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COMBINATORICS OF A CERTAIN IDEAL IN THE SEGRE COORDINATE RING

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ABSTRACT. We focus on a "fat" model of an ideal in the class of the canonical ideal of the Segre coordinate ring, looking at its Rees algebra and related arithmetical questions.

1. INTRODUCTION

Let $\mathfrak{S}$ be the image of the Segre map

$$\sigma = \sigma_{n-1,m-1} : \mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \longrightarrow \mathbb{P}^{nm-1},$$

the so-called Segre variety. As a toric variety, $\mathfrak{S}$ admits $k[t_i s_j]\,(1 \leq i \leq n,\, 1 \leq j \leq m)$ as coordinate ring. This ring can be presented over the polynomial ring $k[\mathbf{X}] = k[X_{ij}]\,(1 \leq i \leq n,\, 1 \leq j \leq m)$ by the ideal $I_2(X_{ij})$ generated by the 2×2 minors of the generic $n \times m$ matrix (X_{ij}). It is well known that the canonical class of the latter is $(m-n)[\mathfrak{K}]$, where $\mathfrak{K} \subset S = k[\mathbf{X}]/I_2(X_{ij})$ is the ideal generated by (the residues of) the entries in the first column of the matrix (X_{ij}) (cf. [BV], (8.4)).

Now, given an integer $d \geq 1$, let $\mathfrak{K}^{[d]}$ denote the ideal generated by the dth powers of the generators of $\mathfrak{K}$. The main purpose of this paper is to investigate the algebraic-combinatorics of the blowup of $\mathfrak{S}$ along the locus of $\mathfrak{K}^{[d]}$. Algebraically, we are therefore looking at the Rees algebra of the ideal $\mathfrak{K}^{[d]}$. Using the toric representation, this algebra is simply the k-subalgebra

$$k[\, t_i s_j,\, (t_1 s_1)^d\, T, \dots , (t_n s_1)^d\, T\,] \subset k[\,\mathbf{t}, \mathbf{s}\,][\,T\,],$$

where $1 \leq i \leq n$, $1 \leq j \leq m$. Since s_1 is fixed in the dth powers, it is not difficult to see that this algebra is isomorphic to the k-algebra $R^{[d]} = k[t_i s_j,\, t_1^d, \dots , t_n^d] \subset k[\mathbf{t}, \mathbf{s}]$.

As it turns out, $R^{[d]}$ is presented over a polynomial ring $A = k[\mathbf{X}, \mathbf{U}]$, with $\mathbf{X} = \{X_{ij}\}$, $\mathbf{U} = \{U_1, \dots , U_n\}$, by a sum of determinantal ideals, each generated by certain 2×2 minors, so our toric variety is a sort of determinantal locus lacking the generic codimension. It can be looked at as the generic version of a few classes of ideals appearing in the recent literature (cf. [Hu], [HuHu], [Sch] and [MoSi]), obtained thereof by specialization and by taking suitable free ring extensions.

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2. A pseudo-determinantal locus

We will fix the following notation:

$I_r(L)$	the ideal generated by $r \times r$ minors of the matrix L.
$\mathbf{t}$, $\mathbf{s}$	sets of (toric) variables $t_1, \ldots, t_n$, $s_1, \ldots, s_m$ over a field k.
S	the coordinate ring $k[x_{ij}] = k[X_{ij}]/I_2(X_{ij})$ of the Segre embedding.
$\mathfrak{K}^{[d]}$	the ideal (row-matrix) in S generated by the dth powers of $x_{11}, \ldots, x_{n1}$.
$R^{[d]}$	the toric ring $k[t_is_j,\, t_1^d, \ldots, t_n^d]$.
$M(\mathbf{Y})$	a monomial in the variables $\mathbf{Y}$.
$M(\mathbf{y})$	the residue of the monomial $M(\mathbf{Y})$ modulo some ideal.
$\mathbb{M}(d, \mathbf{Y})$	the set (row, ideal) of all monomials of degree d in the variables $\mathbf{Y}$.
$\mathbb{M}(d, \mathbf{y})$	the set of residues of $\mathbb{M}(d, \mathbf{Y})$.

2.1. The defining equations.

One needs the following lemmata. In order to save on notation, we set sometimes $\mathbf{X}_i = X_{i1}, \ldots, X_{im}$ and, correspondingly, $\mathbf{x}_i = x_{i1}, \ldots, x_{im}$.

