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p-Groups with few conjugacy classes of normalizers

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Abstract For a group G, denote by $\omega(G)$ the number of conjugacy classes of normalizers of subgroups of G. Clearly, $\omega(G) = 1$ if and only if G is a Dedekind group. Hence if G is a 2-group, then G is nilpotent of class ≤ 2 and if G is a p-group, $p > 2$, then G is abelian. We prove a generalization of this. Let G be a finite p-group with $\omega(G) \leq p + 1$. If $p = 2$, then G is of class ≤ 3; if $p > 2$, then G is of class ≤ 2.

Keywords Conjugacy classes · Normalizers · Finite p-groups · p-Groups of maximal class

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1 Introduction

The study of conjugacy classes of subgroups often plays an important role in determining the structure of the group. For example, let $\nu(G)$ be the number of conjugacy

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classes of non-normal subgroups, Poland and Rhemtulla [8] have shown that for a nilpotent group G which is not a Dedekind group, one has $c(G) \leq \nu(G) + 1$, thus answering a question of the first-named author. In particular, $\nu(G)$ bounds the nilpotency class of G.

Here, for a group G, we consider its normalizers. Clearly, if U is a normalizer in G, then all conjugates of U in G are normalizers. So every normalizer of G gives rise to a full conjugacy class of such subgroups. By $\omega(G)$ we shall denote the number of all G-conjugacy classes of normalizers. Clearly, $\omega(G) = 1$ if and only if every subgroup of G is normal. These groups are precisely the Dedekind groups. Similarly to what happens for $\nu(G)$, it has been proved in [3] that for a finite p-group G where $p \neq 2$, one has $c(G) \leq \omega(G)$.

In this paper we sharpen the latter bound considerably in the case where $\omega(G) \leq p + 1$. Note that the bound on the class is quite uniform:

Theorem *Let G be a finite p-group satisfying $\omega(G) \leq p + 1$.*
Then $c(G) \leq 3$. If $p \neq 2$, then $c(G) \leq 2$.

For every odd prime p, we construct finite p-groups G of class three with $\omega(G) = p+2$ (see the examples in Sect. 6).

In a slightly different direction, La Haye and Rhemtulla [6] proved that if G is a finite p-group with $\nu(G)$ strictly greater than 1, then $\nu(G)$ is at least p, and Brandl (see [2] and Conjugacy classes of non-normal subgroups of finite p-groups, to appear on Israel Journal of Mathematics) determined all finite p-groups with $\nu(G) \leq p + 1$.

There is no analogue of this for the parameter $\omega(G)$. In fact, consider the groups

$$G_m = \langle x, y \mid x^{p^{2m}} = y^{p^m} = 1,\ x^y = x^{1+p^m} \rangle \qquad (m \geq 1,\ p \neq 2).$$

Since $\langle y, x^{p^m} \rangle$ is contained in the normalizer of each subgroup of G_m and $N_{G_m}(y^{p^{m-i}}) = \langle y, x^{p^i} \rangle$ for $i = 0, \ldots, m$, we obtain $\omega(G_m) = m + 1$ (see [7, p. 1174]). Note that the groups G_m are nilpotent of class two, so that we wonder if there is a similar bound when the nilpotency class is strictly greater than two.

All groups considered in this paper are finite. Moreover, p denotes a prime, $\mathbb{F}_p$ the field with p elements and $c(G)$ is the class of a nilpotent group G. Moreover, $\mathrm{Norm}(G)$ is the intersection of all normalizers of subgroups of G.

2 Preliminary results

We start looking for a lower bound for $\omega(G)$, in terms of p, when $c(G) > 2$.

Proposition 2.1 *Let G be a p-group satisfying $\omega(G) \leq p + 1$. Then one of the following holds:*

(i) $c(G) \leq 2$;
(ii) $\omega(G) = p + 1$. *All proper normalizers are maximal subgroups of G. For every $S \leq G$ we have $\omega(S) \leq p + 1$, and all normalizers of subgroups of S have index $\leq p$ in S.*

Proof Let $H_1, \ldots, H_t$ be the maximal normalizers in G with respect to inclusion. Then H_i is normal in G for every $i \in \{1, \ldots, t\}$, since G satisfies the normalizer condition. Let $T = \langle G \setminus (H_1 \cup \cdots \cup H_t) \rangle$.

