mu=\{Y\in TM\mid AY=\mu Y\}$. It follows from the Codazzi equation that the distributions D_λ and D_μ are differentiable, involutive, and that $X(\mu)=Y(\lambda)=0$. Notice that $M^n=\text{int}M_1\cup\text{int }M_2\cup M_3$ where $\text{int}M_3=\emptyset$. Let $x\in\text{int}M_1$ and $Y\in D_\mu$. Then $Y\,[(\lambda^2+1)(r-1)+(\lambda\mu+1)(n-r)]=0$ and this implies that $Y(\mu)=0$ near of x. Since μ is constant near x then $X(\lambda)=0$ (in x), if $X\in D_\lambda$. Similarly, if $x\in\text{int}M_2$, we can see that μ and λ are contant near x. By continuity, λ and μ are constant in M^n, and in this case $f(M^n)=\mathbb{S}^r_{c_1}\times\mathbb{S}^{n-r}_{c_2}$, where $r>1$ and $n-r>1$, which contradicts the fact of that $Ric\,(v)=0$. So $r=1$ or $n-r=1, 1+\lambda\mu=0$ and Theorem 3 (d) follows from Theorem 1.

e) The proof of Theorem 3 (e) is similar to the proof of Theorem 3 (d).

□

Acknowledgements

The authors thanks the referee for his suggestions and corrections.

References

[1] Costa, E.A.: A Ricci inequality for hypersurfaces in the sphere. Arch. Math. **85**, 183–189 (2005)

[2] Derdzinski, A.: On compact Riemannian manifolds with harmonic curvature. Math. Ann. **259**, 145–152 (1982)

[3] Derdzinski, A., Shen, C.L.: Codazzi tensor fields, curvature and Pontryagin forms. Proc. London Math. Soc., III. Ser., **47**, 15–26 (1983)

[4] Hasanis, T., Savas-Halilaj, A., Vlachos, T.: Complete minimal hypersurfaces in a sphere. Monatshefte Math. **145**, 301–305 (2005)

[5] Hu, Z., Li, H.: Classification of Möbius isoparametric hypersurfaces in $\mathbb{S}^4$. Nagoya Math. J. **179**, 147–162 (2005)

[6] Ki, U.-H., Nakagawa, H.: A characterization of the Cartan hypersurface in a sphere. Tôhoku Math. J. **39**(2), 27–40 (1987)

[7] Li, H., Liu, H.-L., Wang, C.-P., Zhao, G.-S.: Möbius isoparametric hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. Acta Math. Sinica, English Series. **18**(3), 437–446 (2002)

[8] Otsuki, T.: Minimal hypersurfaces in a Riemannian manifold of constant curvature. Am. J. Math. **92**, 145–173 (1970)

[9] Wang, Q., Xia, C.: Rigidity theorems for closed hypersurfaces in space forms. Quart. J. Math. **53**, 101–110 (2005)

[10] Wei, G.: Complete hypersurfaces with constant mean curvature in a unit sphere. Monatsh. Math. **149**, 251–258 (2006)

Aldir Brasil Jr
Mathematics Department
Federal University of Ceara
Fortaleza, CE 60455760
Brazil
e-mail: aldir@mat.ufc.br

Ezio de Araujo Costa
Instituto de Matemática
Universidade Federal da Bahia
Salvador, BA 40.170-110
Brazil
e-mail: ezio@ufba.br

Received: September 21, 2010.
Revised: March 10, 2011.
Accepted: March 15, 2011.