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ARTINIAN RINGS IN WHICH ONE SIDED IDEALS ARE QUASI-PROJECTIVE

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Introduction

For a ring R, an R-module M is said to be quasi-injective in case the natural homomorphism $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(K, M)$ is epic for all submodules K of M. Dually M is said to be quasi-projective in case the natural homomorphism $\mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, N)$ is epic for all factor modules N of M. Rings whose left ideals are quasi-injective have been studied by a number of authors ([3], [5], [7]), and for suitable conditions on the ring a number of structure theorems have been obtained ([5], [7]). The main object of this paper is to investigate artinian rings whose left ideals are quasi-projective. These rings include artinian hereditary rings, but many examples exist which are not hereditary. (See Section 4.)

The first three sections are devoted to characterizing artinian rings whose left ideals are quasi-projective. The main theorem (Theorem 3.5) appears in Section 3. There, these rings are characterized in terms of their primitive idempotents and two sided ideals i.e., given a basic set of primitive idempotents and the set of ideals of an artinian ring R, it is possible to determine if R has all left ideals quasi-projective by considering each left R-module $J^{\alpha}e$ where α is a positive integer, J is the Jacobson radical and e is a primitive idempotent. It will be shown that $J^{\alpha}e$ must have a certain decomposition for rings with left ideals quasi-projective, and that with the addition of a suitable hypothesis, this decomposition completely determines such rings.

The final section is devoted to a number of examples to show that the conditions of the structure theorems in 3 are necessary and the best possible.

We shall use the following notation. The ring R is associative with unity. The letter J denotes the Jacobson radical and ${}_RM(M_R)$ signifies that M is a left (right) R-module. The socle of a module M, which is the largest semi-simple submodule of M, will be denoted by $S(M)$. When R is semi-local (i.e., R/J is artinian semi-simple), the semi-simple module M/JM, called the top of M, will be denoted by $T(M)$. Also the notation $M^{(A)}$ means $\oplus \Sigma M_{\alpha}$ where $M_{\alpha} \cong M$.

Preliminaries

A number of concepts will be needed in the development of the results which follow. We begin with the following

Definition. Let P be a projective R-module. Then P is said to be *hereditary* in case every submodule of P is projective.

Clearly any submodule of a hereditary module is again hereditary. Also observe that for a given set $\{P_{\alpha}\}_{\alpha \in A}$ of hereditary modules, the direct sum $\oplus \Sigma P_{\alpha}$ is always hereditary ([6], Proposition 7, page 85). Although many rings with left (right) ideals quasi-projective are not hereditary, a common feature of many of these rings is that they possess a 'large' hereditary left ideal.

We also will need the following lemmas which allow us to simplify some of the proofs in much of the subsequent work. Recall that a module Q is said to be *projective relative* to M if for all factor modules N of M the natural homomorphism $\mathrm{Hom}_R(Q, M) \to \mathrm{Hom}_R(Q, N)$ is epic. The class of modules to which Q is projective is closed under taking submodules, factors, and finite direct sums [8]. From this it is easily seen that $M_1 \oplus M_2$ is quasi-projective if and only if M_i is projective relative to M_j for $i, j = 1, 2$.

1.1. LEMMA. *Let R be a ring. Suppose the module $Re \oplus Re/Ie$ is quasi-projective where e is a primitive idempotent and Ie is a left ideal. Then $Ie = 0$.*

PROOF. By ([8], Proposition 1.2), Re/Ie is projective relative to Re. Thus the map $Re \to Re/Ie$ splits. Since e is primitive, this forces $Ie = 0$.

1.2. LEMMA. *Let R be a ring with every left ideal quasi-projective. Let f be a primitive idempotent and I a left ideal such that $I \cap Rf = 0$. Suppose $fI \neq 0$. Then there exists a monomorphism $\varphi\colon Rf \to I$, given by right multiplication of an element $x \in I$.*

PROOF. Since $fI \neq 0$, there exists an $x \in I$ such that $fx \neq 0$. Let φ be the map given by right multiplication of x. Then $Rf/Kf \cong \mathrm{Im}(\varphi) \subseteq I$. As $I \cap Rf = 0$, $Rf/Kf \oplus Rf$ is isomorphic to a left ideal of R. Hence by 1.1, $Kf = 0$. This shows that φ is monic.