First let $G = T$. For all $g \in G \setminus (H_1 \cup \cdots \cup H_t)$ we have $\langle g \rangle \trianglelefteq G$. Hence $G/C_G(g)$ can be embedded in Aut $\langle g \rangle$, and consequently this factor group is abelian. Thus $G' \leq C_G(g)$ for all $g \in G \setminus (H_1 \cup \cdots \cup H_t)$. So $G' \leq C_G(T) = Z(G)$ and G has class at most 2.

So we may assume that T is a proper subgroup of G. Then we can write $G = H_1 \cup \cdots \cup H_t \cup T$ as a union of proper subgroups and

$$|G| < |H_1| + \cdots + |H_t| + |T|.$$

Since $|H| \leq |G|/p$ for all proper subgroups H of G, it is clear that $t \geq p$. Thus $\omega(G) \geq p + 1$. Hence $\omega(G) = p + 1$. Then $t = p$ and $H_1, \ldots, H_p$ are maximal subgroups of G. In particular, $H_1, \ldots, H_p, G$ are all normalizers of subgroups of G. This proves the first statement of (ii).

Now let $U \leq S$. Then $N_S(U) = N_G(U) \cap S = H_i \cap S$ or $N_S(U) = S$. This proves the claim for S. □

The following proves the first part of our main theorem:

Proposition 2.2 *Let G be a finite p-group. If $\omega(G) \leq p + 1$, then $c(G) \leq 3$.*

Proof By Proposition 2.1, either $c(G) \leq 2$ or all normalizers of G have index $\leq p$. In particular, $G' \leq \mathrm{Norm}(G)$. By [9], we have $\mathrm{Norm}(G) \leq Z_2(G)$, and so we get $c(G) \leq 3$. □

The upper bound for $p = 2$ cannot be improved upon, because the generalized quaternion group G of order 2^4 and class three satisfies $\omega(G) = 3$.

In the remainder of this paper, we sharpen the bound of Proposition 2.2 for p odd. The first result deals with the structure of a possible minimal counterexample.

Lemma 2.3 *Assume that there exists a p-group G, $p \neq 2$, such that $\omega(G) \leq p + 1$ and $c(G) > 2$. Choose such a group G of least possible order. Then:*

(i) $\omega(G) = p + 1$ *and all proper normalizers are maximal subgroups.*
(ii) *G has precisely one minimal normal subgroup M, say.*
(iii) $|M| = p,\ M \leq Z(G),\ \gamma_3(G) = M.$
(iv) $c(G) = 3$ *and $Z(G)$ is cyclic.*

Proof By Proposition 2.1, $\omega(G) = p + 1$ and all proper normalizers $H_1, \ldots, H_t$ are maximal subgroups, so that (i) holds. In particular every subgroup of G has at most p conjugates.

Moreover, if N is a non-trivial normal subgroup of G then we have $\omega(G/N) \leq \omega(G) \leq p + 1$, and $c(G/N) \leq 2$ by minimality. Hence, there exists a unique minimal normal subgroup M of G, as stated in (ii). Note that $|M| = p, M \leq Z(G), \gamma_3(G) = M$, $c(G) = 3$ and $Z(G)$ is cyclic, so that (iii) and (iv) hold. □

3 Metacyclic groups

We restrict to the metacyclic case and we will refer to the paper [5] where we find the following result.

Lemma 3.1 (see [5, Theorem 3.2]) *Let G be a finite metacyclic p-group, p odd. Then:*

$$G \cong \langle a, b \mid a^{p^m} = 1, b^{p^n} = a^{p^{m-s}}, a^b = a^{1+p^{m-c}} \rangle$$

where $m \geq s \geq 0$, $n > 0, m > c$ and one of the following holds:

(i) *Split case:* $0 = s \leq c < min\{n+1, m\}$
(ii) *Non-split case:* $max\{1, m-n+1\} \leq s < min\{c, m-c+1\}$

Proposition 3.2 *Let G be a metacyclic p-group, $p \neq 2$.*
If $\omega(G) \leq p+1$ then $c(G) \leq 2$.

Proof Let G be a counterexample of least possible order. Then Lemma 2.3 applies. In particular, $\Phi(G) \leq \text{Norm}(G)$ and $|\gamma_3(G)| = p$.

By Lemma 3.1, we have to distinguish the following cases.