(2.1.1) Lemma. *For any pair of indices $1 \le i_1, i_2 \le n$, consider the involutive k-algebra automorphism $\Phi = \Phi_{i_1,i_2}$ of the polynomial ring $k[X_{ij}] = k[\mathbf{X}_1, \ldots, \mathbf{X}_n]$ such that*

$$\Phi(X_{i,j}) = \begin{cases} X_{i_2,j} & \text{if } i = i_1, \\ X_{i_1,j} & \text{if } i = i_2, \\ X_{i,j} & \text{otherwise.} \end{cases}$$

Then:

(i) *Φ induces an automorphism of $S = k[X_{ij}]/I_2(X_{ij})$.*
(ii) *For any two monomials $M = M(\mathbf{X}_{i_1})$, $N = N(\mathbf{X}_{i_1}) \in k[\mathbf{X}_{i_1}]$ of the same degree, one has $M\Phi_{i_1,i_2}(N) \equiv N\Phi_{i_1,i_2}(M) \pmod{I_2(X_{ij})}$.*

Proof. (i) Clearly, the ideal $I_2(X_{ij})$ is invariant under Φ. Since Φ is an involution (i.e., $\Phi = \Phi^{-1}$), it then induces an automorphism of S.

(ii) One proceeds by induction on the common degree of M and N. The result is trivial if $M = N$, so assume these are distinct monomials. Now write $M = X_{i_1,j_1}M_1$ and $N = X_{i_1,j_2}N_1$, with $j_1 \ne j_2$. Then, with $\Phi = \Phi_{i_1,i_2}$ and by the inductive hypothesis:

$$\begin{aligned} M\Phi(N) = X_{i_1,j_1}X_{i_2,j_2}M_1\Phi(N_1) &\equiv X_{i_1,j_2}X_{i_2,j_1}M_1\Phi(N_1) \\ &\equiv X_{i_1,j_2}X_{i_2,j_1}N_1\Phi(M_1) = N\Phi(M), \end{aligned}$$

as required. □

(2.1.2) *Remark.* Part (ii) of Lemma (2.1.1) has been used before in different forms (cf., e.g., [Gim, Lemme 5.12.1]).

(2.1.3) Lemma. *The first syzygies of the ideal $\mathfrak{K}^{[d]} \subset S$ are generated by the first syzygies of all pairs $\{x_{i_1,1}^d, x_{i_2,1}^d\}$, $1 \le i_1, i_2 \le n$ and these are generated by those syzygies whose coordinates are terms αM, $\alpha \in k$ and M a monomial.*

Proof. This is a direct consequence of the fact that S is defined by a binomial ideal [EiSt, Corollary 1.7 (b)]. □

Here is the basic technical result of this section:

(2.1.4) Proposition. *Let $d \geq 1$. The ideal $\mathfrak{K}^{[d]} \subset S$ has the following presentation as an S-module:*

$$(\bigwedge^2 S^n)^{\oplus C(m,d)} \xrightarrow{\psi^{[d]}} S^n \xrightarrow{\mathfrak{K}^{[d]}} S,$$

where $C(m,d) = \binom{m-1+d}{d}$, $\mathfrak{K}^{[d]}$ stands for the map given by the row-matrix $(x_{11}^d \dots x_{n1}^d)$ and $\psi^{[d]}$ is given by the matrix

$$\left(\left|\begin{matrix} -\mathbb{M}(d,\mathbf{x}_2) & -\mathbb{M}(d,\mathbf{x}_3) & \dots & -\mathbb{M}(d,\mathbf{x}_n) \\ \mathbb{M}(d,\mathbf{x}_1) & 0 & \dots & 0 \\ 0 & \mathbb{M}(d,\mathbf{x}_1) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \mathbb{M}(d,\mathbf{x}_1) \end{matrix}\right| \left|\begin{matrix} 0 & 0 & \dots & 0 \\ -\mathbb{M}(d,\mathbf{x}_3) & -\mathbb{M}(d,\mathbf{x}_4) & \dots & -\mathbb{M}(d,\mathbf{x}_n) \\ \mathbb{M}(d,\mathbf{x}_2) & 0 & \dots & 0 \\ 0 & \mathbb{M}(d,\mathbf{x}_2) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & \mathbb{M}(d,\mathbf{x}_2) \end{matrix}\right| \begin{matrix} \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \end{matrix} \left|\begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ -\mathbb{M}(d,\mathbf{x}_n) \\ \mathbb{M}(d,\mathbf{x}_{n-1}) \end{matrix}\right|\right).$$

Proof. The containment $\operatorname{Im} \psi^{[d]} \subset \ker \mathfrak{K}^{[d]}$ is a straightforward consequence of Lemma (2.1.1), (ii).