2. The Loewy series decomposition

For a left R-module M, the Loewy series is the sequence of left R-modules

$$M \supset JM \supset \dots \supset J^k M \supset \dots.$$

The k-th Loewy factor is the module $J^{k-1}M/J^kM$. One defines the Loewy series, for right modules in a similar way. The Loewy series will be used to obtain a decomposition for artinian rings whose left ideals are quasi-projective. In light of this, we make the following

DEFINITION. Let R be left artinian and e a primitive idempotent. Let

$$Re \supset Je \supset \dots \supset J^n e \supset 0,$$

be the Loewy series for Re. For each α such $1 \leqq \alpha \leqq n$, $J^\alpha e$ may be decomposed into a direct sum of indecomposables say $J^\alpha e = \oplus \sum_{i_\alpha=1}^{k_\alpha} I_{i_\alpha}$. Then we may express the Loewy series as,

$$Re \supset \oplus \sum^{k_1} I_{i_1} \supset \dots \supset \oplus \sum^{k_n} I_{i_n} \supset 0.$$

The above expression will be called a *Loewy series decomposition* for the module Re.

It will be shown that rings with every left ideal quasi-projective have a particularly nice Loewy series decomposition for each of their principal indecomposable projective modules. This Loewy series decomposition will be used to characterize left artinian rings with every left ideal quasi-projective in terms of the primitive idempotents and two sided ideals of the ring.

REMARK. Note that for each $\alpha>0$, $J^{\alpha}e$ has a unique decomposition using the Krull—Schmidt theorem for artinian rings.

Thus the Loewy series decomposition for each principal indecomposable projective is unique up to isomorphism.

The remainder of this section will be devoted to obtaining the Loewy series decomposition for each Re, where e is a primitive idempotent and R is an artinian ring with every left ideal quasi-projective.

2.1 LEMMA. *Let R be a left artinian ring with every left ideal quasi-projective. Let f be any primitive idempotent and $L\subseteqq Rf$ a left ideal. Then L admits a decomposition $L=P\oplus K$ such that:*

(1) P is projective and $fP=0$.

(2) $K\cong(Rf/If)^{(n)}$ for some two sided ideal I.

Here either P or K may be 0.

PROOF. The left ideal L is quasi-projective, so by ([4], Theorem 1.10),

$$L\cong(Re_1/Ie_1)^{(n_1)}\oplus\ldots\oplus(Re_k/Ie_k)^{(n_k)}$$

where $\{e_j\}_{j=1}^{k}$ are a set of primitive orthogonal idempotents and $Re_i\not\cong Re_j$ when $i\neq j$, and I is a 2-sided ideal in R. As $Re_j\not\cong Rf$ for all j with at most one possible exception, let $P\cong\oplus\Sigma(Re_j/Ie_j)^{(n_j)}$ where $1\leqq j\leqq k$ and $Re_j\not\cong Rf$. Then there exists for each j, a left ideal isomorphic to $Re_j\oplus Re_j/Ie_j$, where $Re_j\not\cong Rf$. By 1.1 $Ie_j=0$. Hence $P\cong\oplus\Sigma(Re_j)^{(n_j)}$. Now suppose $fP\neq0$. Using 1.2 there is an isomorphic copy of Rf contained in $P\subseteqq L$ contradicting R left artinian. Thus $f\cdot P=0$; and $L\cong P$ or $L\cong P\oplus(Rf/If)^{(n)}$ depending on whether there exists $Re_j\cong Rf$ for some $j\leqq k$.