Case 1: Split case.
Then $s = 0$ and $\langle b \rangle \cap \langle a \rangle = 1$. By Proposition 2.1, $a^p \in N_G(\langle b \rangle)$. Thus $[a^p, b] = 1$ and $a^p \in Z(G)$. Hence $G' \leq \langle a^p \rangle \leq Z(G)$, and G is of class two, a contradiction.

Case 2: Non-split case.
Applying Lemma 3.1, we have $0 < s < c$.
Now $\gamma_3(G) = \langle [a^{p^{m-c}}, b] \rangle = \langle a^{p^{2m-2c}} \rangle$. Since $|\gamma_3(G)| = p$, we get $m = 2c - 1$.

As above, we have $a^p \in N_G(\langle b \rangle)$. Thus $b^{a^p} = ba^{-p^c} \in \langle b \rangle$. It follows that $a^{-p^c} \in \langle a \rangle \cap \langle b \rangle = \langle a^{p^{2c-1-s}} \rangle$. Thus $c \geq 2c - 1 - s$ which implies $c - 1 \leq s$ and hence $s = c - 1$. In particular, $m - n + 1 \leq s$ implies $n \geq c + 1$. We thus get:

$$G = \langle a, b \mid a^{p^{2c-1}} = 1, b^{p^n} = a^{p^c}, a^b = a^{1+p^{c-1}} \rangle.$$

Let $H = \langle a^{-1} b^{p^{n-c}} \rangle$. We shall show $H \neq H^{b^p}$. We have

$$a^{b^p} = a^{(1+p^{c-1})^p}$$

and

$$(1 + p^{c-1})^p = 1 + (p^{c-1})^p + \sum_{i=1}^{p-2} \binom{p}{i} (p^{c-1})^{p-i} + p^c \equiv 1 + p^c \pmod{p^{2c-1}}.$$

Hence we obtain $a^{b^p} = a^{1+p^c}$. In particular, $b^p \in Z_2(G)$.

As $|G : N_G(H)| = p$, we must have $b^p \in N_G(H)$, so that $\langle (a^{-1} b^{p^{n-c}})^{b^p} \rangle = \langle a^{-1} b^{p^{n-c}} \rangle$.

Since $(a^{-1}b^{p^{n-c}})^{b^p} = a^{-(1+p^c)}b^{p^{n-c}}$, there exists $\lambda > 0$ such that

$$a^{-1-p^c}b^{p^{n-c}} = (a^{-1}b^{p^{n-c}})^{\lambda}. \tag{1}$$

By [4, p. 253] and using that $b^p \in Z_2(G)$, we have:

$$(a^{-1}b^{p^{n-c}})^{\lambda} = a^{-\lambda}b^{\lambda p^{n-c}}[b^{p^{n-c}}, a^{-1}]^{\binom{\lambda}{2}}.$$

But $[b^{p^{n-c}}, a^{-1}] = a^{p^{n-1}}$, and note that $a^{p^{n-1}} \in Z(G)$.
By (1), we have:

$$a^{\lambda-1}a^{-p^{n-1}\binom{\lambda}{2}} = b^{p^{n-c}(\lambda-1)}.$$

Then $b^{p^{n-c}(\lambda-1)} \in \langle a\rangle \cap \langle b\rangle = \langle b^{p^n}\rangle$, and $p^c \mid (\lambda-1)$. This implies that $\lambda = p^c h+1$, for some positive integer h. By (1), we get

$$a^{-1-p^c}b^{p^{n-c}} = (a^{-1}b^{p^{n-c}})^{p^c h+1} = a^{-hp^c}b^{hp^n}a^{-1}b^{p^{n-c}}.$$

As $b^{p^n} \in Z(G)$ we get $a^{-p^c} = a^{-hp^c}b^{hp^n} = 1$, a contradiction because we are dealing with the non-split case. □

4 Some *p*-groups of maximal class

In this section, we recall some well-known results on normalizers in p-groups of maximal class. mainly due to Blackburn [1].

Lemma 4.1 *Let p be odd and let $E = E_p$ be the non-abelian group of order p^3 and exponent p. Then $\omega(G) = p + 2$. Moreover, every maximal subgroup of E is a normalizer.*

Lemma 4.2 *Let G be a p-group of maximal class. Then:*

(i) *The only normal subgroups of G are the $\gamma_i(G)$ and the maximal subgroups of G. More precisely, if N is a normal subgroup of G of index $p^i \geq p^2$ then $N = \gamma_i(G)$.*
(ii) *If N is a normal subgroup of G of index $\geq p^2$ then also G/N has maximal class.*

The next result is a particular case of Theorem 4 in [3]. For reasons of completeness, we provide here a simple direct proof.