For the reverse inclusion, by Lemma (2.1.3) and by an obvious symmetrical argument we may assume that we are given a relation of the form

$$X_{11}^d M + X_{21}^d N \equiv 0 \pmod{I_2(X_{ij})}, \tag{2-1}$$

where M and N are terms in $k[\mathbf{X}]$.

The crucial point is to establish that $\deg_{\mathbf{X}_2} M \geq d$. At any rate, one has $\deg_{\mathbf{X}_2} M \geq 1$, otherwise by setting to zero all the variables in $\mathbf{X}_2$, it would follow that $X_{11}^d M \in I_2(X_{ij})(i \neq 2)$ which is absurd.

We proceed by induction on d, the assertion for $d = 1$ having just been shown. Thus, let $d \geq 2$ and assume that $\deg_{\mathbf{X}_2} M = d_0 < d$. By the preceding, $d_0 \geq 1$, hence $d - d_0 \leq d - 1$. Write $M = \widetilde{M} M_1$, with $\widetilde{M} \in \mathbb{M}(d_0, \mathbf{X}_2)$. By Lemma (2.1.1), (ii), one has $X_{11}^{d_0} \widetilde{M} \equiv X_{21}^{d_0} \Phi_{12}(\widetilde{M})$, hence (2-1) yields

$$-X_{21}^d N \equiv X_{11}^{d_0} . X_{11}^{d-d_0} \widetilde{M} M_1 \equiv X_{21}^{d_0} . X_{11}^{d-d_0} \Phi_{12}(\widetilde{M}) M_1,$$

from which it follows that $X_{11}^{d-d_0} \Phi_{12}(\widetilde{M}) M_1 + X_{21}^{d-d_0} N \equiv 0$. Then, by the inductive hypothesis we know that $\deg_{\mathbf{X}_2} \Phi_{12}(\widetilde{M}) M_1 \geq d - d_0 > 0$. But since $\Phi_{12}(\widetilde{M}) \in \mathbb{M}(d_0, \mathbf{X}_1)$, we see that $\deg_{\mathbf{X}_2} \Phi_{12}(\widetilde{M}) M_1 = \deg_{\mathbf{X}_2} M_1 = 0$, a contradiction.

Thus, we can write $M = \widetilde{M} M_1$, where $\widetilde{M} \in \mathbb{M}(d, \mathbf{X}_2)$. By Lemma (2.1.1), (ii), we have $X_{11}^d \widetilde{M} \equiv X_{21}^d \Phi_{12}(\widetilde{M})$, from which it follows that $X_{11}^d M \equiv X_{21}^d \Phi_{12}(\widetilde{M}) M_1$. Using (2-1), one then obtains $X_{21}^d (\Phi_{12}(\widetilde{M}) M_1 + N) \equiv 0$, hence $N \equiv -\Phi_{12}(\widetilde{M}) M_1$ because $I_2(X_{ij})$ is a prime ideal. Since $\widetilde{M} \in \mathbb{M}(d, \mathbf{X}_2)$, it follows that $\Phi_{12}(\widetilde{M}) \in \mathbb{M}(d, \mathbf{X}_1)$.

Altogether, one gets

$$\begin{pmatrix} M(\mathbf{x}) \\ N(\mathbf{x}) \end{pmatrix} = M_1(\mathbf{x}) \begin{pmatrix} \widetilde{M}(\mathbf{x}_2) \\ -(\widetilde{M}(\mathbf{x}_1)) \end{pmatrix} \in \operatorname{Im} \psi^{[d]},$$

as was to be shown. $\square$

Here is the main result of this section.

