Congruence-semimodular and congruence-distributive pseudocomplemented semilattices

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Investigations into the structure of the congruence lattices of pseudo-complemented semilattices (*PCS*'s) were initiated in [10]. In this paper¹ we characterize the class of congruence-semimodular *PCS*'s (i.e. *PCS*'s with semimodular lattice of congruences) and the class of congruence-distributive *PCS*'s (i.e. with distributive congruence lattices). We give two characterizations of each class; one of these is a Dedekind-Birkhoff-type characterization which says that the exclusion in a certain sense of a single *PCS* P_6 determines the class of congruence-semimodular *PCS*'s, and the exclusion of the two *PCS*'s P_6 and P_5 (these are defined in the sequel) determines the class of congruence-distributive *PCS*'s. The other characterization shows that each of these classes is strictly elementary and gives explicitly the defining axiom for each class as a universal positive sentence (in the language of *PCS*'s). This paper is a continuation of [10] and borrows the notation and the results from it. For other information see the standard references [6] and [7].

§1. Basic definitions and lemmas

Recall that a pseudocomplemented semilattice (*PCS*) is an algebra $\langle S; \wedge, *, 0 \rangle$ where $\langle S; \wedge, 0 \rangle$ is a $\wedge$ -semilattice with zero and $*$ is the pseudocomplementation and that the class of all *PCS*'s is an equational class. Let S denote an arbitrary *PCS* and $B(S)$ and $N(S)$ denote respectively the set of closed (i.e. $a^{**} = a$) elements and that of non-closed (i.e. $a < a^{**}$) elements of S . It is well-known [5] that $B(S)$ is both a Boolean algebra and a *PCS*-subalgebra of S . $\text{Con } S$ denotes the congruence lattice of S with Δ_s and ∇_s (or simply Δ and ∇) as its least and greatest elements respectively; and the kernel of the homomorphism $** : S \rightarrow S$,

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$s \rightarrow s^{**}$, is denoted by $\Phi(S)$ (or simply Φ). For $a, b \in S$ with $a < b$ (a, b) and (a, b) denote respectively $[a, b] - \{a\}$ and $[a, b] - \{a, b\}$.

Boolean algebras are here thought to be a subvariety of PCS's and the obvious fact that for a BA B the PCS-congruences on B are precisely the BA-congruences on B will be used repeatedly in the sequel.

The following sentences in the first-order language of PCS's are useful later:

- (U) $\forall x \forall y (x < y^{**} \rightarrow (x^* = y^* \text{ or } x \leq y))$,
- (U_p) $\forall x \forall y ((x \wedge y^{**})^* = y^* \text{ or } x \wedge y^{**} \leq y)$,
- (U') $\forall x \forall y ((x = x^{**} \text{ and } x < y^{**}) \rightarrow x < y)$,
- (D) $\forall x \forall y (x < y^{**} \rightarrow (x \leq y \text{ or } y \leq x))$,
- (D_w) $\forall x \forall y (x^* = y^* \rightarrow (x \leq y \text{ or } y \leq x))$,
- (D_p) $\forall x \forall y (x \wedge y^{**} \leq y \text{ or } y \leq x \wedge y^{**})$,
- (D_{w,p}) $\forall x \forall y (x \wedge y^{**} \leq y \wedge x^{**} \text{ or } y \wedge x^{**} \leq x \wedge y^{**})$.

Note that, in the class of PCS's, (U) is equivalent to (U_p) and to (U'), (D) is equivalent to the conjunction of (D_w) and (U'), (D) is equivalent to (D_p) and (D_w) is equivalent to (D_{w,p}).

In Figure 1 the two PCS's P_6 and P_5 (mentioned earlier) are defined (their congruence lattices are also shown).

Observe that if $x \leq a^{**}$ then $x^* = (x \wedge a)^*$. Using this the reader can verify the following theorem which gives a characterization of certain principal congruences and is inspired by a result in Dean and Oehmke [3].

Figure 1a

Figure 1b

THEOREM 1.1. *Let $a, b \in S$ be such that $a \leq b$ and $a^* = b^*$. Then $\theta(a, b) = \theta$ is characterized by*

$$x \equiv y(\theta) \text{ iff either } x = y, \text{ or } x \leq b, y \leq b \text{ and } x \wedge a = y \wedge a.$$

Moreover, $x \equiv y(\theta)$ implies $x^ = y^*$.*

We say that an equivalence relation R on S is a *semilattice-congruence* on S iff R is a congruence on S when S is regarded as a semilattice, ignoring the operation $*$. If $a, b \in S$, we denote by $\theta_{\text{semilat}}(a, b)$ the principal semilattice-congruence on S generated by $\langle a, b \rangle$. For $a, b \in S$ with $a < b$ and $a^* = b^*$, from Theorem 1.1 it follows that $\theta_{\text{semilat}}(a, b)$ is indeed a PCS-congruence on S .

For $c \in B(S)$ we define $D_c(S) = \{x \in S : x^{**} = c\}$ and refer to the elements of $D_c(S)$ as *c-dense*; in particular we write $D(S)$ for $D_1(S)$ whose elements are called simply *dense* elements. It is easy to check that $D_c(S)$, in particular $D(S)$, is a subsemilattice of S (as a semilattice).

