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# HYPERSURFACES OF $\mathbb{S}^{n+1}$ WITH TWO DISTINCT PRINCIPAL CURVATURES

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**Abstract.** The aim of this paper is to prove that the Ricci curvature $\mathrm{Ric}_M$ of a complete hypersurface $M^n$, $n \geq 3$, of the Euclidean sphere $\mathbb{S}^{n+1}$, with two distinct principal curvatures of multiplicity 1 and $n-1$, satisfies $\sup \mathrm{Ric}_M \geq \inf f(H)$, for a function $f$ depending only on $n$ and the mean curvature $H$. Supposing in addition that $M^n$ is compact, we will show that the equality occurs if and only if $H$ is constant and $M^n$ is isometric to a Clifford torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$.

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**1. Introduction.** Let $M^n$ be a $n$-dimensional complete, oriented Riemannian manifold and $\varphi : M \to \mathbb{S}^{n+1}$ a minimal isometric immersion of $M$ into the unit Euclidean sphere $\mathbb{S}^{n+1}$. When $n = 3$, T. Hasanis and D. Koutrofiotis [**5**] proved that $\sup \mathrm{Ric}_M \geq \frac{3}{2}$ and, that if $M^3$ is compact, the equality occurs if and only if $\varphi(M^3)$ is isometric to the Clifford torus $S^1(\sqrt{\frac{1}{3}}) \times S^2(\sqrt{\frac{2}{3}})$. Later, L. Haizhong [**6**] showed that if $M^3$ is compact and $0 \leq \mathrm{Ric}_M \leq \frac{3}{2}$ then $\varphi(M^3)$ is isometric to the Clifford torus $S^1(\sqrt{\frac{1}{3}}) \times S^2(\sqrt{\frac{2}{3}})$. On the other hand, T. Hasanis and T. Vlachos [**4**] proved that $\sup \mathrm{Ric}_M \geq n-2$, for any dimension $n$. Moreover, for even dimension $n = 2m$ they proved that the equality occurs if and only if $\varphi(M^n)$ is isometric to the Clifford torus $S^m(\frac{1}{\sqrt{2}}) \times S^m(\frac{1}{\sqrt{2}})$. In the odd case $n = 2m+1$, the authors obtained a topological result. More precisely, they showed that the universal covering of $M^n$ is homeomorphic to totally geodesic sphere $S^n$.

It is known that the supremum of Ricci curvature of a Clifford torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ with nonnull mean curvature $H$ (constant) is given by

$$\frac{n(n-2)}{n-1}\left[1 + \frac{n}{2(n-1)}H^2 - \frac{1}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}\right], \quad \text{if } r^2 > \frac{n-1}{n},$$

or

$$\frac{n(n-2)}{n-1}\left[1 + \frac{n}{2(n-1)}H^2 + \frac{1}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}\right], \quad \text{if } r^2 < \frac{n-1}{n}.$$

When $H = 0$ we have $r^2 = \frac{n-1}{n}$ and the supremum is $\frac{n(n-2)}{n-1}$.

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Let $k_i$, $i = 1, \dots, n$, denote the principal curvatures of an immersion $\varphi : M^n \to \mathbb{S}^{n+1}$. If there exist smooth functions $\lambda, \mu : M \to \mathbb{R}$ such that

$$\lambda = k_1, \dots, k_m \qquad \mu = k_{m+1}, \dots, k_n,$$

and $\lambda(p) \neq \mu(p)$, for all $p \in M$, we say that $\varphi$ has two distinct principal curvatures of multiplicity $m$ and $n - m$. Clifford tori $S^{n-m}(r) \times S^m(\sqrt{1 - r^2}) \hookrightarrow S^{n+1}$ are examples of these kind of immersions.

We will prove the following result.

THEOREM 1. *Let $M^n$, $n \geq 3$, be a n-dimensional complete, oriented Riemannian manifold, and $\varphi : M^n \to \mathbb{S}^{n+1}$ be an isometric immersion whose mean curvature $H$ is bounded. Suppose that $\varphi$ has two distinct principal curvatures with multiplicity* 1 *and* $n - 1$. *Then*

$$\sup \mathrm{Ric}_M \geq f(\sup |H|), \tag{1}$$

*where*

$$f(x) = \frac{n(n-2)}{n-1} \left[ 1 + \frac{n}{2(n-1)} x^2 - \frac{1}{2(n-1)} \sqrt{n^2 x^4 + 4(n-1)x^2} \right].$$

*Moreover, if $M^n$ is compact, the equality in* (1) *occurs if and only if $H$ is constant and*

$$\varphi(M^n) = S^{n-1}(r) \times S^1(\sqrt{1 - r^2}), \quad r^2 \geq \frac{n-1}{n}.$$

In order to prove the Theorem 1 we will make use of the following result obtained by the author et al. [**1**].

