LETTER

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Friedmann cosmology with decaying vacuum density

Abstract Among the several proposals to solve the incompatibility between the observed small value of the cosmological constant and the huge value obtained by quantum field theories, we can find the idea of a decaying vacuum energy density, leading from high values at early times of universe evolution to the small value observed nowadays. In this paper we consider a variation law for the vacuum density recently proposed by Schützhold on the basis of quantum field estimations in the curved, expanding background, characterized by a vacuum density proportional to the Hubble parameter. We show that, in the context of an isotropic and homogeneous, spatially flat model, the corresponding solutions retain the well established features of the standard cosmology, and, in addition, are in accordance with the observed cosmological parameters. Our scenario presents an initial phase dominated by radiation, followed by a dust era long enough to permit structure formation, and by an epoch dominated by the cosmological term, which tends asymptotically to a de Sitter universe. Taking the matter density equals to half of the vacuum energy density, as suggested by observation, we obtain a universe age given by $Ht = 1.1$, and a decelerating parameter equals to $-1/2$.

1 Introduction

The huge difference between the small cosmological constant inferred from observation and the vacuum energy density resulting from quantum field theories has been, for a long time, a difficult and fascinating problem for cosmologists

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and field theory researchers [1,2]. Observations of the cosmic background radiation indicates that we live in a spatially flat universe [3], with total energy density equals, or approximately equals, to the critical density. On the other hand, gravitational measurements of matter density in the galaxies lead to an average density of matter at the cosmological scale approximately equals to one third of the critical density. Therefore, we are led to the conclusion that two thirds of the total energy are related to other, non-matter, component [4]. Recent observations of type Ia supernovas at high redshift [5] suggest that this dark energy component exerts negative pressure, a natural candidate being then the cosmological constant, which may be associated to the energy density of the vacuum. Its value would then be $\Lambda \approx 10^{-52}$ m^{-2} [4].

On the other hand, when we calculate the energy density associated with the vacuum quantum fluctuations, we obtain a divergent result, which can be regulated by imposing an ultraviolet cutoff of order of, say, the Planck mass (since at the Planck scale our usual description of spacetime breaks down). In this case, we obtain for the vacuum density the absurd figure 10^{122} m^{-2} [1]. Even taking a smaller cutoff, as the energies of the electroweak phase transition or the chiral transition of QCD, we still have a very huge result compared to the observed one. Furthermore, even dismissing the contribution of the vacuum fluctuations, we still should deal with the vacuum expectation value of any field undertaking phase transitions, as the Higgs field, or the QCD condensate [1].

A possible way out of this trouble is to consider a varying cosmological term, which, as long as the universe expands, decays from a huge value at initial times to the small value observed nowadays [6–8]. Such an assumption can be understood on the basis of a renormalization procedure, as follows. The divergent vacuum energy density referred to above is derived by using field theories in flat spacetime. On the other hand, in the flat spacetime the left hand side of Einstein equations is identically zero, which means that its right hand side, that is, the total energy-momentum tensor, is also zero. Therefore, in the Minkowski background the obtained divergent result must be exactly canceled by introducing a bare cosmological constant in the Einstein equations. Then, when we calculate the vacuum energy density in a curved spacetime, we obtain again a divergent result, but a finite, renormalized cosmological constant follows by subtracting the Minkowskian divergent vacuum density. In the case of an expanding background, this finite cosmological term, being dependent on the curvature, should decay with the expansion, being very large for initial times, and very small for an almost flat universe like the ours.

To give a precise form for that reasoning, by making use of quantum field theories in curved spacetimes, is a difficult challenge. Nonetheless, an estimation has recently been made by Schützhold [9], who suggests that the main contribution to the observed vacuum density arises from the QCD trace anomaly, which, as he argues, dominates over the contributions of the other sectors of the standard model of particles interactions. This leads to a vacuum energy density decaying as $\Lambda \approx m^3 H$, where H is the Hubble parameter, and $m \approx 150$ MeV is the energy scale of the chiral phase transition of QCD. By using $H_0 \approx 70$ (km/s)/Mpc [10], it is easy to verify that this scaling law leads to a present value in accordance with observation.