2.2 LEMMA. *Let R be left artinian with every left ideal quasiprojective, and let P and f be as in Lemma 2.1. Then P is hereditary.*

PROOF. Consider $K\subseteqq P$. Then

$$K\cong(Rf_1/If_1)^{(n_1)}\oplus\ldots\oplus(Rf_m/If_m)^{(n_m)}$$

where each f_j is a primitive idempotent and I is a two sided ideal. By 2.1 $fK=0$ which implies that each $Rf_j\not\cong Rf$, $(1\leqq j\leqq m)$. Hence, there exists a left ideal isomorphic to $Rf_j\oplus Rf_j/If_j$. So by 1.1. $If_j=0$ for each j. This shows that K is projective, so P is hereditary.

2.3 LEMMA. *Let R be left artinian with every left ideal quasi-projective. Let K be as in Lemma 2.1. Suppose $K\cong(Rf/If)^{(n)}$ where $n>1$ and f a primitive idempotent, I a two sided ideal. Then*

(1) $f\cdot JK=0$ i.e., JK has no composition factor isomorphic to $T(Rf)$.

(2) JK is hereditary.

PROOF. Suppose that $f\cdot JK\neq0$. Then $f\cdot Jf/If\neq0$. This induces a homomorphism of Rf into Jf/If. Hence there is a factor module of Rf, $N\subseteqq Jf/If$. Since K is a direct sum of at least two copies of Rf/If, there is a submodule of K isomorphic to $N\oplus Rf/If$.

Thus N is projective relative to Rf/If. Using this and that $T(N)\cong T(Rf)$, we have the following diagram,

$$\begin{tikzcd} & N \arrow[dl, "\varphi"'] \arrow[d, "\pi"] & \\ Rf/If \arrow[r] & T(Rf) \arrow[r] & 0 \end{tikzcd}$$

in which the map π can be extended to a map $\varphi\colon N\to Rf/If$. But φ is epic since Jf/If is superfluous in Rf/If. Thus Rf/If is isomorphic to an epimorphic image of N. This contradicts R being left artinian.

To prove (2), we need only note that as JK is quasi-projective $JK\cong\oplus\Sigma Re_\alpha/Ie_\alpha$ where each $Re_\alpha\not\cong Rf$. Thus $Ie_\alpha=0$ for each e_α follows from 1.2. This shows that JK is projective. The proof that JK is hereditary is similar to the proof of 2.2.

Lemma's 2.1, 2.2, 2.3 provide the motivation for the following

Definition. We will say that a left artinian ring R has a *Loewy series decomposition of type qp* if the following conditions hold: For each primitive idempotent f, $J^\alpha f=K_\alpha\oplus P_\alpha$ where P_α is hereditary, and $K_\alpha\cong(Rf/I_\alpha f)^{(n_\alpha)}$ where I_α is some two sided ideal.

The K_α, P_α satisfy,

1. $K_1\supset K_2\supset\ldots\supset K_n=0$ and $JK_\alpha=K_{\alpha+1}\oplus Q_{\alpha+1}$, $Q_{\alpha+1}\subset P_{\alpha+1}$.
2. If $K_1\cong(Rf/I_1f)^{(n_1)}$, $n_1>1$, then $K_\alpha=0, \alpha>1$.
3. If $K_1\cong Rf/I_1f$ then for $\alpha>1$ where $K_\alpha\neq0$, $K_\alpha\cong Rf/I_\alpha f$.

2.4 Proposition. *Let R be a left artinian ring with every left ideal quasi-projective. Then R has a decomposition of type qp.*

Proof. Let f be any primitive idempotent. Then by 2.1 and 2.2, $J^\alpha f=K_\alpha\oplus P_\alpha$ where $K_\alpha\cong(Rf/I_\alpha f)^{(n_\alpha)}$, P_α is hereditary, and I_α is a two sided ideal. To show (1) we use induction to construct $K_{\alpha+1}$ and $P_{\alpha+1}$ from K_α and P_α as follows: Let $J^{\alpha+1}f=J(K_\alpha\oplus P_\alpha)=JK_\alpha\oplus JP_\alpha$. By 2.1 and 2.2, $JK_\alpha=K_{\alpha+1}\oplus Q_{\alpha+1}$ where $Q_{\alpha+1}$ is hereditary and $K_{\alpha+1}\cong(Rf/I_{\alpha+1}f)^{n_{\alpha+1}}$. Clearly $K_{\alpha+1}\subset K_\alpha$. Now $J^{\alpha+1}f=K_{\alpha+1}\oplus Q_{\alpha+1}\oplus JP_\alpha$. Let $P_{\alpha+1}=Q_{\alpha+1}\oplus JP_\alpha$. Then $P_{\alpha+1}$ is hereditary and $Q_{\alpha+1}$ is a direct summand of $P_{\alpha+1}$.

