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Hypersurfaces of the Euclidean sphere with nonnegative Ricci curvature

By

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Abstract. In this paper we prove that a compact oriented hypersurface of a Euclidean sphere with nonnegative Ricci curvature and infinite fundamental group is isometric to an $H(r)$-torus with constant mean curvature. Furthermore, we generalize, whithout any hypothesis about the mean curvature, a characterization of Clifford torus due to Hasanis and Vlachos.

1. Introduction. Let M^n be an n-dimensional hypersurface of the $(n+1)$-dimensional unit Euclidean sphere S^{n+1}. If M^n is compact, minimal and $0 \leqq S \leqq n$, then Simons [16] proved that $S = 0$ or $S = n$, where S is the square of the length of the second fundamental form of M^n. Chern, Do Carmo and Kobayashi [3] and Lawson [11] proved, independently, that the Clifford Tori are the only minimal hypersurfaces with $S = n$. Peng and Terng [15] studied the case where S is constant and $n = 3$, and proved that if $S > 3$, then $S \geqq 6$. Jorge and Mercuri [10] proved that if M^n is minimal with two distinct principal curvatures of multiplicities m and $(n-m)$ and $2 \leqq m \leqq (n-2)$, then M^n is locally $S^m(c_1) \times S^{n-m}(c_2)$. Otsuki in [14] gives necessary conditions for a minimal hypersurface of the sphere to be a product of spheres, namely that the second fundamental form have just two eigenvalues, each one of constant multiplicity. Recently, Hasanis and Vlachos [9] proved that if M^n is minimal and compact with two principal curvatures, one of them has multiplicity 1 and $S \geqq n$, then $S = n$ and M^n is a Clifford Torus. Alencar and do Carmo [1] proved that if M^n is compact with constant mean curvature H and $S - nH^2 \leqq B_H$, where B_H is a constant that depends only on H and n, then $S - nH^2 = 0$ or $S - nH^2 = B_H$. They also proved that the $H(r)$-tori $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ with $r^2 \leqq \frac{n-1}{n}$ are the only hypersurfaces with constant mean curvature $H \neq 0$ and $S - nH^2 = B_H$. These results do not characterize the others tori $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$, with $r^2 \geqq \frac{n-1}{n}$, and $S^{n-k}(r) \times S^k(\sqrt{1-r^2})$ for $2 \leqq k \leqq (n-1)$. We observe that a torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ has nonnegative Ricci curvature and infinite fundamental group, while a torus $S^{n-k}(r) \times S^k(\sqrt{1-r^2})$, for $2 \leqq k < (n-1)$, has

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positive Ricci curvature and finite fundamental group. Hence a natural question is to classify the immersions into the sphere with nonnegative Ricci curvature. The first result we obtained here concerning that problem is a topological-geometrical classification of $H(r)$-tori $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$. We have:

Theorem 1. *Let $f : M^n \to S^{n+1}$, $n \geqq 3$, be a compact oriented hypersurface with nonnegative Ricci curvature. Then, $f(M^n)$ is isometric to an $H(r)$-torus if, and only if, the fundamental group $\pi_1(M)$ is infinite. Furthermore, if $n = 3$ then the universal covering $\widetilde{M}^3$ of M^3 is diffeomorphic to S^3 or $f(M^3)$ is isometric to $S^2(r) \times S^1(\sqrt{1-r^2})$.*

The second theorem stated here is a generalization of the result of Hasanis and Vlachos [9] without any hypothesis about the mean curvature, where we obtain a characterization for $H(r)$-tori $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ with $r^2 \geqq \frac{n-1}{n}$.

Theorem 2. *Let $f : M^n \to S^{n+1}$, $n \geqq 3$, be a compact oriented hypersurface with two distinct principal curvatures λ and μ with multiplicities 1 and $n-1$, respectively, and $S \geqq S(H)$, where*

$$S(H) = n + \frac{n^3 H^2}{2(n-1)} + \frac{n(n-2)|H|}{2(n-1)}\sqrt{n^2H^2 + 4(n-1)}.$$

Then H is constant, $S = S(H)$ and $f(M^n)$ is isometric to an $H(r)$-torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ with $r^2 \geqq \frac{n-1}{n}$.