In this paper we investigate a cosmological scenario with such a varying cosmological constant. We consider an isotropic and homogeneous flat space, filled with matter and a cosmological term proportional to H, obeying the equation of state of the vacuum. We will show that the well established features of the standard big-bang model are preserved, and that, in addition, the model is in accordance with recent measurements of cosmological parameters like the universe age, the deceleration parameter, and the relative density between matter and vacuum energy. The universe evolution has three distinct phases: a radiation dominated era, with the same expansion rate as in standard cosmology; a phase dominated by dust, long enough to permit structure formation; and a later epoch dominated by the cosmological term, which tends asymptotically to a de Sitter universe. As an additional feature of the model, we show that the conservation of the total energy, contained in the Einstein equations, leads to a process of matter production, at the expenses of the decaying vacuum energy. This process does not affect the primordial nucleosynthesis, and, for late times like the ours, it is too small to be detected.

2 Cosmological solutions with varying Λ

Following the Schützhold suggestion, we will consider the decaying vacuum energy density

$$\rho_\Lambda = \Lambda = \sigma H, \tag{1}$$

where σ is a positive constant of the order of m^3, and we have made $8\pi G = c = \hbar = 1$. Furthermore, we will take for the vacuum the equation of state

$$p_\Lambda = -\rho_\Lambda. \tag{2}$$

This is a natural choice: As the vacuum has the symmetry of the background, its energy-momentum tensor has the form $T_\Lambda^{\mu\nu} = \Lambda g^{\mu\nu}$, where Λ is an invariant function of the coordinates (in a homogeneous and isotropic space, it is just a function of time). In comoving coordinates, this corresponds to a perfect fluid with energy density $\rho_\Lambda = \Lambda$, and pressure $p_\Lambda = -\Lambda$.

As the matter component of the cosmic fluid, we will consider a perfect fluid with equation of state

$$p = (\omega - 1)\rho, \tag{3}$$

where p and ρ are the pressure and energy density, respectively. For dust matter ($p = 0$) we have $\omega = 1$, while for radiation ($p = \rho/3$) one has $\omega = 4/3$.

In the case of an isotropic and homogeneous, spatially flat universe, the Einstein equations can be written as [11]

$$\rho_T = 3H^2, \tag{4}$$

$$\dot{\rho}_T + 3H(\rho_T + p_T) = 0, \tag{5}$$

where the dot means derivation with respect to the cosmological time, and

$$\rho_T = \rho + \rho_\Lambda, \tag{6}$$

$$p_T = p + p_\Lambda \tag{7}$$

are the total energy density and total pressure, respectively.

Equation (4), as well known, is the Friedmann equation in the spatially flat case. In what concerns Eq. (5), it is the continuity equation for the total energy. Using (1), (2), (6), and (7), it can be rewritten as

$$\dot{\rho} + 3H(\rho + p) = -\dot{\Lambda}. \tag{8}$$

In the case of a constant Λ, we recover the continuity equation for matter. The above equation shows us that, in order to satisfy the energy conservation, a decaying vacuum term necessarily leads to matter production [12]. The variation in the number of particles is a known property of non-stationary backgrounds. Its microscopic description is, in general, a difficult task, for it involves quantum field calculations in curved spacetimes. Here we will consider matter production from a macroscopic perspective, obtaining its rate from the Einstein equations.

The set (1)–(7) leads us to the differential equation

$$2\dot{H} + 3\omega H^2 - \sigma\omega H = 0, \tag{9}$$

which determines the time evolution of the Hubble parameter. Apart from an integration constant related to the choice of the origin of time, its general solution is given by

$$t = \frac{2}{\sigma\omega} \ln\left|\frac{3H}{3H - \sigma}\right|. \tag{10}$$

From Eqs. (1), (4), and (6), it is possible to verify that

$$\rho = (3H - \sigma)H. \tag{11}$$

As the weak energy condition requires $\rho \geq 0$, and since, in an expanding universe, we have $H = \dot{a}/a \geq 0$, it follows that $3H - \sigma \geq 0$. Therefore, the solution (10) can simply be written as

$$t = \frac{2}{\sigma\omega} \ln\left(\frac{H}{H - \sigma/3}\right), \tag{12}$$

leading to

$$H = \frac{\sigma/3}{1 - \exp(-\sigma\omega t/2)}. \tag{13}$$

Integrating once more with respect to time, we obtain the scale factor

$$a = C[\exp(\sigma\omega t/2) - 1]^{\frac{2}{3\omega}}, \tag{14}$$

where C is an integration constant.