(2.1.5) Theorem. *Let $d \geq 1$ be an integer and let $\mathfrak{K}^{[d]} \subset S = k[X_{ij}]/I_2(X_{ij})$ as before stand for the ideal generated by the dth powers of the generators of the ideal*

$\mathfrak{K}$ *of* S*. Also let* $R^{[d]} = k[t_i s_j, t_1^d, \ldots, t_n^d] \subset k[\mathbf{t}, \mathbf{s}]$ $(1 \le i \le n,\ 1 \le j \le m)$*. Then:*

(i) *The ideal* $\mathfrak{K}^{[d]}$ *is of linear type.*

(ii) *There is a presentation*

$$R^{[d]} \simeq k[\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n, \mathbf{U}] / \sum_{1 \le i_1 < i_2 \le n} I_2(L_{i_1,i_2})$$

where

$$L_{i_1,i_2} = \begin{pmatrix} X_{i_1,1} \ldots X_{i_1,m} & U_{i_1} \cdot \mathbb{M}(d-1, \mathbf{X}_{i_2}) \\ X_{i_2,1} \ldots X_{i_2,m} & U_{i_2} \cdot \mathbb{M}(d-1, \mathbf{X}_{i_1}) \end{pmatrix},$$

with $U_{i_l} \cdot \mathbb{M}(d-1, \mathbf{X}_{i_l})$ *designating the row whose entries are the entries of* $\mathbb{M}(d-1, \mathbf{X}_{i_l})$ *multiplied by the variable* U_{i_l}.

Proof. (i) We show that the generators $x_{11}^d, \ldots, x_{n1}^d$ of $\mathfrak{K}^{[d]}$ form a d-sequence. For that, we use the characterization of such sequences as given in [HSV, Section 6] to the effect that

$$((x_{11}^d, \ldots, x_{s1}^d) : x_{s+1\,1}^d) \cap \mathfrak{K}^{[d]} = (x_{11}^d, \ldots, x_{s1}^d) \quad \text{for } 0 \le s \le n-1.$$

By Proposition (2.1.4), one sees that

$$((x_{11}^d, \ldots, x_{s1}^d) : x_{s+1\,1}^d) = (\,\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2), \ldots, \mathbb{M}(d, \mathbf{x}_s)\,),$$

hence we are to prove that

$$(\,\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2), \ldots, \mathbb{M}(d, \mathbf{x}_s)) \cap (\,x_{11}^d, \ldots, x_{n1}^d\,) \subset (\,x_{11}^d, \ldots, x_{s1}^d).$$

Set $J_1 = (\,\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2), \ldots, \mathbb{M}(d, \mathbf{x}_s))$ and $J_2 = (\,x_{11}^d, \ldots, x_{n1}^d\,)$.

To compute the above intersection of monomial ideals modulo the binomial ideal $I_2(X_{ij})$ we follow the prescription given in [EiSt, Proof of Corollary 1.6]: choose a monomial order on the polynomial ring $A = k[X_{ij}]$ and take the standard monomials $\bmod I_2(X_{ij})$; then, $\mathcal{M}(\mathbf{x}) \subset A/I_2(X_{ij})$, the set of residues of the standard monomials, is a vector space basis of $A/I_2(X_{ij})$; next, one takes a vector space basis $\mathcal{J}_1$ (resp $\mathcal{J}_2$) of J_1 (resp. J_2) $\bmod I_2(X_{ij})$ which is contained in $\mathcal{M}(\mathbf{x})$; at the outset, $\mathcal{J}_1 \cap \mathcal{J}_2$ is a vector space basis of the ideal $J_1 \cap J_2$.

Now, in the present case, choosing a suitable order, the 2×2 minors already form a Gröbner basis of the ideal $I_2(X_{ij})$ (cf., e.g., [Stu]). Therefore, a monomial in $\mathcal{M}(\mathbf{x})$ is characterized by the property that it involves indeterminates belonging to one and only one row or to one and only one column of the matrix (x_{ij}). It follows from this that

$$\mathcal{J}_1 = \left(\bigcup_{\substack{1 \le i \le s \\ r \ge d}} \mathbb{M}(r, \mathbf{x}_i) \right) \cup \ \{x_{ij}^d M(\mathbf{x}^j) \,|\, 1 \le i \le s, 1 \le j \le m\}$$

is a vector basis of J_1, where $M(\mathbf{x}^j)$ designates a monomial involving only variables along the jth column.

By a similar token,

$$\mathcal{J}_2 = \{\, x_{i1}^d M(\mathbf{x}^j),\, x_{i1}^d M(\mathbf{x}_i) \,|\, 1 \le i \le n\,\}$$

is a vector basis of J_2, where $M(\mathbf{x}_i)$ designates a monomial involving only variables along the ith row. One clearly has $\mathcal{J}_1 \cap \mathcal{J}_2 = \{x_{i1}^d M(\mathbf{x}^j),\, x_{i1}^d M(\mathbf{x}_i) \,|\, 1 \le i \le s\,\}$. Therefore, the *ideal* $J_1 \cap J_2$ is generated by $\{x_{i1}^d, \,|\, 1 \le i \le s\}$, as was to be shown.