Proposition 4.3 *Let G be a p-group of maximal class of order p^n, where p is odd and $n \geq 4$. Then every maximal subgroup of G is a normalizer. In particular, $\omega(G) \geq p+2$.*

Proof Let M be a maximal subgroup of G. We start with $n = 4$, then $|M| = p^3$. Assume $\exp(M) \geq p^2$. Then M has a cyclic normal subgroup U of order p^2 and $M \leq N_G(U)$. On the other hand, G has an elementary abelian normal subgroup of order p^2 which is the unique normal subgroup of G of order p^2. Hence U cannot be normal in G. It follows $M = N_G(U)$. Now assume $\exp(M) = p$. Then M has at least two elementary abelian normal subgroup, U_1, U_2, of order p^2 and they cannot be both normal in G. Thus $M = N_G(U_i)$ for some $i \in \{1, 2\}$. Now let $n > 4$. Then G contains a normal subgroup R such that G/R is of order p^4 and of maximal class. Every maximal subgroup M of G contains R. So, by the first part of the proof, there exists $U/R \leq G/R$ such that $M/R = N_{G/R}(U/R)$ and we obtain $M = N_G(U)$. □

5 Proof of the main result

In this section, we shall prove our theorem for $p \neq 2$. Let G be a counterexample of least possible order. Lemma 2.3 provides us with some information about G. In particular, $c(G) = 3$ and $Z(G)$ is cyclic.

We shall split the proof into two parts according to the existence or non-existence of abelian normal subgroups of rank ≥ 3. The following result of N. Blackburn is crucial for the case of small ranks:

Lemma 5.1 (see [4, 12.4 and 12.5]) *Let G be a p-group with p odd. If every abelian normal subgroup of G is 2-generated, then one of the following holds:*

(I) *G is metacyclic.*
(II) $G \cong \langle x, y, z \mid x^p = y^p = z^{p^{n-2}} = 1, [x, z] = [y, z] = 1, y^x = yz^{p^{n-3}} \rangle$, *for* $n \geq 3$.
(III) $G \cong \langle x, y, z \mid x^p = y^p = z^{p^{n-2}} = 1, [y, z] = 1, y^x = yz^{sp^{n-3}}, z^x = yz \rangle$, *for* $n \geq 4$ *and* $s = 1$ *or* s *is a nonsquare mod* p.
(IV) $p = 3$ *and G is of maximal class.*

Lemma 5.2 *Let G be a p-group with p odd and assume $\omega(G) \leq p + 1$. If every abelian normal subgroup of G is 2-generated, then $c(G) \leq 2$.*

Proof We consider the cases displayed in Lemma 5.1.

If G is as in Case (IV) then either $c(G) \leq 2$ and we are done or $\omega(G) \geq p + 2$, by Proposition 4.3, and G does not satisfy our hypothesis. Let G be as in Case (I). Then, the result follows by Proposition 3.2. The groups of Case (II) are nilpotent of class two and we are also done. So let G be as in Case (III). Then $G/\langle z^p \rangle$ is the non abelian group of order p^3 and exponent p. Then, $\omega(G/\langle z^p \rangle) = p + 2$ and hence $\omega(G) \geq p + 2$. □

We finally deal with the case when our minimal counterexample G contains abelian normal subgroups of large rank:

Proof of Theorem Let G be as in Lemma 2.3. By Lemma 5.2, we may assume that G contains an abelian normal subgroup A of rank ≥ 3. Refining the normal series $1 < \Omega_1(A) < G$, we can choose an elementary abelian normal subgroup N of G of rank 3. Let $Q = \overline{G} = G/C_G(N)$. Then Q embeds into a subgroup of $\mathrm{Aut}(N) = GL(3, p)$.

As Q is a p-group, we can identify Q with a subgroup of the group $UT(3, p)$ of unitriangular matrices of the form

$$m(a, b, c) = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

where $a, b, c \in \mathbb{F}_p$.