We observe that a hypersurface $M^n \to S^{n+1}$, $n \geqq 4$, is conformally flat if and only if f has a principal curvature with multiplicity at least $(n-1)$ (see Theorem 7.11 of [7]). So, we have the following consequence of the Theorem 1:

Corollary 1. *Let $f : M^n \to S^{n+1}$, $n \geqq 3$, be a compact, oriented conformally flat hypersurface in such way that M^n has nonnegative Ricci curvature.*

(i) *If $n = 3$, then the universal covering $\widetilde{M}^3$ is homeomorphic to S^3 or $f(M^3)$ is isometric to an $H(r)$-torus $S^2(r) \times S^1(\sqrt{1-r^2})$ and f has constant mean curvature.*
(ii) *If $n \geqq 4$, then M^n is homeomorphic to S^n or $f(M^n)$ is isometric to a $H(r)$-torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ and f has constant mean curvature.*

Recently, Cheng [7] obtained the following result:

Theorem 3 [Q. M. Cheng, [7]]**.** *Let M^n be a compact oriented conformally flat n-dimensional Riemannian manifold with constant scalar curvature. If the Ricci curvature of M^n is nonnegative, then M^n is isometric to a space form or to a Riemannian product $S^{n-1} \times S^1$.*

A consequence of the Cheng's result above and Theorem 1 of Noronha [13] is that similars statements of Corollary 1 can be obtained if we assume that M^n is only a compact, oriented and conformally flat manifold with nonnegative Ricci curvature.

2. Preliminaries. Let M^n be an n-dimensional smooth, oriented Riemannian manifold and $f : M^n \to S^{n+1}$ be an isometric immersion. Denote the standard connection of S^{n+1} by $\bar{\nabla}$, the Riemannian connection of M^n by ∇, and the second fundamental form of the immersion by B. For tangent vectors X and Y of M^n we have the Gauss formula

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y),$$

as well as the Weingarten formula

$$\bar{\nabla}_X N = -A_N(X),$$

where A_N is the shape operator associated with the normal vector field N. It is well known that $\langle A_N(X), Y\rangle = \langle B(X, Y), N\rangle$. Moreover, for any unit tangent vector X

$$\mathrm{Ric}(X) = (n-1) + tr(A_N)\langle A_N(X), X\rangle - |A_N(X)|^2.$$

In particular, for a minimal immersion we have

$$\mathrm{Ric}(X) = (n-1) - |A_N X|^2.$$

Now, we consider the parallel hypersurfaces of M^n given by the map

$$f_\theta = \cos\theta f + \sin\theta N.$$

If $\cot\theta$ is not a principal curvature of f, then f_θ is an isometric immersion if we endow M^n with the pullback metric $\langle,\rangle_\theta$ via f_θ. Moreover, if the Riemannian manifold $(M^n, \langle,\rangle)$ is complete, then $(M^n, \langle,\rangle_\theta)$ is complete. Let $\lambda_1, \ldots, \lambda_n$ be the principal curvatures of f and let us suppose that $\lambda_i \neq \cot\theta$, for all $i = 1, \ldots, n$. In that case, the shape operator A_θ of the immersion f_θ with respect to the unit normal vector $N_\theta = \cos\theta N - \sin\theta f$ is given by

$$A_\theta = [(\cot\theta)A_N + Id][(\cot\theta)Id - A_N]^{-1}$$

and the principal curvatures of f_θ are

$$\lambda_i(\theta) = \frac{\cot\theta\lambda_i + 1}{\cot\theta - \lambda_i}.$$

If the shape operator A of f is not singular, for $\theta = \frac{\pi}{2}$, the immersion $f_{\frac{\pi}{2}}$ is the Gauss map of f with associated shape operator $(-A^{-1})$. We note that the pullback metric $\langle\ ,\ \rangle_*$ via N is given by

$$\langle X, Y\rangle_* = \langle A_N X, A_N Y\rangle, \quad X, Y \in TM,$$

and the principal curvatures of $f_{\frac{\pi}{2}}$ are $-\frac{1}{\lambda_1}, \ldots, -\frac{1}{\lambda_n}$.