(ii) By part (i), the canonical surjection $\mathcal{S}(\mathfrak{K}^{[d]}) \to \mathcal{R}(\mathfrak{K}^{[d]})$ is an isomorphism, where $\mathcal{S}(\mathfrak{K}^{[d]})$ and $\mathcal{R}(\mathfrak{K}^{[d]})$ denote the symmetric and the Rees algebra of the ideal $\mathfrak{K}^{[d]}$, respectively. On the other hand, by Proposition (2.1.4), $\mathcal{S}(\mathfrak{K}^{[d]})$ admits the presentation that is being proposed for $R^{[d]}$. Therefore, it suffices to show that $R^{[d]}$ is isomorphic to $\mathcal{R}(\mathfrak{K}^{[d]})$. Clearly,

$$\mathcal{R}(\mathfrak{K}^{[d]}) \simeq S[\mathfrak{K}^{[d]}T] \simeq k[t_is_j,\, (t_1s_1)^dT, \dots, (t_ns_1)^dT] \subset k[\mathbf{t}, \mathbf{s}][T].$$

Since s_1^d is a common factor throughout the terms $t_i^d s_1^d T$ and these have a fixed degree, we see that there is an isomorphism $k[t_is_j, t_1^d, \dots, t_n^d] \simeq k[t_is_j,\, (t_1s_1)^dT, \dots, (t_ns_1)^dT]$. □

2.2. Hilbert function data of $R^{[d]}$. The reader is referred to [HUT] and [STV] for the background needed in this portion. Again, one considers the Segre ring $S = k[\mathbf{X}]/I_2(\mathbf{X})$, which will be thought of as the current base ring. By Theorem (2.1.5), $R^{[d]}$ is isomorphic to the Rees algebra of the ideal $\mathfrak{K}^{[d]} \subset S$ and, moreover, as such, it has a natural structure of standard bigraded k-algebra, its presentation ideal over S being bihomogeneous with respect to the two sets of variables $\mathbf{X} = \{X_{ij}\}$ and $\mathbf{U} = \{U_1, \dots, U_n\}$.

Consider an $\mathbb{N}^{n+1}$-gradation on $S[\mathbf{U}]$ by setting

$$S[\mathbf{U}]_{(a_0,a_1,\dots,a_n)} := S_{a_0} U_1^{a_1} \cdots U_n^{a_n}.$$

Let $\succeq$ be the graded lexicographic order on the monoid $\mathbb{N}^{n+1}$. It induces a filtration $\mathcal{F}$ on $S[\mathbf{U}]$, with $\mathcal{F}_{\mathbf{a}} := \oplus_{\mathbf{b}\succeq\mathbf{a}} S[\mathbf{U}]_{\mathbf{b}}$, hence also on the residue ring $R^{[d]} \simeq S[\mathbf{U}]/\mathcal{J}$ which we still denote by $\mathcal{F}$. Letting $\mathcal{J}^*$ denote the ideal generated by the initial forms of $\mathcal{J}$, one has $\mathrm{gr}_{\mathcal{F}}((R^{[d]})) \simeq S[\mathbf{U}]/\mathcal{J}^*$ as bigraded k-algebras.

By Proposition (2.1.4) (or by the proof of Theorem (2.1.5), (i)) and [HUT, Lemma 1.1], one obtains

$$\mathcal{J}^* = (\mathbb{M}(d, \mathbf{x}_1)\, U_2,\, (\mathbb{M}(d, \mathbf{x}_1), \mathbb{M}(d, \mathbf{x}_2))\, U_3,\, \dots,\, (\mathbb{M}(d, \mathbf{x}_1), \dots, \mathbb{M}(d, \mathbf{x}_{n-1}))\, U_n)\,.$$

(2.2.1) Proposition. *With the preceding notation and considering $R^{[d]}$ and $\mathrm{gr}_{\mathcal{F}}((R^{[d]}))$ as $\mathbb{N}$-graded rings (via the homomorphism $\mathbb{N}^2 \to \mathbb{N}, (a,b) \mapsto a+b$), one has:*

(i) *$R^{[d]}$ and $\mathrm{gr}_{\mathcal{F}}((R^{[d]}))$ admit the same Hilbert function.*

(ii) *The multiplicity of $R^{[d]}$ is*

$$e(R^{[d]}) = \sum_{j=0}^{n-1} d^j \binom{m+n-j-2}{n-j-1}.$$

Proof. (i) This is easy and holds quite generally.