LEMMA 1.2. *Let $b \in B(S)$ and $A \subseteq D_b(S)$. Suppose $\langle x, y \rangle \in \theta(A \times A)$ with $x \neq y$. Then there exist h and h' in A such that $x \leq h$ and $y \leq h'$.*

Proof. If a_0 is an arbitrary fixed element of A , we can write $\theta(A \times A) = \bigvee_{a \in A} \theta(a_0, a)$. Then $\langle x, y \rangle \in \theta(A \times A)$ implies that there exist $x = x_0, x_1, \dots, x_n = y$ in S and $a_1, \dots, a_n$ in A such that $\langle x, x_1 \rangle \in \theta(a_0, a_1)$, $\langle x_1, x_2 \rangle \in \theta(a_0, a_2)$, $\dots$, $\langle x_{n-1}, y \rangle \in \theta(a_0, a_n)$. We may assume $x \neq x_1$ and $x_{n-1} \neq y$. Since $\theta(a_0, a_1) = \theta(a_0, a_0 \wedge a_1) \vee \theta(a_0 \wedge a_1, a_1)$, and $a_0^* = a_1^*$ by hypothesis, we can see by Theorem 1.1 (using also the fact that $a_0^* = (a_0 \wedge a_1)^* = a_1^*$) that $x \leq a_0$ or $x \leq a_1$; similarly one sees that $y \leq a_0$ or $y \leq a_n$. From this the lemma follows.

LEMMA 1.3. *Let $c \in B(S)$. Then*

$$\theta(D_c(S) \times D_c(S)) = \bigvee_{a \in D_c(S)} \theta(c, a) = \bigcup_{a \in D_c(S)} \theta(c, a)$$

Proof. It is clear that $\theta(D_c(S) \times D_c(S)) = \bigvee_{a \in D_c(S)} \theta(c, a)$. Since $\{\theta(c, a) : a \in D_c(S)\}$ is directed upwards (a common upper bound for $\theta(c, a_1)$ and $\theta(c, a_2)$ is $\theta(c, a_1 \wedge a_2)$ if $a_1, a_2 \in D_c(S)$), the equality of the join and the union is immediate.

§2. Atoms in congruence lattices

To achieve our characterizations we need to examine first the atoms in the congruence lattices.

EXAMPLE 2.1.

Figure 2

Recall from [10] that $\langle x, y \rangle \in \hat{e}$ iff $x \wedge e = y \wedge e$. Observe that, in the preceding example, $\theta(h, g)$ is an atom in $\text{Con } S$ while $\hat{f}$ is not an atom in $\text{Con } S'$. Observe also that $\hat{e}$ is an atom in $\text{Con } S$ and has an additional property, namely $\langle x, y \rangle \in \hat{e}$ and $x \neq y$ imply $x \leq e$ or $y \leq e$, whereas $\hat{f}$ fails to have this property. These considerations lead us to the following theorem. In the sequel we use $\prec$ and $\succ$ for the covering relation and its dual respectively.

THEOREM 2.2. *Let $\alpha \in \text{Con } S$. Then α is an atom in $\text{Con } S$ iff α is one or the other of the following two types:*

- (T1) $\alpha = \{\langle a, b \rangle, \langle b, a \rangle\} \cup \Delta$ for some $a, b \in S$ such that $a^* = b^*$ and $a \prec b$, (i.e. $a \prec b$ and $\forall x (x < b \rightarrow x \leq a)$),
- (T2) $\alpha = \hat{e}$ for some $e \in B(S)$ with $e \prec 1$ and α has the following additional property:
 (*) If $\langle x, y \rangle \in \alpha$ and $x \neq y$ then $x \leq e$ or $y \leq e$.

Proof. First note that if a relation is of type (T1) or of type (T2) then it is indeed an atom in $\text{Con } S$.

To prove the reverse, suppose α is an atom in $\text{Con } S$. Then α is $\vee$ -irreducible. From [10, Theorem 2.6] we have $\alpha = ([1]_\alpha)^\wedge \vee (\alpha \vee \Phi)$ and hence it follows that $\alpha = ([1]_\alpha)^\wedge$ or $\alpha \subseteq \Phi$. We show that if the former holds then α is an atom of type (T2), whereas the latter implies α is an atom of type (T1).

CASE 1. Let $\alpha = ([1]_\alpha)^\wedge$; then from $\alpha \neq \Delta$ we easily see that there exists an element $e \in [1]_\alpha \cap B(S)$ with $e \neq 1$. Since $\Delta \subset \hat{e} \subseteq \alpha$ and α is an atom, we get $\alpha = \hat{e}$. Clearly $e \prec 1$ in S and $\alpha \cap \Phi = \Delta$. We now show that α has the property (*). Let $\langle x, y \rangle \in \alpha$ and $x \neq y$. Thus $x^* \neq y^*$ as $\alpha \cap \Phi = \Delta$, and $\langle x^*, y^* \rangle \in (\hat{e})_B$. Then it can be easily seen by an argument for Boolean algebras that $x^{**} \leq e$ or $y^{**} \leq e$ from which it follows that α is an atom of type (T2).