3. Proof of Theorem 1. Let $i : S^{n+1} \to R^{n+2}$ be the inclusion map and $\bar{f} : M^n \to R^{n+2}$ be the isometric immersion $\bar{f} = i \circ f$. Since $\pi_1(M^n)$ is infinite, applying the Theorem 1 of [12] to $\bar{f}$, we conclude that M^n has nonnegative secctional curvature. On the other hand, by Aubin (see [2], p. 397), for every $x \in M$, there exists $v \in T_x M$, so that $\|v\| = 1$ and $\mathrm{Ric}(v) = 0$. Since $K(v, w) \geqq 0$ for every $w \neq v$, then $K(v, w) = 0$. Since the Ricci curvature attains its absolute extrema at principal directions, we can choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ in a neighborhood of $p \in M^n$ satisfying $Ae_i = \lambda_i e_i$, $i = 1, \ldots, n$, and $\mathrm{Ric}(e_1) = 0$. Then the nonnegativity of sectional curvatures of M^n implies $K(e_1, e_j) = 0$ for all $j \geqq 2$. Therefore $1 + \lambda_1 \lambda_j = 0$. We conclude that f has only two distinct principal curvatures λ e μ with multiplicities 1 and $n - 1$, respectively.

If $n \geqq 4$, by Dajczer [8] (Theorem 7.11, p. 118), M^n is conformally flat. If $n = 3$, M^3 is conformally flat if and only if holds the Codazzi condition

$$(\nabla_X \gamma)(Y) = (\nabla_Y \gamma)(X), \tag{1}$$

where $\gamma : TM \to TM$ is given by $\gamma(X) = Q(X) - \frac{\tau}{4}X$, Q is the Ricci tensor and τ is the scalar curvature of M^n.

Let us prove that the tensor γ satisfies (1). We observe that $Q(e_i) = \mathrm{Ric}(e_i)e_i$. By Gauss equation we have

$$\mathrm{Ric}(e_1) = 2 + 2\lambda\mu,$$
$$\mathrm{Ric}(e_i) = 2 + \lambda\mu + \mu^2\,, \; i = 2, 3,$$

and, therefore,

$$Q(e_1) = (2 + 2\lambda\mu)e_1,$$
$$Q(e_i) = (2 + \lambda\mu + \mu^2)e_i\,, \; i = 2, 3.$$

Since

$$\tau = \mathrm{Ric}(e_1) + 2\mathrm{Ric}(e_2) = 6 + 4\lambda\mu + 2\mu^2\,,$$

we obtain

$$\gamma(e_1) = \left[\frac{1}{2} + \mu\left(\lambda - \frac{\mu}{2}\right)\right] e_1$$

and

$$\gamma(e_i) = \left[\frac{1}{2} + \frac{1}{2}\mu^2\right] e_i, \quad i = 2, 3.$$

Consequently,

$$\gamma = \frac{1}{2}I + \mu\left(A - \frac{\mu}{2}I\right).$$

Since μ has constant multiplicity bigger than 1, it follows from the Theorem 4.4, p. 139 of [6] that $e_2(\mu) = e_3(\mu) = 0$. Using this fact and the Codazzi equation

$$(\nabla_{e_i} A)(e_j) = (\nabla_{e_j} A)(e_i),$$

we obtain (1). Hence M^3 is conformally flat.

Therefore, for any $n \geqq 3$, M^n is an orientable, compact manifold, conformally flat with nonnegative Ricci curvature. Since $\pi_1(M^n)$ is infinite, we can use the same arguments of the proof of the Theorem 1 of [14] to conclude that the universal covering $\widetilde{M}^n$ of M^n is isometric to $R \times S_c^{n-1}$. Consequently, $\widetilde{M}^n$ and M^n have constant scalar curvature. But the scalar curvature of M^n is given by

$$\begin{aligned}\tau &= 2\sum_j K(e_1, e_j) + \sum_{\substack{i \neq j \\ i,j \neq 1}} K(e_i, e_j) \\ &= (n-1)(n-2)(1+\mu^2)\,.\end{aligned}$$

We conclude that μ is constant. Since $\lambda\mu = -1$, we have that λ is also constant. Hence f has constant mean curvature and M^n is isometric to a torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$.