For statement (2), we note that it follows easily from 2.3. For (3) let $K_1\cong Rf/I_1f\subseteqq Jf$. Suppose $K_\alpha\cong Rf/I_\alpha f$ and $K_{\alpha+1}\cong(Rf/I_{\alpha+1})^{(n_{\alpha+1})}$ where $n_{\alpha+1}\geqq1$. Then $Rf\to K_\alpha\to0$ whence $Jf\to JK_\alpha\to0$. Now using that Jf/J^2f has one isomorphic copy of $T(Rf)$ we have $T(K_{\alpha+1})\cong T(Rf)$. So $n_{\alpha+1}=1$.

Remark. In the future the terminalogy $K_{\text{subscript}}$, $P_{\text{subscript}}$ will be used to stand for the modules K_α, P_α when $J^\alpha f=K_\alpha\oplus P_\alpha$ whenever R has a decomposition of type qp and f is a primitive idempotent.

3. Left artinian rings whose left ideals are quasi-projective

The Loewy series decomposition of type qp will now be used to characterize the rings of this section. An additional property must be satisfied by the above decomposition in order to completely determine the structure of these rings. This is indicated by the following

3.1 LEMMA. *Let R be as in Lemma 2.1. Suppose $J^\alpha f = K_\alpha \oplus P_\alpha$ where $K_\alpha \cong (Rf/If)^{(n)}$. Then for any indecomposable projective left ideal P, $P \cong Re$, e a primitive idempotent, such that $P \cap K_\alpha = 0$, we have $e \cdot K_\alpha \neq 0$ if and only if there exists an isomorphic copy of P contained in K_α.*

PROOF. If $e \cdot K_\alpha \neq 0$, then by 1.2 K_α contains a copy of Re whenever $Re \not\cong Rf$. Otherwise 1.1 applies and K_α contains a copy of Rf, a contradiction to R artinian.

We now examine the left ideals of rings possessing a decomposition of type qp.

3.2 LEMMA. *Let R be a left artinian ring with a decomposition of type qp. Then for any left ideal $L \subseteq Rf$, f a primitive idempotent, $L \cong M_\alpha \oplus N$, where N is hereditary, and $M_\alpha \cong (Rf/I_\alpha f)^{(n_\alpha)}$ where M_α is a direct summand of K_α and $J^\alpha f = K_\alpha \oplus P_\alpha$.*

PROOF. Since $L \subseteq Rf$, there exists α_1 such that $L \subseteq J^{\alpha_1} f$, $L \not\subseteq J^{\alpha_1+1} f$. Since $J^{\alpha_1} f = K_{\alpha_1} \oplus P_{\alpha_1}$ the restriction to L of the canonical projection of $J^{\alpha_1} f$ onto P_{α_1} maps L onto a submodule L_{α_1} of P_{α_1}. As P_{α_1} is hereditary, L_{α_1} is projective, hence $L \cong L_{\alpha_1} \oplus M_{\alpha_1}$ where $L_{\alpha_1} \subseteq P_{\alpha_1}$, $M_{\alpha_1} \subseteq K_{\alpha_1}$.