Substituting (13) into (1) and (11), we obtain, respectively, the cosmological term and the matter density as functions of time,

$$\Lambda = \frac{\sigma^2/3}{1 - \exp(-\sigma\omega t/2)}, \tag{15}$$

$$\rho = \frac{\sigma^2}{12} \sinh^{-2}(\sigma\omega t/4). \tag{16}$$

Therefore, the ratio between the vacuum and matter densities scales as

$$\Omega \equiv \frac{\Lambda}{\rho} = \exp(\sigma \omega t / 2) - 1. \tag{17}$$

With the help of (12) or (13), it is possible to derive from the above equation the relation

$$\sigma = \frac{3H\Omega}{\Omega + 1} \tag{18}$$

(which can also be found by using (1), (4), and (6)). This equation allows one to obtain a precise value for σ from the observed current values of H and Ω.

The vacuum and matter densities can also be expressed as functions of the scale factor. With the help of (14), we rewrite Eq. (16) in the form

$$\rho = \frac{\sigma^2}{3} \left(\frac{C}{a}\right)^{3\omega/2} \left[1 + \left(\frac{C}{a}\right)^{3\omega/2}\right], \tag{19}$$

while for Λ we have

$$\Lambda = \frac{\sigma^2}{3} \left[1 + \left(\frac{C}{a}\right)^{3\omega/2}\right]. \tag{20}$$

From (14) one can deduce the deceleration factor, $q = \ddot{a}a/\dot{a}^2$. It is given by

$$q = \frac{3\omega}{2} \exp(-\sigma \omega t / 2) - 1, \tag{21}$$

which, by using (17), can also be written as

$$q = \frac{3\omega}{2(\Omega + 1)} - 1. \tag{22}$$

3 The radiation era

For the radiation epoch, we have $\omega = 4/3$. In this case, Eq. (14) is written as

$$a = C[\exp(2\sigma t / 3) - 1]^{1/2}. \tag{23}$$

In the limit of small time ($\sigma t \ll 1$), this expression reduces to

$$a \approx \sqrt{2C^2 \sigma t / 3}. \tag{24}$$

On the other hand, expressions (19)–(20) turn out to be

$$\rho = \frac{\sigma^2 C^4}{3a^4} + \frac{\sigma^2 C^2}{3a^2}, \tag{25}$$

$$\Lambda = \frac{\sigma^2}{3} + \frac{\sigma^2 C^2}{3a^2}. \tag{26}$$

In the limit $a \to 0$, they reduce to

$$\rho = \frac{\sigma^2 C^4}{3a^4} = \frac{3}{4t^2}, \tag{27}$$

$$\Lambda = \frac{\sigma^2 C^2}{3a^2} = \frac{\sigma}{2t} \tag{28}$$

(where we have also used (24)).

From (24) and (27) one can see that the scale factor and the matter density have the same time dependence as in the standard model, and that the radiation density scales as a^{-4}, as should be [11]. Alternatively, comparing (28) to (1) we obtain $Ht = 1/2$, and, from (21), with $t \to 0$ and $\omega = 4/3$, we have $q \approx 1$, again in accordance with the standard model. Furthermore, comparing (27) to (28), we see that, for $a \to 0$, the radiation density diverges faster than the cosmological term, leading to $\Omega \approx 0$. In other words, at early times the expansion is completely dominated by radiation.

The first term of Eq. (25) gives the usual scaling of the radiation density. The second term is owing to the process of matter production, resulting from the decay of the vacuum energy density, as already discussed. Since, in the limit of small time, the first term dominates, we expect that the matter production does not interfere on processes taking place at early times, as nucleosynthesis, for example. Let us show that this is the case.

The rate of matter production can be defined in this context as

$$T_\gamma = \frac{1}{\rho a^4} \frac{d}{dt}(\rho a^4) \tag{29}$$

(in the case of a genuine cosmological constant, or in the absence of any cosmological term, ρa^4 is a constant, and this rate is equal to zero). With the help of (17), (18), (23), and (25) (with $\omega = 4/3$), it is possible to verify that

$$T_\gamma = 2\sigma/3 = \frac{2\Omega}{1+\Omega} H. \tag{30}$$

We then obtain a constant rate. However, what is really important in what concerns processes like nucleosynthesis is the ratio between the rate of matter production and the expansion rate, that is, the ratio

$$T_\gamma/H = \frac{2\Omega}{1+\Omega}. \tag{31}$$

We have seen that, for early times, $\Omega \approx 0$, leading to $T_\gamma/H \approx 0$. As the ratio between the nucleosynthesis rate and the expansion rate is finite at the time of primordial nucleosynthesis, we conclude that the matter production does not affect this process, as expected.