Consider the case $n = 3$. By Theorem 1.2 of Hamilton [8], M^3 is diffeomorphic either S^3 or a quocient of S^3, $S^2 \times R$ or R^3 by a fixed-point free group of isometries of the standard metric in each case. Then the universal covering $\widetilde{M}^3$ of M^3 is diffeomorphic to either S^3, $S^2 \times R$ or R^3. If $\widetilde{M}^3$ is diffeomorphic to $S^2 \times R$ or R^3, then M^3 is non-compact and so $\pi_1(M^3)$ is infinite. In this case, $f(M^3)$ is isometric to $S^2(r) \times S^1(\sqrt{1-r^2})$, which finishes the proof of theorem. □

4. Proof of Theorem 2. Note that

$$\lambda + (n-1)\mu = nH \tag{2}$$

and

$$S = \lambda^2 + (n-1)\mu. \tag{3}$$

In particular, $\lambda \neq 0$ and $\mu \neq 0$, otherwise (2) and (3) imply $S = \frac{n^2}{n-1}H^2$ if $\lambda = 0$ or $S = n^2H^2$ if $\mu = 0$, which contradicts $S \geqq S(H)$. Then, the shape operator of f is not singular. Consequently, we can define the Gauss map of f and $\langle\ ,\ \rangle_*$ is a metric on M.

Now let us write

$$S(H) = \frac{n}{n-1}\left\{(n-1) + \frac{n^2}{2}H^2 + \frac{n-2}{2}\alpha(H)\right\},$$

where $\alpha(H) = \sqrt{n^2H^4 + 4(n-1)H^2}$.

Setting $\lambda\mu = Z$, from equations (2) and (3), we get

$$n^2Z^2 + 2n(S - 2nH^2)Z + (S - n^2H^2)\left(S - \frac{n^2H^2}{n-1}\right) = 0.$$

The solutions of this equation are

$$\frac{-(S-2nH^2) \pm (n-2)\sqrt{\frac{n}{n-1}}|H|\sqrt{S-nH^2}}{n}.$$

Since $S \geqq S(H)$ the solutions above are less that -1. Hence $\lambda\mu \leqq -1$. Therefore, the seccional curvatures K_* of the Gauss map of f with respect to the planes generated by e_1 and e_j, $j > 1$, satisfy $K_*(e_1, e_j) = 1 + \frac{1}{\lambda\mu} \geqq 0$. On the other hand, we have $K_*(e_i, e_j) = 1 + 1/\mu^2 > 1$, for $j > i > 1$. Hence, the Ricci curvature of the immersion $f_{\frac{\pi}{2}}$ is nonnegative. On the other hand, for $n \geqq 4$, since $f_{\frac{\pi}{2}}$ has only two principal curvatures of multiplicities 1 and $n-1$, we have that $(M^n, \langle\ ,\ \rangle_*)$ is conformally flat (see Theorem 7.11 of [7]). Hence, since $f_{\frac{\pi}{2}}$ has not umbilical points, we may apply Corollary 2.6 of [5] (for immersions in S^{n+1}) to conclude that M^n is homeomorphic to a product of spheres $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$. Therefore, $\pi_1(M^n)$ is infinite. For $n = 3$, the same conclusion of Corollary 2.6 of [5] holds, because $f_{\frac{\pi}{2}}$ has only two principal curvatures and has not umbilical points. Consequently, applying the Theorem 1 for $f_{\frac{\pi}{2}}$, we conclude that $f_{\frac{\pi}{2}}(M)$ is isometric to an $H(r)$-torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$. In particular, we have that the principal curvatures $1/\lambda$ and $1/\mu$ of $f_{\frac{\pi}{2}}$ are constant. Consequently, we conclude that $f(M^n)$ is a torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$. Since $S = S(H)$, we have $r^2 \geqq (n-1)/n$, which completes the proof of theorem.

5. Proof of Corollary 1. (i) Let $n = 3$. If $\pi_1(M^3)$ is finite, then the universal covering of M^3 is compact and simply connected. By a theorem of N. Kuiper (see Corollary 7.9 of [7]), $\widetilde{M}^3$ is homeomorphic to S^3. If $\pi_1(M^3)$ is infinite, the corollary is a consequence of the Theorem 1.

(ii) Let $n \geqq 4$. If $\pi_1(M^n)$ is finite, by Theorem 1.10 of [5], M is homeomorphic to S^n. If $\pi_1(M^n)$ is infinite, the result is a consequence of Theorem 1.

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