THEOREM 2. *Let $\varphi : M^n \to \mathbb{S}^{n+1}$, $n \geq 3$, be a closed and orientable hypersurface. If the Ricci curvature of $M^n$ is nonnegative and the fundamental group $\pi_1(M^n)$ of $M$ is infinite, then $\varphi(M^n)$ is isometric to a Clifford torus $S^{n-1}(r) \times S^1(\sqrt{1 - r^2})$.*

**2. Preliminaries.** Let $M^n$ be a $n$-dimensional and oriented Riemannian manifold. We consider an isometric immersion $\varphi : M^n \to \mathbb{S}^{n+1}$ of $M^n$ into the unit Euclidean sphere $\mathbb{S}^{n+1}$. We denote by $N$ the unit normal field to $\varphi$. The Gauss mapping $\eta : M^n \to \mathbb{S}^{n+1}$ of $\varphi$ is defined as follows: for each $p \in M^n$, $\eta(p)$ is the end point of the vector obtained by translating $N(p)$ parallel in $R^{n+2}$ so as its initial point is the origin of $R^{n+2}$. Identifying $M^n$ and $\varphi(M^n)$ locally, we have, for tangent vectors $X$ to $M^n$, that $(\nabla_X N)^\top = -AX$, where $\nabla$ is the connection of $\mathbb{S}^{n+1}$, $A$ is the Weingarten operator of $\varphi$ and $v^\top$ denote the tangent component to $M^n$ of a vector $v$ tangent to $\mathbb{S}^{n+1}$. We can see easily that $d\eta(X) = -AX$. If $A$ is nonsingular, then the map $\eta : M^n \to \mathbb{S}^{n+1}$ is an isometric immersion when we endow $M^n$ with the metric $\langle\ ,\ \rangle_*$ given by

$$\langle X, Y \rangle_* = \langle AX, AY \rangle,$$

where $\langle\ ,\ \rangle$ denote the induced metric of $M^n$ by $\varphi$. Moreover, the Weingarten operator of the immersion $\eta$ is $A^{-1}$ and the equalities

$$\langle A^{-1}X, Y \rangle_* = \langle X, AY \rangle = \langle AX, Y \rangle$$

imply that $\varphi$ and $\eta$ have the same principal directions. More precisely, if $\{e_1, \ldots, e_n\}$ is an orthonormal basis which diagonalizes $A$, then $\{\frac{e_1}{\lambda_1}, \ldots, \frac{e_n}{\lambda_n}\}$ is an orthonormal basis with respect to metric $\langle\ ,\ \rangle_*$ that also diagonalizes $A^{-1}$, where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of $\varphi$. Hence the principal curvatures of $\eta$ are $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ and the sectional curvatures $k_*$ of $(M^n, \langle\ ,\ \rangle_*)$ with respect to the 2-planes spanned by principal directions are given by

$$k_*(e_i, e_j) = 1 + \frac{1}{\lambda_i\lambda_j}, \quad i, j = 1, \ldots, n, \quad i \neq j.$$

LEMMA 1. *Let $\varphi : M^n \to \mathbb{S}^{n+1}$ be an oriented hypersurface of $\mathbb{S}^{n+1}$ with bounded mean curvature. Suppose there exists a constant $\alpha$, with $\alpha < n-1$, so that the Ricci curvature of $M^n$ satisfies everywhere $\mathrm{Ric}_M \leq \alpha\langle\ ,\ \rangle$. Then, the principal curvatures of $M^n$ satisfy $|\lambda_i| \geq \beta$, for some positive constant $\beta$. It follows that the Gauss mapping $\eta$ of the immersion $\varphi$ is an isometric immersion and that if $M^n$ is complete in the induced metric by $\varphi$, then $\langle X, Y\rangle_* = \langle AX, AY\rangle$ is also a complete metric on $M^n$.*

The proof of the Lemma 1 can be found in the paper of T. Hasanis and D. Koutroufiotis [**5**].

**3. Proof of Theorem 1.** Let us put $\sup \mathrm{Ric}_M = \alpha$ and suppose, by contradiction, that $\alpha < f(\sup|H|)$. Then, since $f(x)$ is decreasing for $x \geq 0$, we have

$$\alpha < f(\sup|H|) \leq \frac{n(n-2)}{n-1} < n-1. \tag{2}$$

Consequently, we can apply the Lemma 1 to conclude that the principal curvatures $\lambda$ and $\mu$ of $\varphi$ are non-zero and that $(M^n, \langle\ ,\ \rangle_*)$ is complete. We will denote by $e_1, \ldots, e_n$ the principal directions with respect the principal curvatures $\lambda_1 = \lambda$ and $\lambda_2, \ldots, \lambda_n = \mu$, respectively. Since

$$\mathrm{Ric}_M(e_i) = n - 1 + nH\lambda_i - \lambda_i^2$$

and $\mathrm{Ric}_M \leq \alpha$, it follows that

$$\lambda_i \geq \frac{n}{2}H + \sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha}$$

or

$$\lambda_i \leq \frac{n}{2}H - \sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha}.$$