Now we consider two cases:

Case 1: $\alpha_1 = 1$, $K_1 \cong (Rf/I_1 f)^{(n_1)}$, $n_1 > 1$. Consider the restriction to M_1 of the canonical projection π of K_1 onto each of the indecomposable summands $I \cong Rf/I_1 f$ of K_1. If the restriction is epic for one of the indecomposable direct summands I, $M_1 \subseteq K_1$ and I quasi-projective imply that $M_1 \cong I \oplus M_2$ where $M_2 \subseteq K_1$. Now apply the same argument to M_2 as was done to M_1 in case one of the projections onto an indecomposable direct summand of K_1 is epic when restricted to M_2. Since K_1 is a finite direct sum of indecomposable quasi-projective modules, continue the process until

$$M_1 \cong (Rf/I_1 f)^{(s)} \oplus M_{s+1}, \quad s \leqq n_1$$

and $M_{s+1} \subseteq K_1$ has the property that for each $\pi: K_1 \to I$, when restricted to M_{s+1} is not epic. This means that $\pi(M_{s+1}) \subseteq JI$ for all indecomposable I in the direct sum decomposition of K_1. Therefore,

$$M_{s+1} \subseteq JK_1 \subseteq J(K_1 \oplus P_1) = J^2 f.$$

But by property 2 of the Loewy series decomposition of type qp, $J^2 f = P_2$, P_2 hereditary. Hence M_{s+1} is hereditary. Setting $N = M_{s+1} \oplus L_1$, we have $L \cong N \oplus M_s$ where $M_s \cong (Rf/I_1 f)^{(s)}$. Thus the conditions of the lemma are satisfied.

Case 2: $K_{\alpha_1} \cong Rf/I_{\alpha_1} f$. If $M_{\alpha_1} = K_{\alpha_1}$ there is nothing to prove. Otherwise

$$M_{\alpha_1} \subseteq JK_{\alpha_1} \subseteq P_{\alpha_2} \oplus K_{\alpha_2} = J^{\alpha_2} f$$

where $\alpha_2 = \alpha_1 + 1$. The projection $\pi: J^{\alpha_2} f \to P_{\alpha_2}$ maps M_{α_1} onto a hereditary submodule L_{α_2} of P_{α_2}. Hence $M_{\alpha_1} \cong L_{\alpha_2} \oplus M_{\alpha_2}$, $M_{\alpha_2} \subseteq K_{\alpha_2}$. If $K_{\alpha_2} = 0$, we are through.

Otherwise, property 3 of a Loewy series of type qp implies that $K_{\alpha_2}\cong Rf/I_{\alpha_2}f$. If $K_{\alpha_2}=M_{\alpha_2}$ there is nothing more to prove. Otherwise, using that R is artinian, we can continue the process s number of times until we obtain M_{α_s} such that M_{α_s} is hereditary or $M_{\alpha_s}\cong L_{\alpha_{s+1}}\oplus M_{\alpha_{s+1}}$, $L_{\alpha_{s+1}}$ hereditary, $M_{\alpha_{s+1}}\cong Rf/I_{\alpha_{s+1}}f$. Let $N=\oplus\sum_{i=1}^{s+1} L_{\alpha_i}$. Then $L\cong N$ or $L\cong N\oplus M_{\alpha_{s+1}}$. In either case the lemma is satisfied. This completes the proof.

3.3. Lemma. *Suppose R has a decomposition of type qp and satisfies the conclusion of* 3.1. *Then K_α is projective relative to P where P is a projective left ideal such that $K_\alpha\cap P=0$. Thus $K_\alpha\oplus P$ is quasi-projective.*

Proof. Clearly the last statement follows from the first and the remark before 1.1. Recall that $K_\alpha\cong(Rf/If)^{(n)}$ where K_α is a direct summand of $J^\alpha f$, and f is a primitive idempotent. Now $P=\oplus\sum^k P_i$, $P_i\cong Re_i$, where each $e_i\in R$ is a primitive idempotent. We show that K_α is projective relative to P by first showing that it is projective relative to each P_i. So let g be a map $g\colon K_\alpha\to P_i/K_i$, P_i/K_i a factor module of P_i, and π a map $\pi\colon P_i\to P_i/K_i$ which is epic. Now consider the module

$$H=\{x\in P_i\colon\ \pi(x)\in\operatorname{Im}(g)\}.$$

By 3.2,

$$H=H_1\oplus\ldots\oplus H_{t-1}\oplus(H_t)^{(s)} \tag{1}$$

where $H_j\cong Rf_j$, $1\leqq j\leqq t-1$, $f_j\in R$ a primitive idempotent, and $H_t=Re_i/Ie_i$, H_t quasi-projective. In the following discussion set $e_i=f_t$.