To close this section let us comment that, although Λ tends to infinity for small times, we do not have inflation in this model (because the radiation density tends to infinity faster). Nevertheless, let us recall that, in his derivation of $\Lambda = \sigma H$ [9], Schützhold has used the approximation that the cosmological time scale, given by H^{-1}, is very large compared to the time scale of the vacuum quantum fluctuations. This approximation does not hold for very early times, when inflation occur.

4 Dust era and the limit of late times

Let us now discuss the phase dominated by dust matter, that is, the case $\omega = 1$. Now, the scale factor (14) has the form

$$a = C[\exp(\sigma t/2) - 1]^{2/3} \tag{32}$$

(evidently, the integration constant is not the same as in Eqs. (23)–(24)).

For small times (small compared to the present time), it can be approximated by

$$a = C(\sigma t/2)^{2/3}, \tag{33}$$

which has the same time dependence as in the standard flat model with dust [11]. Therefore, the radiation era is followed by an epoch with decelerating expansion, as necessary in order to allow structure formation. As we shall see, the varying cosmological term starts dominating just at the present time, which guarantees a large enough dust era.

From expressions (19)–(20) (with $\omega = 1$), we obtain for the matter density

$$\rho = \frac{\sigma^2 C^3}{3a^3} + \frac{\sigma^2 C^{3/2}}{3a^{3/2}}, \tag{34}$$

while for Λ one has

$$\Lambda = \frac{\sigma^2}{3} + \frac{\sigma^2 C^{3/2}}{3a^{3/2}}. \tag{35}$$

The first term in (34) gives the usual scaling of dust matter. The second term is related, as before, to the production of matter, at the expenses of the vacuum decay.

In the limit of large times, that is, $\sigma t \gg 1$ and $a \to \infty$, Eqs. (32), (34), and (35) lead to

$$a = C \exp(\sigma t/3), \tag{36}$$

$$\Lambda = \sigma^2/3, \tag{37}$$

$$\rho \approx 0. \tag{38}$$

That is, as long as the cosmological term tends asymptotically to a genuine cosmological constant, our solution tends to a de Sitter universe, with $H = \sqrt{\Lambda/3} = \sigma/3$. The same result can be seen from Eq. (21): In the limit $t \to \infty$, it gives $q = -1$, which characterizes the de Sitter solution [11].

Let us now obtain the universe age, i.e., the value of the cosmological time at present. For any time, by using (12) and (18) (with $\omega = 1$) it is easy to show that

$$tH = \frac{2(\Omega + 1)}{3\Omega} \ln(\Omega + 1). \tag{39}$$

The observations suggest that the present ratio between the vacuum and matter densities is $\Omega_0 \approx 2$ [4]. With this value we have, from the above equation, $t_0 H_0 \approx \ln 3 \approx 1.1$. This is inside the current limits for the universe age, $0.8 \lesssim t_0 H_0 \lesssim 1.3$, and in good accordance with the best estimation $t_0 H_0 \approx 1$ [13].

From Eq. (22), we can also obtain the present value of the deceleration parameter. Taking $\omega = 1$ and $\Omega_0 \approx 2$, one has $q_0 \approx -1/2$. This is the same result

obtained if we consider a genuine cosmological constant, and it is consistent with supernova observations.

As commented above, the epoch dominated by dust, with decelerating expansion, must be sufficiently large to allow the formation of large structures. Let us verify that this is indeed the case. The time t_Λ for which the cosmological term starts dominating is given by the condition $\Lambda = \rho$. Then, equating (15) to (16) (with $\omega = 1$), and using (18), it is not difficult to derive

$$t_\Lambda = \frac{2 \ln 2}{3H_0} \left(\frac{1 + \Omega_0}{\Omega_0} \right).$$

(40)

By using $\Omega_0 \approx 2$ and our previous result $t_0 H_0 \approx \ln 3$, we obtain $t_\Lambda \approx 0.6\, t_0$.