CASE 2. Let $\alpha \subseteq \Phi$. Now $\alpha \neq \Delta$ implies that there exist a, b in S such that $\langle a, b \rangle \in \alpha$ and $\alpha \neq b$. Then $a^* = b^*$; and since α is an atom, it follows that $\theta(a, b) = \alpha$. Now $\langle a, a \wedge b \rangle \in \alpha$ and since $a \neq b$, we may assume that $a \neq a \wedge b$ and hence $\theta(a, a \wedge b) = \alpha$ which implies that $\langle a, b \rangle \in \theta(a, a \wedge b)$. Then from Theorem 1.1 we see that $b < a$. If $b \not\prec a$ then there exists an element u in S such that $u < a$ and $u \neq b$, so $\langle u \wedge b, u \rangle \in \theta(a, b)$ and hence $\theta(u \wedge b, u) \subseteq \theta(a, b)$. However $\theta(a, b) \not\subseteq \theta(u \wedge b, u)$. Thus $b \prec a$ and so α is an atom of type (T1).

Remark 2.3. Let $\alpha = \hat{e}$ be an atom of type (T2) and let $\langle x, y \rangle \in \alpha$ with $x \neq y$. Then (i) $x^* \neq y^*$, (ii) $x \prec y$ or $y \prec x$, and (iii) $x = x^{**} \wedge y$ or $y = y^{**} \wedge x$.

§3. Semimodular congruence lattices

In this section we shall be concerned with the characterization of the PCS's whose congruence lattices are semimodular. It is easily verified that all PCS's with at most 5 elements have semimodular congruence lattices, while $\text{Con } P_6$ is not semimodular. The special role played by P_6 in this section is implicit in the following lemma.

LEMMA 3.1. *Suppose there are two incomparable elements b and d in S such that $b \in B(S)$ and d is dense in S . Then there exists a quotient algebra S' of S with the following properties:*

- (1) $[b] \in B(S')$ and $[d]$ is dense in S' ,
- (2) $[b]$ and $[d]$ are incomparable,
- (3) $[b] \prec [1]$, $[b \wedge d] \prec [b]$, $[d] \prec [1]$ and $[b \wedge d] \prec [d]$,
- (4) $N(S') = \{[b \wedge d], [d]\}$.

Proof. First we note that property (1) holds in any quotient algebra of S . We construct a sequence of PCS's $S = S_0, S_1, S_2, \dots, S_5$ such that $S_i = S_{i-1}/\psi_i$ where ψ_i is a suitably chosen congruence on S_{i-1} , $i = 1, \dots, 5$, and S' will be the quotient S_5 . For $s \in S$ we define $[s]_1 = [s]\psi_1$ and $[s]_i = [s_{i-1}]\psi_i$ and we write $[1]$ for $[1]_i$, $i = 1, \dots, 5$. The congruences ψ_i will be so chosen that:

- (i) $[b]_i$ and $[d]_i$ are incomparable, for $i = 1, \dots, 5$,
- (ii) $[b]_i \prec [1]$ in S_i , $i \geq 2$,
- (iii) $[b \wedge d]_i \prec [b]_i$ in S_i , $i \geq 3$,
- (iv) $[b \wedge d]_i \prec [d]_i \prec [1]$ in S_i , $i \geq 4$, and
- (v) $N(S_5) = \{[b \wedge d]_5, [d]_5\}$.

STEP 1. Let $[b, 1]_B = [b, 1] \cap B(S)$. Then $[b, 1]_B$ is a Boolean lattice. Let U be an ultrafilter in $[b, 1]_B$. Define $\psi_1 = (\langle U \rangle)^\wedge$ where $\langle U \rangle$ is the filter in S generated by

U and let $S_1 = S/\psi_1$. Then it is clear that $[[b]_1, [1]]_B = \{[b]_1, [1]\}$ and hence if $[x]_1 > [b]_1$ then $[x]_1$ is dense in S_1 . We claim that $[d]_1 \neq [1]$, for, if $\langle d, 1 \rangle \in (\langle U \rangle)^\wedge$ then $d \wedge f = f$ for some $f \in \langle U \rangle$. Then $f \geq u$ for some $u \in U$ and hence $d \wedge u = u$, implying $d \geq u \geq b$ which is a contradiction. Thus $[d]_1 \neq [1]$, and similarly $[b \wedge d]_1 \neq [b]_1$. If $\langle b \wedge d, d \rangle \in \psi_1$ then $\langle (b \wedge d)^{**}, d^{**} \rangle \in \psi_1$, i.e. $\langle b, 1 \rangle \in \psi_1$, so $[b]_1 = [1]$ which is impossible – thus $[b]_1, [d]_1, [b \wedge d]_1$ and $[1]$ are distinct from each other.