Let $H=M_1\oplus M_2$ where M_2 is the direct sum of all the indecomposable modules in (1) contained in the ker (π). Hence for each indecomposable module $H_j\subseteq M_1$, K_α has a composition factor isomorphic to $T(Rf_j)$. This implies that $f_jK_\alpha\neq 0$ for each $H_j\subseteq M_1$. Clearly each $H_j\cap K_\alpha=0$ for $1\leqq j\leqq t-1$. Thus for each $H_j\subseteq M_1$, $(1\leqq j\leqq t-1)$ there is an isomorphic copy of H_j contained in K_α since R satisfies the conclusion of 3.1. By the same argument, if $H_t\subseteq M_1$, $P_i\cap K_\alpha=0$ implies that K_α contains an isomorphic copy of P_i. These two statements imply that K_α is projective relative to M_1. Thus it is possible to extend g to M_1 (and hence to H). So K_α is projective relative to Re_i for each i, and is therefore projective relative to P.

3.4 Lemma. *Let R be a left artinian ring. Suppose R has a decomposition of type qp and satisfies the conclusion of* 3.1. *If $1=\sum^k e_i$, where $\{e_i\}$ is a set of primitive orthogonal idempotents, then for any left ideal $L\subseteq R$, L is quasi-projective and $L\cong\oplus\sum^k(M_{\alpha_i}\oplus N_i)$ where $M_{\alpha_i}\oplus N_i\subseteq Re_i$ and N_i is hereditary, M_{α_i} a direct summand of K_{α_i}, K_{α_i} as in* 3.2.

Proof. We first show that any left ideal of the form $L=L_1e_1\oplus\ldots\oplus L_ke_k$ is quasi-projective where L_i, $i=1,\ldots,k$, are left ideals.

By 3.2, $L_ie_i=M_{\alpha_i}\oplus N_i$ where $M_{\alpha_i}\cong(Re_i/Ie_i)^{(n_{\alpha_i})}$ and N_i is hereditary. Thus $L=\oplus\Sigma M_{\alpha_i}\oplus N_i$. Using 3.3, M_{α_i} is projective relative to N_j for all $1\leqq j\leqq k$. Since $M_{\alpha_i}\subseteq Re_i$, and $Re_i\cap Re_j=0$ for $j\neq i$, 3.3 implies that M_{α_i} is projective relative to Re_j. Since M_{α_i} is a direct sum of factor modules of Re_j, M_{α_i} is projective relative to

M_{α_j} $(i \neq j)$. Now using the remark before 1.1 and the quasi-projectivity of each M_{α_i}, it is easily seen that $\oplus \Sigma L_i e_i \cong \oplus \Sigma M_{\alpha_i} \oplus N_i$ is quasi-projective.

We need only show that $L = \oplus \Sigma L_i e_i$ for suitably chosen left ideals L_i, $i=1, \dots, k$. As $L \subseteqq \oplus \Sigma L e_i$, and $\oplus \Sigma L e_i$ is quasi-projective by the previous remarks, Le_1 is projective relative to L by the remark before 1.1. Thus, the canonical epimorphism $\pi_1 : L \to Le_1$ given by right multiplication by e_1 splits. Hence $L \cong Le_1 \oplus L_2$ where $L_2 \subseteqq L$, and $L_2 e_1 = 0$. Now there exists a canonical epimorphism π_2 of L_2 onto $L_2 e_2$. Using that $L_2 \subseteqq \oplus \Sigma L_2 e_i$ quasi-projective, we can apply the same argument on L_2 as on L. Thus $L_2 \cong L_2 e_2 \oplus L_3$ where $L_3 \subseteqq L_2$ and $L_3 e_2 = 0$. By the application of this argument for at most k times, L can be expressed as