If there exists g such that $|C_N(g)| = p$ then the theory of the Jordan canonic form tells us that N has a basis $\{n_1, n_2, n_3\}$ with $n_1^g = n_1 n_2, n_2^g = n_2 n_3, n_3^g = n_3$. Consider the subgroup $S = \langle N, g \rangle$ of G. As $p \neq 2$, the order of $\overline{g}$ is p, so that $g^p \in Z(S)$. Thus, $\langle n_3, g^p \rangle \lhd S$ and $S^* = S/\langle n_3, g^p \rangle$ is the group E_p of Lemma 4.1. By Lemma 2.3, we must have $\omega(S) \leq p + 1$. But $p + 2 = \omega(E_p) \leq \omega(S^*) \leq \omega(S) = p + 1$, a contradiction.

Case 1. $Q = \overline{G}$ contains an element $\overline{g} = m(a, b, c)$ with $ab \neq 0$. Then $|C_N(\overline{g})| = p$.

Case 2. Q contains elements $\overline{g_i} = m(a_i, b_i, c_i)$, $(i = 1, 2)$ with $a_1 = 0, b_1 \neq 0$ and $a_2 \neq 0, b_2 = 0$. Then $\overline{g} := \overline{g_1}\, \overline{g_2} = m(a_2, b_1, *)$. As $a_2 b_1 \neq 0$, we are in the case 1.

Case 3. $Q \leq \{m(a, 0, c) \mid a, c \in \mathbb{F}_p\}$. Here $\langle n_2, n_3 \rangle \leq C_N(Q) \leq Z(G)$. But $Z(G)$ is cyclic, a contradiction.

Case 4. $Q \leq \{m(0, b, c) \mid b, c \in \mathbb{F}_p\}$. If $Q = \overline{G} = \langle \overline{g} \rangle$ is cyclic. As $Z(G)$ is cyclic, we have $|C_N(\overline{g})| = p$, so that as above, we arrive at a contradiction. Thus, assume that Q is a non-cyclic subgroup of $UT(3, p)$. Then $Q = \{m(0, b, c) \mid b, c \in \mathbb{F}_p\}$. We shall construct $p + 2$ pairwise non-conjugate normalizers arising as normalizers of cyclic subgroups of N. Let $\{n_1, n_2, n_3\}$ a basis of N and assume that $N \cap Z(G) = \langle n_3 \rangle$. Let $U_c = \langle n_1 - n_2 c \rangle \leq N$, the subgroup of N generated by the vector of components $(1, -c, 0)$, and $M_c = N_G(U_c)$, $(0 \leq c \leq p - 1)$. Choose $g_c \in G$ with $\overline{g_c} = m(0, 1, c)$. Then $g_c \in M_c$. Suppose that M_{c_1} and M_{c_2} are conjugate for some $c_1 \neq c_2$. By Lemma 2.3, both M_{c_1} and M_{c_2} are maximal subgroups of G, so that $M_{c_1} = M_{c_2}$. This implies $\langle g_{c_1}, g_{c_2} \rangle \leq M_{c_1} = M_{c_2} = N_G(U_{c_1})$. Clearly, $C_G(N) \leq N_G(U_{c_1})$, so that $H := \langle C_G(N), g_{c_1}, g_{c_2} \rangle \leq N_G(U_{c_1})$. But $\overline{H} = \langle \overline{g}_{c_1}, \overline{g}_{c_2} \rangle = Q$, and so $H = G \leq N_G(U_{c_1}) = C_G(U_{c_1})$. This, however, implies $U_{c_1} \leq Z(G) \cap N = \langle n_3 \rangle$, a contradiction. We thus have found p pairwise non-conjugate proper normalizers. An analogous argument using $U_\infty = \langle (0, 1, 0) \rangle$ and $\overline{g}_\infty = m(0, 0, 1)$ yields another normalizer not encountered before. We arrive at the final contradiction $\omega(G) \geq p + 2$.

6 Examples of finite p-groups G with $\omega(G) = p + 2$

Finally, we list some examples for p-groups G of class 3 and $\omega(G) = p + 2$. We start with an example which is already known.

Example 6.1 (see [7, p. 1173]) Let $G = \langle x, y \mid x^{p^3} = y^{p^2} = 1, x^y = x^{1+p} \rangle$ where $p > 2$. Then $c(G) = 3$ and G has $p + 2$ conjugacy classes of normalizers, corresponding to the whole group, the subgroups $\langle yx^i, x^p \rangle$ for $i = 0, \ldots, p - 1$, which are maximal (and in particular normal) in G, and finally to $\langle y, x^{p^2} \rangle$.