(ii) We apply [HUT, Theorem 1.4] (or rather, its recipe), for which we first check its hypotheses. In the present situation, they boil down to the equalities

$$\dim S/I_j = \dim S - j, \quad 1 \le j \le n-1,$$

where $I_j = (\mathbb{M}(d, \mathbf{x}_1), \dots, \mathbb{M}(d, \mathbf{x}_j))$. To verify these, we show that $\mathrm{ht}\, I_j = j$ for $1 \le j \le n-1$ (recalling that S is Cohen–Macaulay). For every such j, consider the prime ideal

$$\begin{aligned} P_j &= (\{X_{kl} \,|\, 1 \le k \le j, 1 \le l \le m\}) \,+\, I_2(\{X_{k'l} \,|\, j+1 \le k' \le n, 1 \le l \le m\}) \\ &= (\{\mathbf{X}_1, \dots, \mathbf{X}_j\}) \,+\, I_2(\mathbf{X} \setminus \{\mathbf{X}_1, \dots, \mathbf{X}_j\}) \subset k[\mathbf{X}]. \end{aligned}$$

Clearly, P_jS is a prime as well and contains I_j. It follows that

$$\begin{aligned}\operatorname{ht} I_j &\leq \operatorname{ht} P_jS = \operatorname{ht} P_j - \operatorname{ht} I_2(\mathbf{X})\\ &= \operatorname{ht}(\{\mathbf{X}_1,\dots,\mathbf{X}_j\}) + \operatorname{ht} I_2(\mathbf{X}\setminus\{\mathbf{X}_1,\dots,\mathbf{X}_j\}) - (n-1)(m-1)\\ &= jm + (n-j-1)(m-1) - (n-1)(m-1) = j\end{aligned}$$

On the other hand, it is easy to see that every prime ideal of S containing I_j already contains P_jS. This leads to $\operatorname{ht} I_j = j$, as required.

We now compute the multiplicity $e(S/I_j)$ by the associativity formula. By the above calculation, this formula reduces to

$$e(S/I_j) = \ell(S_{P_jS}/I_{j\,P_jS})e(S/P_jS).$$

To simplify the notation, set $P = P_j$, $I = I_j$. Observe that the ideal PS_{PS}/I_{PS} is generated by the images of the variables $X_{11},\dots,X_{j1}$. Indeed, typically, $X_{k1}X_{nl} - X_{kl}X_{n1} \equiv 0 \pmod{I_2(\mathbf{X})}$. Since X_{n1} is invertible, the image of X_{kl} belongs to the ideal generated by the image of X_{k1}, for $1 \leq k \leq j$. The above length is then given by the number of monomials $\{X_{11}^{a_1}\cdots X_{j1}^{a_j} \mid 0 \leq a_k \leq d-1,\ 1 \leq k \leq j\}$. This number is clearly d^j.

Next, one has $S/P_jS = k[\mathbf{X}]/P \simeq k[\mathbf{X}\setminus\{\mathbf{X}_1,\dots,\mathbf{X}_j\}]/I_2(\mathbf{X}\setminus\{\mathbf{X}_1,\dots,\mathbf{X}_j\})$, which is a Segre ring of size $(n-j)\times m$. Therefore, $e(S/P) = \binom{m+n-j-2}{n-j-1}$ by a well-known formula (cf., e.g., [STV, Remark 2.5]).

To piece everything together, [HUT, Theorem 1.4] tells us that $e(R^{[d]}) = \sum_{j=0}^{n-1} e(S/I_j)$, hence we are through. $\square$

(2.2.2) *Remark.* By Proposition (2.2.1), (i), it is in principle possible to compute the Hilbert function of $R^{[d]}$, but it is hardly the case that it may be of any uselfuness here. Thus, for example, $\dim R^{[d]} = m+n$ follows directly from the fact that $R^{[d]}$ is a Rees algebra of an ideal in the $m+n-1$-dimensional domain S.