$$L \cong Le_1 \oplus L_2 e_2 \oplus \dots \oplus L_k e_k \oplus L_{k+1}$$

where $L_i \cong L_i e_i \oplus L_{i+1}$, $L_{i+1} \subseteqq L_i$, and $L_{i+1} e_i = 0$. Since $L \supseteqq L_2 \supseteqq \dots \supseteqq L_k \supseteqq L_{k+1}$ and $L_{i+1} e_i = 0$, we have $L_{k+1} e_i = 0$ $(1 \leqq 1 \leqq k)$. So $L_{k+1} = 0$. Therefore $L \cong Le_1 \oplus \oplus L_2 e_2 \oplus \dots \oplus L_k e_k$. By the remarks at the beginning of the proof, L is quasi-projective,

Now the following theorem can be proved which completely characterizes the left artinian rings whose left ideals are quasi-projective.

3.5 THEOREM. *Let R be a left artinian ring. Then R has every left ideal quasi-projective if and only if R satisfies the following conditions:*

(1) *For each primitive idempotent f, Rf has a decomposition of type qp.*

(2) *For each K_α such that $J^\alpha f = K_\alpha \oplus N_\alpha \subseteqq Rf$, and indecomposable projective left ideal P, $P \cong Re$, e a primitive idempotent, such that $P \cap K_\alpha = 0$; either $e \cdot K_\alpha = 0$ or K_α contains an isomoprhic copy of P.*

PROOF. $\Rightarrow$ follows from 2.4 and 3.1. $\Leftarrow$ is a consequence of 3.4.

4. Examples

This section presents a number of examples of rings which serve to illustrate the main features of the decomposition used to characterize rings with every left ideal quasi-projective. The first two examples show that such rings cannot be completely characterized by their Loewy decomposition for each principal indecomposable module — we really need to know the two sided ideals of the ring. The following notation will be used. The 2-sided ideal $l_R(M) = \{x \in R : xM = 0\}$ is the left annihilator of the module M. It is known that for left artinian rings M is quasi-projective if and only if M is projective over $R/l_R(M)$ [2].

1. Let F be a field and R the ring of matrices of the form,

$$R = \left\{ \begin{vmatrix} \alpha & \lambda_4 & \lambda_3 & \lambda_2 \\ 0 & \gamma & 0 & \lambda_1 \\ 0 & 0 & \gamma & \lambda_1 \\ 0 & 0 & 0 & \gamma \end{vmatrix} \right\} \quad (\lambda_i \in F,\ i = 1, \dots, 4,\ \alpha, \gamma \in F)$$

with primitive idempotents

$$e_1=\begin{bmatrix}1&&&\\&0&&0\\0&&0&\\&&&0\end{bmatrix},\quad e_2=\begin{bmatrix}0&&&0\\&1&&\\&&1&\\0&&&1\end{bmatrix}.$$

Then $Je_2\cong T(Re_1)\oplus T(Re_1)\oplus K$ where K is a uniserial left ideal with $T(K)\cong$ $\cong T(Re_2)$, $S(K)\cong T(Re_1)$. So the Loewy series decomposition for Re_2 is

$$\begin{array}{r|c|c|}\multicolumn{1}{r}{}&\multicolumn{2}{|l}{\cong T(Re_2)}\\ \hline \begin{matrix}T(Re_2)\\T(Re_1)\end{matrix}&T(Re_1)&T(Re_1)\end{array}$$

However, the decomposition for Re_2 is not of type qP. For R does not have every left ideal quasi-projective as the uniserial left ideal

$$K=\left\{\begin{bmatrix}0&0&0&\lambda_2\\&0&0&\lambda_1\\&&0&\lambda_1\\&&&0\end{bmatrix}\right\}$$

is not quasi-projective, since K is not projective over $R/l_R(K)=R/K$.

The next example gives a ring with every left ideal quasi-projective and with a Loewy series decomposition the same as the ring in 1.