STEP 2. Consider the set $E = ([b]_1, [1])$ in S_1 ; from step 1 every element of E is dense in S_1 , so E is a filter in S_1 . Define $\psi_2 = \hat{E}$ and $S_2 = S_1/\psi_2$. If $\langle [x]_1, [1] \rangle \in \psi_2$ then $[x]_1 \wedge [e]_1 = [e]_1$ for some $[e]_1 \in E$ and hence $[e]_1 \leq [x]_1$ which implies that $[x]_1 \in E$. From this it is immediate that $[b]_2 \neq [1]$ and $[d]_2 \neq [1]$. Thus we get $[b]_2 - [1]$ and therefore the closed element $[b]_2$ and the dense element $[d]_2$ are incomparable giving $[b]_2 \neq [b \wedge d]_2$ and $[d]_2 \neq [b \wedge d]_2$.

STEP 3. Let $A = D_{[b]_2}(S_2) - \{[b]_2\}$ where $D_{[b]_2}(S_2) = \{[x]_2 : [x]_2^{**} = [b]_2\}$, and note that $[b \wedge d]_2 \in A$. Consider $\psi_3 = \theta(A \times A)$ and define $S_3 = S_2/\psi_3$. From Lemma 1.2 it follows that if $\langle [x]_2, [y]_2 \rangle \in \psi_3$ and $[x]_2 \neq [y]_2$ then $[x]_2 < [b]_2$ and $[y]_2 < [b]_2$ which leads us to conclude that $[b \wedge d]_3 \neq [b]_3, [b \wedge d]_3 \neq [d]_3, [b]_3 \neq [1]$ and $[d]_3 \neq [1]$ – thus showing that $[b]_3$ and $[d]_3$ are incomparable. We also have that $[b \wedge d]_3 \rightarrow [b]_3$ and $D_{[b]_3}(S_3) = \{[b \wedge d]_3, [b]_3\}$ which implies that if $[x]_3$ is dense in S_3 then $[x]_3 > [b \wedge d]_3$ since $[x \wedge b \wedge d]_3^{**} = [b]_3^{**}$.

STEP 4. Let $M = ([b \wedge d]_3, [1]) - \{[b]_3\}$ and let $S_4 = S_3/\psi_4$ where $\psi_4 = \theta(M \times M)$. We note that if $[m]_3 \in M$ then $([m]_3)^{**} \geq ([b \wedge d]_3)^{**} = ([b]_3)^{**}$ and hence $[m]_3$ is dense in S_3 since $[M]_3 > [b \wedge d]_3$ – thus $M = D(S_3) - \{[1]\}$. Then, since $\psi_4 \subseteq \Phi(S_3)$, we have $[b]_4 \neq [1]$; from Lemma 1.2 it follows that $[d]_4 \neq [1]$ and hence $[b]_4$ and $[d]_4$ are incomparable. In S_4 we then have $D(S_4) = \{[d]_4, [1]\}$ and $[b \wedge d]_4 \rightarrow [d]_4 \rightarrow [1]$.

STEP 5. Letting $K = B(S_4) - \{[b]_4, [1]\}$, define $S_5 = S_4/\psi_5$ where $\psi_5 = \bigvee_{[k]_4 \in K} \theta(D_{[k]_4}(S_4) \times D_{[k]_4}(S_4))$. By Lemma 1.3 we get $\psi_5 = \bigcup_{\substack{[a]_4 \in D_{[c]_4} \\ [k]_4 \in K}} \theta([a]_4, [k]_4)$. Then $[b]_5 \neq [1]$ and $[d]_5 \neq [1]$, hence $[b]_5$ and $[d]_5$ are incomparable and $N(S_5) = \{[b \wedge d]_5, [d]_5\}$. Of course, we have $[b \wedge d]_5 \rightarrow [b]_5 \rightarrow [1]$ and $[b \wedge d]_5 \rightarrow [d]_5 \rightarrow [1]$. Thus $S' = S_5$ is a quotient of S having all the desired properties (1)–(4).

THEOREM 3.2. *Suppose there are two incomparable elements b and d in S such that $b \in B(S)$ and d is dense in S . Then there exists in $\text{Con } S$ an interval $[\alpha, \beta]$ isomorphic with N_5 .*

Proof. Using Lemma 3.1 we obtain a quotient algebra S' of S with the properties (1)–(4) of the lemma. Let us now define four congruences θ_i , $i = 1, \dots, 4$ on S' as $\theta_1 = ([b])^\wedge$, $\theta_2 = \theta([b \wedge d], [b])$, $\theta_3 = ([d])^\wedge$ and $\theta_4 = ([b \wedge d])^\wedge$. By Theorem 2.2 θ_1 and θ_2 are atoms; it is easily seen then that the congruences $\Delta_{S'}$, θ_1 , θ_2 , θ_3 , θ_4 form the interval $[\Delta_{S'}, \theta_4]$ in $\text{Con } S'$ isomorphic with N_5 . If α and β are congruences on S corresponding to $\Delta_{S'}$ and θ_4 , then it follows that $[\alpha, \beta] \cong N_5$ – proving the theorem.

We are now ready to give a necessary condition on S in order for $\text{Con } S$ to be upper-semimodular.