On the other hand, we can also calculate the time t_q for which $q = 0$, that is, when the expansion changes from the decelerating phase to an accelerating one. Equating to zero the expression (21), with $\omega = 1$, and using again (18), we obtain

$$t_q = \frac{2 \ln(3/2)}{3H_0} \left(\frac{1 + \Omega_0}{\Omega_0} \right).$$

(41)

With $t_0 H_0 \approx \ln 3$, one has $t_q \approx 0.4\, t_0$. Thus, the characteristic times t_Λ and t_q are close to the present time, indicating that we are living just at the end of the dust era.

Finally, let us investigate the production of matter at late times. Analogously to the case of radiation (see Eq. (29)), the rate of matter production is now defined as

$$T = \frac{1}{\rho a^3} \frac{d}{dt}(\rho a^3)$$

(42)

(when $T = 0$, we have $\rho a^3 = $ constant, as should be for conserved dust matter). It can be calculated with the help of Eqs. (17), (18), (32), and (34) (with $\omega = 1$), leading to

$$T = \sigma/2 = \frac{3\Omega}{2(1 + \Omega)} H.$$

(43)

In the limit of large times we have $\Omega \to \infty$, and so

$$T_\infty = 3H/2.$$

(44)

For the present time, on the other hand, we have $\Omega \approx 2$, and then

$$T_0 \approx H_0.$$

(45)

These results (which, actually, are both equals to $\sigma/2$) are smaller than the rate characteristic of the old stead-state cosmology, given by $T = 3H$. They are beyond the current possibilities of direct observation.

5 Concluding remarks

The proposal of a cosmological term scaling with H, which follows from quantum field estimations of the vacuum energy density in an expanding background [9], besides to give the small Λ observed nowadays, leads to a cosmological scenario in accordance with well based features of modern cosmology, as an initial phase dominated by radiation, followed by a dust epoch long enough to allow structure formation, and by an accelerated expansion at late times. In addition, by using as input the present ratio between the vacuum and matter energy densities, we obtain a universe age in accordance with the observed limits, and a deceleration parameter that does not differ from the case of a constant cosmological term.

We have also verified that the matter production characteristic of the model does not affect the primordial nucleosynthesis, and that it is not directly observable nowadays. Nevertheless, one could ask about indirect evidences of this process. As the time scaling of the energy density during the radiation and dust phases is the same as in the standard recipe, we do not expect important changes in what concerns the present temperature and spectrum of the cosmic microwave background. But it would be an interesting line of investigation to look for signatures of radiation or matter production in CMB.

Signatures of matter production may also be found in the relative abundances of light elements, owing to possible baryon production originated from the vacuum decay throughout the whole expansion. On the other hand, the observed abundances may also be used to establish limits to the decay of vacuum into baryonic matter. Note, however, that this would not impose limits on the vacuum decay at all. While we do not have a complete, microscopic description of the process, we do not know what kind of particles are produced: baryons, dark matter or any other else. The absence of a definite quantum field theory for the vacuum decay in curved backgrounds is also related to another open problem, namely, where the new matter is generated. Has it some correlation with the existing matter, or is it produced throughout the entire space? These questions are common to any cosmological model with matter production.

As a last point, we have to note that the Schützhold proposal does not explain a second important problem related to the cosmological constant, namely the cosmic coincidence, that is, the approximate coincidence observed today between the vacuum and matter densities. This second problem may, in principle, be solved in other kinds of model with vacuum decay, which consider a distinct evolution law for Λ, varying with H^2 instead of a linear relation [14]. These models admit solutions in which, in the limit of large times, the ratio between the matter and vacuum energy densities tends to a finite constant. The problem is that they lead to a different dynamics at early times, and so are limited by nucleosynthesis and CMB observations. Another problem is that, in order to get the observed ratio between the matter and vacuum densities, we obtain a universe age very high compared to observational bounds [15].

As discussed in other works [15,16], cosmological models with decaying vacuum density admit late time solutions characterized by a constant and correct ratio between vacuum and matter densities, and in good accordance with the observed universe age and deceleration parameter. In this case, however, we need also to suppose a time variation of the gravitational constant, a hypothesis that depends on observational confirmation, and that is outside the scope of the present paper.

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