2. Let F be a field and R the ring of matrices of the form,

$$R=\left\{\begin{bmatrix}\alpha&\lambda_3&\lambda_2&&&\\&\gamma&\lambda_1&&0&\\&&\gamma&&&\\&&&\alpha&\lambda_3&\lambda_4\\&0&&&\gamma&\lambda_1\\&&&&&\gamma\end{bmatrix}\right\}$$

with primitive idempotents

$$e_1=\begin{bmatrix}1&&&&&\\&0&&&&\\&&0&&&\\&&&1&&\\&&&&0&\\&&&&&0\end{bmatrix},\quad e_2=\begin{bmatrix}0&&&&&\\&1&&&&\\&&1&&&\\&&&0&&\\&&&&1&\\&&&&&1\end{bmatrix}.$$

It is easily checked that

$$K=\left\{\begin{bmatrix} 0 & 0 & \lambda & & & \\ & 0 & \lambda_1 & & 0 & \\ & & 0 & & & \\ & & & 0 & 0 & \lambda \\ & 0 & & & 0 & \lambda_1 \\ & & & & & 0 \end{bmatrix}\right\}$$

where K is generated by the element

$$x=\begin{bmatrix} 0 & 0 & 0 & & & \\ & 0 & 1 & & 0 & \\ & & 0 & & & \\ & & & 0 & 0 & 0 \\ & 0 & & & 0 & 1 \\ & & & & & 0 \end{bmatrix}.$$

Thus $Je_2 \cong T(Re_1) \oplus T(Re_1) \oplus K$ where $K=Rx$ is a uniserial module such $T(K) \cong \cong T(Re_2)$, $S(K) \cong T(Re_1)$.

So the Loewy series decomposition is of form,

$$\begin{array}{c|c|c|}
 & \multicolumn{2}{|l}{T(Re_2)} \\ \hline
\begin{matrix} T(Re_2) \\ T(Re_1) \end{matrix} & T(Re_1) & T(Re_1)
\end{array}.$$

It is easily checked that K is projective over $R/l(K)$ and in fact that every left ideal is quasi-projective.

3. This example gives a ring with every left ideal quasi-projective, with a Loewy series decomposition for a principle indecomposable Re_2 such that Je_2/J^2e_2 has more than one copy of $T(Re_2)$. Let K be a field, and R the set of matrices of the form.

$$R=\left\{\begin{bmatrix} \alpha & \lambda_3 & \lambda_2 & & & \\ & \gamma & \lambda_1 & & 0 & \\ & & \gamma & & & \\ & & & \alpha & \lambda_3 & \lambda_5 \\ & 0 & & & \gamma & \lambda_4 \\ & & & & & \gamma \end{bmatrix}\right\} \quad (\alpha, \gamma \in K,\ \lambda_i \in K,\ i=1, \dots, 5).$$

with primitive idempotents

$$e_2 = \begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 & & & & & \\ & 0 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix}.$$

It is easily checked that R has every left ideal quasi-projective and that $Je_2 \cong T(Re_1) \oplus \oplus K_1 \oplus K_2$ where $K_1 \cong K_2$ and $T(K_1) \cong T(Re_2)$, $S(K_1) \cong T(Re_1)$.

So the Loewy series decomposition for Re_2 is

$$\begin{array}{c|c|c|} \multicolumn{2}{c|}{} & T(Re_2) \\ \cline{2-3} T(Re_2) & T(Re_2) & \\ T(Re_1) & T(Re_1) & T(Re_1) \end{array}$$

4. This example shows that condition (2) of 3.5 is necessary by exhibiting a ring with a Loewy series of type qp without having all left ideals quasi-projective.

Let S be any local uniserial ring with a composition series of length 2, so that $JS \cong T(S)$. Define R to be the matrix ring $M_n(S)$, n an integer such that $n>1$. Then for any primitive idempotent $e \in R$, $T(Re) \cong S(Re)$. So it is easily seen that R has a Loewy series decomposition of type qp. But for f any primitive idempotent such that $Rf \cap Re = 0$, we must have $fJe \neq 0$. Thus condition 2 does not hold. Clearly R does not have every left ideal quasi-projective since $Re \oplus T(Re)$ is not quasi-projective.

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