3. The defining equations of the special algebra

As above, let $I = I^{[d]} \subset k[\mathbf{X},\mathbf{U}]$ denote the presentation ideal of the k-algebra $R^{[d]}$ and let $\tilde{I} = IS[\mathbf{U}] \subset S[\mathbf{U}]$, an ideal generated in bidegree $(d,1)$. We consider the Rees algebra $\mathcal{R}_{S[\mathbf{U}]}(\tilde{I})$: geometrically, one is looking at the blowup of the product $\mathfrak{S}\times\mathbb{P}^{n-1}_{\mathbf{U}}$ along the subvariety $\mathcal{B}\ell_{\mathcal{K}}(\mathfrak{S})$, where $\mathcal{K}$ denotes the subvariety of $\mathfrak{S}$ defined by the ideal $\mathfrak{K}^{[d]}$.

The *special algebra* (or *fiber cone algebra*) of an ideal (resp. homogeneous ideal) $\mathfrak{a}$ in a local (resp. positively graded) ring A is the residue ring $\mathcal{F}(\mathfrak{a}) := \mathcal{R}_A(\mathfrak{a})/\mathfrak{m}\mathcal{R}_A(\mathfrak{a})$, with $\mathfrak{m}$ standing for the maximal (resp. maximal graded) ideal of A.

We will take $A = S[\mathbf{U}]$ and $\mathfrak{a} = \tilde{I}$. As it will turn out, $\mathcal{F}(\tilde{I})$ is a nice determinantal locus which, in the case where $n = 2$, is the coordinate ring of a Veronese variety. The reason for that is a far more reaching principle which may have an independent interest outside the scope of the present work.

(3.1) Theorem. *Let $\mathbf{X},\mathbf{Y}$ be mutually independent sets of variables over a field k of characteristic zero, with $\mathbf{X}$ and $\mathbf{Y}$ having the same number of elements, and let $f_1,\dots,f_r$ be homogeneous polynomials in the $\mathbf{X}$-variables, of the same degree. Let U,V be two additional variables and set $A = k[f_1V-\Phi(f_1)U,\dots,f_rV-\Phi(f_r)U] \subset$*

$k[\mathbf{X},\mathbf{Y},U,V]$, *where* Φ *as in Lemma (2.2.1) denotes the involutive* k*-isomorphism* $X_i \mapsto Y_i$. *Then*

$$k[f_1,\ldots,f_r] \simeq A/A \cap I_2(\mathbf{X},\mathbf{Y})\, k[\mathbf{X},\mathbf{Y},U,V]$$

as graded k*-algebras, where* $I_2(\mathbf{X},\mathbf{Y})$ *denotes the ideal of* $k[\mathbf{X},\mathbf{Y}]$ *generated by the* 2×2 *minors of the generic matrix whose rows are* $\mathbf{X}$ *and* $\mathbf{Y}$.

Proof. Let $T_1,\ldots,T_r$ be presentation variables over k for both algebras. It will suffice to show that they have the same presentation ideal. We show, namely, that any homogeneous polynomial relation of one of the two algebras is a polynomial relation of the other. We need the notion of *polarization*.

Consider a polynomial ring $k[\mathbf{T},\mathbf{U}]$ in two sets of indeterminates $\mathbf{T} = T_1,\ldots,T_r$ and $\mathbf{U} = U_1,\ldots,U_r$. Clearly, $k[\mathbf{T},\mathbf{U}]$ is a free $k[\mathbf{U}]$-module with basis the monomials in $\mathbf{T}$.

(3.2) Definition. The *polarization* of $\mathbf{T}$ by $\mathbf{U}$ is the (unique) $k[\mathbf{U}]$-homomorphism P of the $k[\mathbf{U}]$-module $k[\mathbf{T},\mathbf{U}]$ such that $P(1)=0$ and

$$P(\mathbf{T}^a) = \sum_{a_j\neq 0} a_j U_j T_1^{a_1}\cdots T_j^{a_j-1}\cdots T_r^{a_r}$$

for $\mathbf{T}^a = T_1^{a_1}\cdots T_r^{a_r}$.

One sets $P_0(\mathbf{T}^a) = \mathbf{T}^a$ and $P_l(\mathbf{T}^a) = P_{l-1}(P(\mathbf{T}^a))$. Next, consider the k-algebra homomorphism $\Psi' : k[\mathbf{T},\mathbf{U}] \to k[f_1,\ldots,f_r,\Phi(f_1),\ldots,\Phi(f_r)]$ such that $\Psi'(T_j) = f_j, \Psi'(U_j) = \Phi(f_j)$, and let Ψ denote the restriction of Ψ' to $k[\mathbf{T}]$.