Example 6.2 Assume $p = 3$ and let $G = A\langle x\rangle$, where $A = \langle a_1\rangle \times \langle a_2\rangle$ is abelian, $a_1^{p^2} = a_2^p = 1$, $a_1^x = a_1a_2$, $a_2^x = a_2a_1^{-p}$, $x^p = a_1^p$. Then $c(G) = 3$, every maximal subgroup of G is a normalizer and $\omega(G) = p + 2$.

Proof Note that $|G| = p^4$ and $c(G) = 3$ so that G is of maximal class. Then, by Proposition 4.3, $\omega(G) \geq p + 2$. Let U be a non normal subgroup of G. If $|U| = p$ then $U \leq \Omega_1(G) \leq A$ and $N_G(U) = C_G(U) = A$ is a maximal subgroup. Let $|U| = p^2$, by the normalizer condition its normalizer is a maximal subgroup. Hence $\omega(G) = p + 2$. □

Next we present an infinite series of examples.

Example 6.3 Let p be a prime and $r \geq 1$. Assume $p \geq 5$ or $p = 3$ and $r \geq 2$. Let $G = A\langle x\rangle$ where $A = \langle a_1, a_2, a_3\rangle$ is elementary abelian of order p^3, and $a_1^x = a_1$, $a_2^x = a_1a_2$, $a_3^x = a_2a_3$, $x^{p^r} = a_1$. Then $\omega(G) = p + 2$, and the normalizers are precisely the maximal subgroups of G.

Proof We have $G = \langle x, a_3\rangle$, and A is normal in G. The element x induces on A an automorphism of order p, so that $x^p \in Z(G)$. The relations imply that $G/\langle x^p\rangle$ is the non-abelian group of order p^3 and exponent p. In particular, $G^p = \langle x^p\rangle$. Moreover, every maximal subgroup of G is a normalizer, and so $\omega(G) \geq p + 2$.

We show that $\langle a_2, x^p\rangle \leq \mathrm{Norm}(G)$. Indeed, as $x^p \in Z(G)$, we have $x^p \in \mathrm{Norm}(G)$, so we need to prove $a_2 \in \mathrm{Norm}(G)$. For this, it suffices to show that a_2 normalizes every cyclic subgroup U of G. Let $N = \langle A, x^p\rangle$. Then N is abelian and $G/N = \langle xN\rangle$ is of order p.

If $U \leq N$, then clearly a_2 normalizes U. So let $U \not\leq N$. By the above, we may assume $U = \langle ax\rangle$ for some $a \in A$. We get

$$[a_2, U] = \langle [a_2, ax]\rangle = \langle [a_2, x]\rangle = \langle a_1\rangle. \tag{2}$$

If $p \geq 5$, then G is regular, so that we have $|U| \geq p^2$. Thus $1 \neq U^p \leq G^p = \langle x^p\rangle$. As $\langle a_1\rangle$ is the unique subgroup of order p of $\langle x^p\rangle$, we deduce that $a_1 \in U$. From (2) we deduce that $[a_2, U] \leq U$ whence $a_2 \in N_G(U)$. We arrive at $\langle a_2, x^p\rangle \leq \mathrm{Norm}(G)$.

Now let $p = 3$ and $r \geq 2$. Here we have $(ax)^3 = \tilde{a}x^3$ for some element $\tilde{a} \in A$. If $(ax)^3 = 1$, then we would get $x^3 \in A$. But this contradicts $r \geq 2$. As before, we get $\langle a_2, x^p\rangle \leq \mathrm{Norm}(G)$.

In all cases, we have shown $[G : \mathrm{Norm}(G)] \leq p^2$. Now $\mathrm{Norm}(G)$ is properly contained in the abelian subgroup N, so that it clearly cannot be a normalizer. Hence all normalizers are of index $\leq p$ in G. The result follows. □

The list of the above examples is by no means complete. In fact, using GAP, among the groups of order p^5 ($p \geq 3$) we found three groups G with $\omega(G) = p + 2$ and class 3.

We do not know the answer to the following

Question Let G be a p-group with $\omega(G) \leq p + 2$. Do we have $c(G) \leq 3$ if $p > 2$ and $c(G) \leq 4$ for $p = 2$?

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