Recall that a lattice L is upper-semimodular iff $a \succ a \wedge b$, $b \succ a \wedge b$ and $a \neq b$ implies $a \vee b \succ a$ and $a \vee b \succ b$, $a, b \in L$. (Lower-semimodularity is defined dually.)

THEOREM 3.3. *If $\text{Con } S$ is upper-semimodular then S satisfies:*

$$(U) \quad \forall x \forall y (x < y^{**} \rightarrow (x^* = y^* \text{ or } x \leq y)).$$

Proof. Assume that (U) fails in S . Then there are x and y in S such that $x < y^{**}$, $x^* \neq y^*$ and $x \not\leq y$. The first two of these relations imply $x^{**} < y^{**}$ and hence $y \not\leq x^{**}$, while it follows from the last that $x^{**} \not\leq y$ – thus we see that x^{**} and y are incomparable, hence that $y < y^{**}$. Consider the quotient $T = S/(y^{**})^\wedge$. Clearly $[y]$ is dense in T and $[x^{**}]$ is closed in T . If $[y] \leq [x^{**}]$ then $\langle y, y \wedge x^{**} \rangle \in (y^{**})^\wedge$, i.e. $y \wedge x^{**} \wedge y^{**} = y \wedge y^{**}$, so $y \leq x^{**}$ which is false, proving that $[y] \not\leq [x^{**}]$. Similarly it can be shown that $[x^{**}] \not\leq [y]$ and so we obtain that $[x^{**}]$ and $[y]$ are incomparable. With $b = [x^{**}]$ and $d = [y]$ in Theorem 3.2 we see that $\text{Con } T$, and hence $\text{Con } S$, contains an interval isomorphic with N_5 . Since N_5 is not upper-semimodular, $\text{Con } S$ is not upper-semimodular, proving the theorem.

The following lemma is implicit in Lemma 3.2.

LEMMA 3.4. *(U) holds in S iff $P_6 \notin SH(S)$.*

Proof. Suppose (U) fails in S , then as in the proof of Theorem 3.3 we let $T = S/(y^{**})^\wedge$, $b = [x^{**}]$ and $d = [y]$. Therefore there is a quotient T' of T with the properties (1)–(4) of the Lemma 3.1 and hence $[b]^* < [d]$ in T' . Then the elements $[1]$, $[b]$, $[d]$, $[b \wedge d]$, $[b]^*$ and $[0]$ form a subalgebra of T' isomorphic to P_6 . Since $T' \in H(S)$, $P_6 \in SH(S)$. Conversely, assume (U) holds in S and $P_6 \in SH(S)$. Now (U) is equivalent to the positive universal sentence (U_p) for PCS's and hence it follows from well-known preservation properties of first-order sentences that (U) is true in P_6 . But with $x = b$ and $y = d$ (U) fails in P_6 and we have a contradiction.

We shall now study the PCS's with the property (U).

LEMMA 3.5. Let S satisfy (U) and let α be an atom of type (T2) in $\text{Con } S$. Then $\alpha = (\alpha)_B \cup \Delta_{S-B(S)}$.

Proof. Let $\langle x, y \rangle \in \alpha$ with $x \neq y$. Then using Remark 2.3 one has that $x^* \neq y^*$ and $x \prec y$, say. Now applying (U) for x^{**} and y we get $x^{**} < y$ and thus $x = x^{**}$, whence $y = y^{**}$.

We shall now give a “strong” converse of Theorem 3.4. Recall that a lattice L is semimodular iff for $a, b \in L$, $a \succ a \wedge b \rightarrow a \vee b \succ b$.

THEOREM 3.6. If S satisfies (U) then $\text{Con } S$ is semimodular.