Let $F(\mathbf{T}) = \sum_a \alpha_a \mathbf{T}^a \in k[\mathbf{T}]$ be a homogeneous polynomial of degree t, with $a = (a_1,\ldots,a_r)$, $|a| = t$ and $\mathbf{T}^a = T_1^{a_1}\cdots T_r^{a_r}$, and let s denote the common degree of the f's. We claim that $X_1^s\Psi'(P(F(\mathbf{T}))) \equiv t\,Y_1^s\Psi(F(\mathbf{T})) \pmod{I_2(\mathbf{X},\mathbf{Y})}$. Indeed, it follows from Lemma (2.2.1) that, for a given term $\alpha_a\mathbf{T}^a$ of $F(\mathbf{T})$ ($\alpha_a \neq 0$), one has

$$\Psi'(P(\mathbf{T}^a)) \equiv (a_{i(a)}+\ldots+a_r)\Phi(f_{i(a)})f_{i(a)}^{a_{i(a)}-1}f_{i(a)+1}^{a_{i(a)+1}}\cdots f_r^{a_r},$$

where $a_{i(a)}\neq 0, a_i = 0$ $(i<i(a))$ $\pmod{I_2(\mathbf{X},\mathbf{Y})}$. By summing up over all terms of $F(\mathbf{T})$, one obtains

$$\Psi(P(F(\mathbf{T}))) \equiv t\sum_a \alpha_a\Phi(f_{i(a)})f_{i(a)}^{a_{i(a)}-1}f_{i(a)+1}^{a_{i(a)+1}}\cdots f_r^{a_r} \pmod{I_2(\mathbf{X},\mathbf{Y})}.$$

Again by Lemma (2.2.1), one has $X_1^s\Phi(f_j) = Y_1^s f_j$. Substituting yields the desired result.

Next, by iterating the polarization, one easily gets

$$X_1^{ls}\Psi'(P_l(F(\mathbf{T}))) \equiv \frac{t!}{(t-l)!}\,Y_1^{ls}\Psi(F(\mathbf{T})) \pmod{I_2(\mathbf{X},\mathbf{Y})} \tag{3-1}$$

where $P_l(F(\mathbf{T})) = 0$ if $l>t$.

On the other hand, a computation yields

$$F(f_1V-\Phi(f_1)U,\ldots,f_rV-\Phi(f_r)U) = \sum_{l=0}^{t}(-1)^l\Psi'(P_l(F(\mathbf{T})))\,V^{t-l}U^l.$$

Using (3-1) with $l=t$, one gets

$$X_1^{ls}F(f_1V-\Phi(f_1)U,\ldots,f_rV-\Phi(f_r)U) \equiv g\,\Psi(F(\mathbf{T})) \pmod{I_2(\mathbf{X},\mathbf{Y})},$$

where

$$g = \sum_{l=0}^{t} (-1)^l \frac{t!}{(t-l)!} X_1^{(t-l)s} Y_1^{ls} V^{t-l} U^l \notin I_2(\mathbf{X}, \mathbf{Y})[\mathbf{X}, \mathbf{Y}, U, V].$$

One concludes that $F(f_1 V - \Phi(f_1)U, \dots, f_r V - \Phi(f_r)U) \in I_2(\mathbf{X}, \mathbf{Y})[\mathbf{X}, \mathbf{Y}, U, V]$ if and only if $\Psi(F(\mathbf{T})) \in I_2(\mathbf{X}, \mathbf{Y})[\mathbf{X}, \mathbf{Y}, U, V] \cap k[\mathbf{X}] = (0)$.

This finishes the proof. ☐

(3.3) Corollary. *Notation as in the beginning of the section. Moreover, let $n = 2$. Then $\mathcal{F}(\tilde{I})$ is isomorphic to the homogeneous coordinate ring of the duple Veronese model of $\mathbb{P}^{m-1}$. In particular, $\mathcal{F}(\tilde{I})$ is normal and Cohen–Macaulay.*

Proof. By Proposition (2.2.3), $\mathcal{F}(\tilde{I}) \simeq k[M_\alpha]$, where M_α runs through the monomials of degree d in the variables $\mathbf{X}$. ☐

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