Since $\lambda + (n-1)\mu = nH$, by changing the orientation of $M$, if necessary, we may assume that

$$\lambda \geq \frac{n}{2}H + \sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha} \tag{3}$$

and

$$\mu \leq \frac{n}{2}H - \sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha}. \tag{4}$$

On the other hand, we have

$$\begin{aligned}\lambda &= nH - (n-1)\mu \\ &\geq nH - \frac{n(n-1)}{2}H + (n-1)\sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha} \\ &= -\frac{n(n-3)}{2}H + (n-1)\sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha}.\end{aligned} \tag{5}$$

Then, (4) and (5) yield

$$\begin{aligned}\lambda\mu &\leq \left(-\frac{n(n-3)}{2}H + (n-1)\sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha}\right) \\ &\quad \times \left(\frac{n}{2}H - \sqrt{\frac{n^2}{4}H^2 + n - 1 - \alpha}\right) \\ &= -g(H),\end{aligned} \tag{6}$$

where

$$g(x) = (n-1)^2 - (n-1)\alpha + \frac{n^2(n-2)}{2}x^2 - n(n-2)x\sqrt{\frac{n^2}{4}x^2 + n - 1 - \alpha}.$$

It is obvious that $g(x)$ is decreasing everywhere. Moreover, it satisfies

$$g(\sup|H|) > 1. \tag{7}$$

In fact, this inequality is equivalent to

$$(n-1)^2\alpha^2 - n(n-2)[2(n-1) + n(\sup|H|)^2]\alpha + n^2(n-2)^2[1 + (\sup|H|)^2] > 0.$$

This is true, since the minor root is $f(\sup|H|)$ and $\alpha < f(\sup|H|)$.

The sectional curvature $k_*$ of the Gauss mapping $\eta$ of the immersion $\varphi$, with respect to the plane generated by $e_1$ and $e_j$ ($j > 1$), taking into account of (7), satisfies

$$k_*(e_1, e_j) = 1 + \frac{1}{\lambda\mu} \geq 1 - \frac{1}{g(H)} \geq \frac{g(\sup|H|) - 1}{g(\sup|H|)} = \delta > 0,$$

where $\delta$ is a positive constant. On the other hand, for $i > j > 1$ we have

$$k_*(e_i, e_j) = 1 + \frac{1}{\mu^2} > 1.$$

Since the sectional curvature of hypersurface of a space form attains its absolute extrema at planes spanned by principal directions, the sectional curvatures of $(M^n, \langle\ ,\ \rangle_*)$ are bounded from below by a positive constant. Hence we may apply Bonnet-Myers Theorem to conclude that $M^n$ is compact and its fundamental group $\pi_1(M^n)$ is finite.

For $n \geq 4$, since $\eta$ has only two principal curvatures of multiplicity 1 and $n-1$, we conclude that $(M^n, \langle\ ,\ \rangle_*)$ is conformally flat (see [**3**, Theorem 7.11]) and without umbilical points. Since $M^n$ is compact, we may apply Theorem 1.4 of M. do Carmo et al. [**2**], to derive that $M^n$ is homeomorphic to a product $S^{n-1}(r_1) \times S^1(r_2)$. Therefore, $\pi_1(M^n)$ is infinite, which implies a contradiction. For $n=3$, we obtain the same conclusion since $\eta$ is conformally flat ([**1**]) without umbilical points. This proves the first part of the theorem.

Now, we will suppose that $M^n$ is compact and $\sup \mathrm{Ric}_M = f(\sup |H|)$, i.e.,

$$\mathrm{Ric}_M(X) \leq \alpha \leq f(\sup |H|), \quad \forall X \in TM, \quad |X| = 1.$$

Hence, we have $\alpha < n-1$ and in an analogous way to the first part of proof, we conclude

$$k_*(e_1, e_j) = 1 + \frac{1}{\lambda\mu} \geq \frac{g(\sup |H|) - 1}{g(\sup |H|)} \geq 0, \quad j > 1.$$

However, we note that it can happen that $g(\sup |H|) - 1 = 0$ since now $\alpha \leq f(\sup |H|)$. On the other hand we have

$$k_*(e_i, e_j) = 1 + \frac{1}{\mu^2} > 1, \quad j > i > 1. \tag{8}$$

It follows that the Ricci curvature of $\eta$ is nonnegative. Since $M$ is compact and $\eta$ has two distinct principal curvatures of multiplicity 1 and $n-1$, we can show by the same argument as the first part of the proof that $\pi_1(M)$ is infinite. Then we can apply Theorem 2 for $\eta$ to conclude that $\eta(M^n)$ is a Clifford torus $S^{n-1}(r_0) \times S^1(\sqrt{1-r_0{}^2})$ with constant mean curvature. In particular, we have that the principal curvatures $1/\lambda$ and $1/\mu$ of $\eta$ are constants. Hence, $\lambda$, $\mu$ and $H$ are constants. Consequently, $\varphi(M^n)$ is a Clifford torus $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$. Since $\sup \mathrm{Ric}_M = f(H)$, it follows that $r^2 \geq (n-1)/n$, which completes the proof of the theorem. □

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