Proof. Let $\beta, \delta \in \text{Con } S$ such that $\delta \succ \beta \wedge \delta$. Consider the quotient $S_1 = S/\beta \wedge \delta$ and let β_1, δ_1 be the congruences on S_1 corresponding to β, δ respectively. Then $\beta_1 \wedge \delta_1 = \Delta_{S_1}$ and δ_1 is an atom in $\text{Con } S_1$. In view of [10, Theorem 4.9] we can assume that δ_1 is of type (T2). Let us write Φ_1 for $\Phi(S_1)$. From [10, Theorem 2.6] it follows that $\beta_1 \vee \Phi_1 = ([1]_{\beta_1})^\wedge \circ \Phi_1 \circ ([1]_{\beta_1})^\wedge$, then Lemma 3.5 gives $\delta_1 \wedge (\beta_1 \vee \Phi_1) = \Delta_{S_1}$; and we have by [10, Theorem 4.7] that $[\Delta_{S_1}, \delta_1] \cong [\beta_1 \vee \Phi_1, \beta_1 \vee \Phi_1 \vee \delta_1]$. Now we claim that $(\beta_1 \vee \delta_1) \wedge (\beta_1 \vee \Phi_1) = \beta_1$. In fact, from [10, Theorem 2.8] we know $\beta_1 \vee \delta_1 = \delta_1 \circ \beta_1 \circ \delta_1$ and as noted earlier $\beta_1 \vee \Phi_1 = ([1]_{\beta_1})^\wedge \circ \Phi_1 \circ ([1]_{\beta_1})^\wedge$. Thus we only need to show that $(\delta_1 \circ \beta_1 \circ \delta_1) \wedge ([1]_{\beta_1})^\wedge \circ \Phi_1 \circ ([1]_{\beta_1})^\wedge \subseteq \beta_1$. Let $\langle x, y \rangle \in \delta_1 \circ \beta_1 \circ \delta_1$ and $\langle x, y \rangle \in ([1]_{\beta_1})^\wedge \circ \Phi_1 \circ ([1]_{\beta_1})^\wedge$ with $x \neq y$. Then there exist elements s, t, u, w in S such that $x\delta_1 s\beta_1 t\delta_1 y$ and $x([1]_{\beta_1})^\wedge u\Phi_1 w([1]_{\beta_1})^\wedge y$. If x, y are closed then $\langle x, y \rangle \in (\beta_1 \vee \Phi_1)_B = (\beta_1)_B \subseteq \beta_1$. Next suppose x and y are non-closed. In this case we get from $x\delta_1 s\beta_1 t\delta_1 y$ that $x = s$ and $t = y$, implying that $\langle x, y \rangle \in \beta_1$. Finally let us assume that x is closed and y is not closed. Then $x\delta_1 s\beta_1 t = y$, i.e. $x\delta_1 s\beta_1 y$, and s must be closed. Thus we get $s\beta_1 y^{**}$. On the other hand, from the other relations we get $x([1]_{\beta_1})^\wedge y^{**} = w^{**}([1]_{\beta_1})^\wedge y^{**}$ which simplifies to $x([1]_{\beta_1})^\wedge y^{**}$ and hence $x\beta_1 y^{**}$. Thus we have $x\beta_1 y^{**}\beta_1 s\beta_1 y$, i.e. $x\beta_1 y$, proving the claim. Then from [10, Theorem 4.7] we get $[\beta_1, \beta_1 \vee \delta_1] \cong [\beta_1 \vee \Phi_1, \beta_1 \vee \Phi_1 \vee \delta_1]$ and the latter, as was noted earlier in the proof, is isomorphic to $[\Delta_{S_1}, \delta_1]$. Thus $[\Delta_{S_1}, \delta_1] \cong [\beta_1, \beta_1 \vee \delta_1]$, yielding $\beta_1 \vee \delta_1 \succ \beta_1$ from which it is immediate that $\beta \vee \delta \succ \beta$ in S , proving the theorem.

We have thus proved the following theorem, which includes the two characterizations mentioned in the introduction.

THEOREM 3.7. For any PCS S the following are equivalent.

- (1) $\text{Con } S$ is upper-semimodular,
- (2) S satisfies (U),

- (3) $\text{Con } S$ is semimodular,
- (4) $P_6 \notin SH(S)$,
- (5) S satisfies (U_p) .

Moreover, if $\text{Con } S$ is of finite length, each of the above conditions is equivalent to

- (6) $\text{Con } S$ is a graded lattice.

Proof. (1) implies (2) by Theorem 3.3, (2) implies (3) by Theorem 3.6, that (3) implies (1) is obvious, (4) and (2) are equivalent by Lemma 3.4 and $(2) \leftrightarrow (5)$ was noted earlier. If $\text{Con } S$ is of finite length then from the proof of Theorem 3.3 it follows that (6) implies (2), while $(1) \leftrightarrow (6)$ is well-known.

In view of the fact that the sentence (U_p) is both universal and positive and that universal sentences are preserved under the formation of substructures while positive sentences are preserved under homomorphic images, the following corollary is immediate.

COROLLARY 3.8. *The class of PCS's S such that $\text{Con } S$ is semimodular is closed under formation of subalgebras and quotients.*

We would like to point out that even if S is of finite length, $\text{Con } S$ need not be of finite length as is shown by the following example.

EXAMPLE 3.9.

If S has, as shown in Figure 3 as infinite antichain consisting of a_i ($i \in \mathbb{Z}$), then $\text{Con } S$ contains an infinite Boolean sublattice generated by the congruences $\theta(0', a_i)$, $i \in \mathbb{Z}$ and hence is not of finite length.

In this connection, therefore, the following theorem is of some interest.

Figure 3

THEOREM 3.10. If $\text{Con } S$ is of finite length l then S is finite.

Proof. The hypothesis implies that $[\Phi, \nabla]$ is of finite length. Hence $[\Phi, \nabla]$ is finite because it is isomorphic with $\text{Con } B(S)$ which, it is known, is finite iff it is of finite length. This implies that $B(S)$ is finite. Let $c \in B(S)$ be arbitrarily fixed. From the hypothesis it can be easily argued that every chain in D_c is finite and every antichain in D_c has width at most l . Then the fact that D_c is finite follows from Dilworth's result: If P is a poset of width m , then $P = \bigcup_{i=1}^m C_i$, where C_i is a chain in P , $1 \leq i \leq m$. Now $S = \bigcup \{D_c : c \in B(S)\}$. Since each D_c is finite and $B(S)$ is finite, it follows that S is finite.

Using an argument similar to the one in the above theorem, it can also be shown that if $\text{Con } S$ satisfies the ascending chain condition or descending chain condition then $\text{Con } S$ is finite. Thus we have the following.

THEOREM 3.11. *T.f.a.e.*

- (1) $\text{Con } S$ is of finite length,
- (2) $\text{Con } S$ satisfies A.C.C.,
- (3) $\text{Con } S$ satisfies D.C.C.,
- (4) $\text{Con } S$ is finite,
- (5) S is finite.

§4. Congruence-distributive PCS's

In this section we shall give the characterizations of the class of congruence-distributive PCS's. The general program to arrive at these characterizations follows closely the one carried out in section 3. Accordingly the proofs of the results in this section are similar to those of the corresponding theorems of section 3 and hence the presentation in this section will be sketchy, leaving most of the details to the reader (or see [9]). Use Lemma 1.3 to prove the following.

LEMMA 4.1. Let $K \subseteq B(S)$ have the property: If $k_1 \leq k_2$ and $k_2 \in K$ then $k_1 \in K$. Then

$$\bigvee_{k \in K} \theta(D_k(S) \times D_k(S)) = \bigcup_{\substack{k \in K \\ a \in D_k(S)}} \theta(k, a).$$

COROLLARY 4.2.

$$\Phi = \bigcup_{\substack{b \in B(S) \\ a \in D_b(S)}} \theta(b, a).$$

The role of P_5 will be clear from the following lemma which in many respects is similar to Lemma 3.1.

LEMMA 4.3. *Suppose there are two incomparable elements d and h both dense in S . Then there exists a quotient algebra S'' of S with the following properties:*

- (1) $[d]$ and $[h]$ are dense in S'' ,
- (2) $[d]$ and $[h]$ are incomparable,
- (3) $[d \wedge h] \rightarrow d$, $[d \wedge h] \rightarrow h$, $[d] \rightarrow 1$ and $[h] \rightarrow 1$,
- (4) $N(S'') = \{[d], [h], [d \wedge h]\}$.

Proof. We shall briefly indicate how to construct a sequence of PCS's $S = S_0, S_1, \dots, S_5$ such that $S_i = S_{i-1}/\xi_i$ where ξ_i is a suitably chosen congruence on S_{i-1} , $i = 1, \dots, 5$, and S_5 will have the properties (1)–(4) and hence will qualify to be S'' . Let $[s]_i$ denote the congruence class of $[s]_{i-1}$ modulo ξ_i , with $[s]_0 = s$ for $s \in S$, and $[1]$ denotes $[1]_i$, $i = 1, \dots, 5$.

STEP 1. The set $M = \{x \in D(S) : x > d \text{ and } x > h\} = (\{d, h\})^u$ is a filter in S . With $\xi_1 = \hat{M}$ and $S_1 = S/\hat{M}$ one verifies that $[d]_1$ and $[h]_1$ are incomparable and $[1]$ is the only common upper bound of $[d]_1$ and $[h]_1$.

STEP 2. Let $K = D(S_1) - ([d]_1, [1]) \cup ([h]_1, [1])$, $\xi_2 = \bigvee_{[k]_1 \in K} \theta([k]_1, [d \wedge h]_1)$ and $S_2 = S_1/\xi_2$. Then using Lemma 1.2 it follows that $[d]_2$ and $[h]_2$ are incomparable, $[d \wedge h]_2 \rightarrow [d]_2$, $[d \wedge h]_2 \rightarrow [h]_2$ and $D(S_2) = ([d]_2, [1]) \cup ([h]_2, [1]) \cup \{[d \wedge h]_2\}$.

STEP 3. Let $H = ([d]_2, [1])$ and define $\xi_3 = \bigvee_{[x]_2 \in H} \theta([x]_2, [d]_2)$ and $S_3 = S_2/\xi_3$. Then again using Lemma 1.2 one sees that $[d]_3$ and $[h]_3$ are incomparable, $[d]_3 \rightarrow [1]$ and $D(S_3) = ([h]_3, [1]) \cup \{[d]_3, [d \wedge h]_3\}$.

STEP 4. Let $G = ([h]_3, [1])$, let $\xi_4 = \bigvee_{[g]_3 \in G} \theta([g]_3, [h]_3)$ and let $S_4 = S_3/\xi_4$. Then $[d]_4$ and $[h]_4$ are incomparable, $[h]_4 \rightarrow [1]$ and $D(S_4) = \{[1], [d]_4, [h]_4, [d \wedge h]_4\}$.

STEP 5. Let $T = \{[t]_4 \in B(S_4) : [t]_4 \neq [1]\}$. Define $S_5 = S_4/\xi_5$ where $\xi_5 = \bigvee_{[t]_4 \in T} \theta(D_{[t]_4}(S_4) \times D_{[t]_4}(S_4))$. Then using Lemma 4.1 it is easy to see that $[d]_5$ and $[h]_5$ are incomparable, and $N(S_5) = \{[d]_5, [h]_5, [d \wedge h]_5\}$. Thus $S'' = S_5$ is the quotient algebra that was sought, proving the lemma.

THEOREM 4.4. *Suppose there are two incomparable elements d and h both dense in S . Then there exists an interval in $\text{Con } S$ which is isomorphic with the following lattice.*

Figure 4

Proof. By Lemma 4.3 there exists a quotient S'' of S with the properties (1)–(4) mentioned in the lemma. We now define four congruences $\psi_a, \psi_b, \psi_w, \psi_r$ by $\psi_a = \theta([d] \wedge [h], [d])$, $\psi_b = \theta([d] \wedge [h], [h])$, $\psi_w = ([d])^\wedge$ and $\psi_r = ([h])^\wedge$. Then it is clear, in view of Theorem 2.2, that the interval $[\Delta_{S''}, \psi_w \vee \psi_r]$ of $\text{Con } S''$ is isomorphic with the lattice of Figure 4 and so there is an interval in $\text{Con } S$ isomorphic with the lattice of Figure 4 – this proves the theorem.

THEOREM 4.5. *If $\text{Con } S$ is lower-semimodular then S satisfies (D).*

Proof. From the proof of Theorem 3.3 S satisfies (U) and therefore, it is sufficient to show that S satisfies (D_w) .

Let us suppose S fails to satisfy (D_w) . Then there is a closed element c in S such that $D_c(S)$ is not totally ordered and hence there are elements s, t in $D_c(S)$ which are incomparable. Now consider $S_1 = S/\hat{c}$ and let $d = [s]$ and $h = [t]$. Then using Theorem 4.4 one can conclude that $\text{Con } S_1$ is not lower-semimodular and hence $\text{Con } S$ also is not lower-semimodular.

We shall now give a model-theoretic condition which is equivalent to (D). $SH(S)$ denotes the class of all subalgebras of homomorphic images of S .

THEOREM 4.6. *(D) holds in S iff $P_6 \notin SH(S)$ and $P_5 \notin SH(S)$.*

Proof. (D) holds in S iff (U) and (D_w) hold in S . If (U) holds in S then by Theorem 3.4 $P_6 \notin SH(S)$. If (D_w) is true in S and if $P_5 \in SH(S)$ then $(D_{w,p})$ would be true in P_5 which is impossible, thus $P_5 \notin SH(S)$. For the converse observe that if (D_w) fails in S then $0, 1, d, h, d \wedge h$ form a subalgebra of S_1 (see the proof of the preceding theorem) isomorphic to P_5 and then apply Theorem 3.4.

THEOREM 4.7. *If S satisfies (D) then $\text{Con } S$ is distributive.*

Proof. Let S satisfy (D) and let $x \equiv y(\theta \wedge (\theta_1 \vee \theta_2))$ for $\theta, \theta_1, \theta_2 \in \text{Con } S$. It suffices to show that $x \equiv y((\theta \wedge \theta_1) \vee (\theta \wedge \theta_2))$. Now there are $u_1, \dots, u_n$ in S such that $u_i < x$, $i = 1, \dots, n$ and $x\theta_1 u_1 \theta_2 u_2 \theta_1 \cdots \theta_2 y$. If y is closed, we may suppose that x is closed, but then the result follows since $\text{Con } B(S)$ is distributive. So assume y is not closed. If $x^* = y^*$ then use an argument similar to [8], and if $x^* \neq y^*$ then use the arguments in the above two cases to conclude the result.

Thus we have the following theorem which includes the two characterizations mentioned in the introduction (see Varlet [14] for $(4) \rightarrow (5)$).

THEOREM 4.8. *T.f.a.e.:*

- (1) $\text{Con } S$ is lower-semimodular,
- (2) S satisfies (D),
- (3) $\text{Con } S$ is distributive,
- (4) $\text{Con } S$ is modular,
- (5) $\text{Con } S$ is a Heyting lattice,
- (6) $P_6 \notin SH(S)$ and $P_5 \notin SH(S)$.

Note that the class of all PCS's whose congruence lattices are distributive is closed under formation of subalgebras and quotients.

We have seen in [10] that the interval $[\Phi, \nabla]$ is distributive and also that the interval $[\Delta, \Phi]$ satisfies the condition (B). Also observe that although $\text{Con } P_6$ is not distributive, the interval $[\Delta, \Phi]$ is distributive, and that P_6 satisfies (D_w) . This motivates the following theorem whose proof is left to the reader.

THEOREM 4.9. *T.f.a.e.*

- (1) $[\Delta, \Phi]$ is lower-semimodular,
- (2) S satisfies (D_w) ,
- (3) $[\Delta, \Phi]$ is distributive,
- (4) $[\Delta, \Phi]$ is modular,
- (5) $P_5 \notin SH(S)$.

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Added in proof: Methods of this paper also yield the equivalence of v -semidistributivity and distributivity on Con S . For further related results see *Mathematica Slovaca* 29 (1977), 381–395; *Proceedings of the Third Brazilian Conference on Mathematical Logic*, Sociedade Brasileira de Logica, São Paulo, 1980, pp. 281–307 and *Mathematica Japonica* 25 (1980), 519–521.